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Extended Fractional Hypergeometric Function and Applications

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Outline

Abstract

Eulers beta, gamma functions (hypergeometric functions) are of the important members of the family of special functions and they plays a vital role in the whole theory of special functions. These hypergeometric functions together with their extension have many applications in research fields such as engineering, chemical, statistics, fractional calculus, and physical problems. In this talk, our discuses have been focus on the extended Eulers beta function, which is developed by using the 2-parameter Mittag-Leffler function as the kernel.

Abstract

We discuss various basic properties and formulas of the extended Eulers beta function such as integral representations, transformation formulas, and summation formulas. We also introduce the logarithmic convexity and some important inequalities for this extended Eulers beta function. Then by using this extended Eulers beta function as kernel, we have generalized hypergeometric functions and study various properties of these extended hypergeometric functions. For application point of view, we have also derived some relations between this extended Eulers beta function and extended fractional derivative operators such as Caputo fractional derivative operator and Riemann-Liouville fractional operators.

Keywords

Eulers beta function, Gamma function, Hypergeometric function, Mittag-Leffler function, Caputo fractional derivative operator, and Riemann-Liouville fractional derivative operator.

Introduction and Preliminaries

The study of the extension of the special function began in early 1990. Chaudhry and Zubair extended the classical gamma function [1] in 1994 for the first time, as defined in the defined in the following.

$$\Gamma(x_1; r) = \int_0^\infty t^{x_1 - 1} e^{-t - rt^{-1}} dt, \qquad \Re(x_1) > 0, r > 0. \tag{1}$$

In the sequence, Chaudhry et al. extended the classical Euler beta function [2] in 1997 defined as follows:

$$B(x_1, x_2; r) = \int_0^1 t^{x_1 - 1}(1 - t)^{x_2 - 1} e^{\frac{-r}{t(1-t)}} dt, \tag{2}$$

where $\min\{\Re(x_1), \Re(x_2)\} > 0$ **and** $\Re(r) > 0$.

Introduction and Preliminaries

In the above extension, Chaudhry and Zubair have used an exponential function as kernel to extend the classical Euler beta function and gamma function in terms of integrals. After few years, many researchers have used many known Special function in order to generalize the classical Euler beta function and gamma function. In 2011, Özergin et al.[3] have used the confluent hypergeometric function ${}_1F_1$ to generalize the classical Euler beta function and gamma function and later In 2018, Shadab et al. [4] have adopted as similar method used by Chaudhry and Özergin [2, 3] to extend the classical Euler beta function by using one parameter Mittag-Leffler function $E_{r_1}(z)$.

Extension of beta function is justify not only by the fact that most of properties of beta function is carried over simply but also by the fact that this function is related to other special function for particular values of the variables.

Main Results

Motivated by the above work [2, 3, 4], In 2021 [5], we extend the classical euler beta function by using the 2**-parameter Mittag-leffler function** $E_{r_1,r_2}(z)$ **(Wiman's function).**

The 2**-parameter Mittag-Leffler function(known as Wiman's function) is defined as follows[6].**

$$E_{r_1,r_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(kr_1 + r_2)}, \qquad \Re(r_1) \geq 0, \Re(r_2) \geq 0, z \in C. \quad (3)$$

Extension of Classical Beta function

Definition

An extension of extended beta function $B^{(r)}_{(r_1,r_2)}(x_1, x_2)$ with $r \geq 0$ is defined by the following:

$$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt, \tag{4}$$

where $\min\{\Re(x_1), \Re(x_2)\} > 0$, $\Re(r_1) > 0$, $\Re(r_2) > 0$ and $E_{r_1,r_2}(z)$ is the 2-parameter Mittag-Leffler function.

Remark

Remark

If we set $r_1 = 1 = r_2$ *and* $r = 0$ *in* (4)*, then we obtain the classical Euler Beta function.*

$$B^{(0)}_{(1,1)}(x_1, x_2) = B(x_1, x_2). \tag{5}$$

Properties of Extended Beta function

Theorem

Let $r \geq 0, \min\{\Re(x_1), \Re(x_2)\} > 0, \Re(r_1) > 0$ *and* $\Re(r_2) > 0$*, then extended beta function defined as* (4) *satisfies the Symmetric relation.*

$$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = B^{(r)}_{(r_1,r_2)}(x_2, x_1). \tag{6}$$

Properties of Extended Beta function

Proof of Theorem 1.

From (4) we have

$$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt.$$

By substituting $t = (1-x)$ in the above and interchanging cross-ponding variables, we obtain the following.

$$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = \int_0^1 (1-x)^{x_1-1}x^{x_2-1}E_{r_1,r_2}\left(\frac{-r}{x(1-x)}\right)dx.$$

Then, by definition of an extension of the extended beta function, we obtain our desired result.

$$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = B_{(r_1,r_2)}^{(r)}(x_2,x_1).$$

□

Properties of Extended Beta function

Theorem

Let $r \geq 0, \min\{\Re(x_1), \Re(x_2)\} > 0, \Re(r_1) > 0$ *and* $\Re(r_2) > 0$, *then the extended beta function defined as* (4) *satisfies the Functional relation.*

$$B_{(r_1,r_2)}^{(r)}(x_1, x_2) = B_{(r_1,r_2)}^{(r)}(x_1, x_2 + 1) + B_{(r_1,r_2)}^{(r)}(x_1 + 1, x_2). \tag{7}$$

Properties of Extended Beta function

Proof of Theorem 2.

We apply the definition of (4) to right hand side of the (7) and perform some manipulations with the terms. Then, we obtain our desired result.

$$B_{(r_1,r_2)}^{(r)}(x_1, x_2+1) + B_{(r_1,r_2)}^{(r)}(x_1+1, x_2)$$
$$\int_0^1 t^{x_1-1}(1-t)^{x_2} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt$$
$$+\int_0^1 t^{x_1}(1-t)^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt \tag{8}$$

Properties of Extended Beta function

Proof of Theorem 2.

$$
\begin{aligned}
&B^{(r)}_{(r_1,r_2)}(x_1, x_2+1) + B^{(r)}_{(r_1,r_2)}(x_1+1, x_2)\\
&= \int_0^1 [t^{-1} + (1-t)^{-1}] t^{x_1}(1-t)^{x_2} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt.\\
&= \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt.\\
&= B^{(r)}_{(r_1,r_2)}(x_1, x_2).
\end{aligned} \tag{9}
$$

□

Properties of Extended Beta function

Theorem

Let $r \geq 0, \Re(x_1 + s) > 0, \Re(x_2 + s) > 0, \Re(r_1) > 0, \Re(r_2) > 0$ *and* $\Re(s) > 0$*, then the Mellin Transform of an extended beta function defined as* (4) *is given by the following.*

$$\int_0^\infty r^{(s-1)} B_{(r_1,r_2)}^{(r)}(x_1, x_2) dr = B(x_1 + s, x_2 + s) \Gamma_0^{(r_1,r_2)}(s). \tag{10}$$

Properties of Extended Beta function

Proof of Theorem 6.

Upon multiplying an extended beta function defined as (4) by $r^{(s-1)}$ and integrating it with respect to r from limit $r = 0$ to $r = \infty$, we obtain the following.

$$\begin{aligned}
&\int_0^\infty r^{(s-1)} B^{(r)}_{(r_1,r_2)}(x_1, x_2) dr \\
&= \int_0^\infty r^{(s-1)} \left(\int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt \right) dr. \\
&= \int_0^1 t^{x_1-1}(1-t)^{x_2-1} \left(\int_0^\infty r^{(s-1)} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dr \right) dt.
\end{aligned} \tag{11}$$

Properties of Extended Beta function

Proof of Theorem 6.

Substitute the above $v = \frac{r}{t(1-t)}$, then we obtain the following.

$$\int_0^\infty r^{(s-1)} B^{(r)}_{(r_1,r_2)}(x_1,x_2)dr = \int_0^1 t^{x_1+s-1}(1-t)^{x_2+s-1}\left(\int_0^\infty v^{s-1}E_{r_1,r_2}(-v)dv\right)dt. \tag{12}$$

Then, we have

$$\int_0^\infty r^{(s-1)} B^{(r)}_{(r_1,r_2)}(x_1,x_2)dr = B(x_1+s,x_2+s)\Gamma_0^{(r_1,r_2)}(s). \tag{13}$$

Inequalities of Extended Beta function

Theorem

Assume that:

- x, y, x_1, y_1 *are non-zero and non-negative numbers such that* $(x - x_1)(y - y_1) \geq 0$,
- $r_1 \in [0, 1]$ *and* $r_2 \in [0, 1]$.

Then,

$$B^{(r)}_{(r_1,r_2)}(x, y_1) \cdot B^{(r)}_{(r_1,r_2)}(x_1, y) \leq B^{(r)}_{(r_1,r_2)}(x, y)B^{(r)}_{(r_1,r_2)}(x_1, y_1). \quad (14)$$

Inequalities of Extended Beta function

Corollary

Assuming that $x, y > 0$, $r_1 \in [0, 1]$ *and* $r_2 \in [0, 1]$, *then:*

$$\left[B^{(r)}_{(r_1,r_2)}(x, y)\right]^2 \geq B^{(r)}_{(r_1,r_2)}(x, x) \cdot B^{(r)}_{(r_1,r_2)}(y, y). \tag{15}$$

Inequalities of Extended Beta function

Theorem

The map $(x, y) \longmapsto B^{(r)}_{(r_1,r_2)}(x, y)$ *is logarithmically convex on* $\mathbb{R}^+ \times \mathbb{R}^+ \ \forall\ u \geq 0$*, with* $r_1 \in [0, 1]$ *and* $r_2 \in [0, 1]$*. Moreover:*

$$\left[B^{(r)}_{(r_1,r_2)}\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\right]^2 \leq B^{(r)}_{(r_1,r_2)}(x_1, y_1) \cdot B^{(r)}_{(r_1,r_2)}(x_2, y_2). \tag{16}$$

Extension of Hypergeometric functions

Then, later Jain et al. [7], introduced a new extension of Gauss hypergeometric function and confluent hypergeometric function by using extended beta function given in (4).

Definition

A *new extended Gauss hypergeometric function* $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$ is defined as follows:

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_1 + n, p_2 - p_1)}{B(p_1, p_2 - p_1)} (p_0)_n \frac{z^n}{n!}, \quad (17)$$

where $\Re(p_2) > \Re(p_1) > 0,\ \Re(r_1) > 0, \Re(r_2) > 0,\ r \geq 0,\ | z |< 1$ and $B^{(r)}_{(r_1,r_2)}(x_1, x_2)$ extended *beta function*.

Extension of Hypergeometric functions

Definition

A *new extended confluent hypergeometric function* $\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z)$ is defined as follows:

$$\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z) = \sum_{n=0}^{\infty} \frac{B^{(r)}_{(r_1,r_2)}(p_1 + n, p_2 - p_1)}{B(p_1, p_2 - p_1)} \frac{z^n}{n!}, \tag{18}$$

where $\Re(p_2) > \Re(p_1) > 0,\ \Re(r_1) > 0, \Re(r_2) > 0,\ r \geq 0$ and $B^{(r)}_{(r_1,r_2)}(x_1, x_2)$ extended beta function.

Properties of Extended Hypergeometric functions

Theorem

Consider $F_{(r_1,r_2)}^{(r)}(p_0, p_1, p_2; z)$ and $\Phi_{(r_1,r_2)}^{(r)}(p_1, p_2; z)$ functions. Then, the following functional relations hold:

(1)

$$F_{(r_1,r_2)}^{(r)}(p_0, p_1, p_2; z) = \frac{p_1}{p_2} F_{(r_1,r_2)}^{(r)}(p_0, p_1 + 1, p_2 + 1; z) + \frac{(p_2 - p_1)}{p_1} F_{(r_1,r_2)}^{(r)}(p_0, p_1, p_2 + 1; z), \quad (19)$$

where $\Re(p_2) > 0, \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0, \ r \geq 0$ and $\mid z \mid < 1$.

(2)

Properties of Extended Hypergeometric functions

Theorem

Consider $F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z)$ and $\Phi^{(r)}_{(r_1,r_2)}(p_1,p_2;z)$ functions. The following Summation relations hold:
(1)

$$F^{(r)}_{(r_1,r_2)}(p_0,p_1,p_2;z) = (p_2-p_1) \sum_{k=0}^{\infty} \frac{(p_1)_k}{(p_2)_{k+1}} F^{(r)}_{(r_1,r_2)}(p_0,p_1+1,p_2+k+1;z), \tag{21}$$

where $\Re(p_2)>0, \Re(p_1)>0, \Re(r_1)>0, \Re(r_2)>0,\ r\geq 0$ *and* $\mid z \mid < 1$.

Properties of Extended Hypergeometric functions

Theorem

Consider $\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z)$ function . The following Summation relation hold:
(2)

$$\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z) = (p_2 - p_1) \sum_{k=0}^{\infty} \frac{(p_1)_k}{(p_2)_{k+1}} \Phi^{(r)}_{(r_1,r_2)}(p_1 + k, p_2 + k + 1; z), \tag{22}$$

where $\Re(p_2) > 0, \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0$ *and* $r \geq 0.$

Properties of Extended Hypergeometric functions

Theorem

Consider $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$ *function. The following integral representation hold:*
(1)

$$F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z) = \frac{1}{B(p_1, p_2 - p_1)}$$

$$\int_0^1 t^{p_1-1}(1-t)^{p_2-p_1-1}(1-zt)^{-p_0} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt, \quad (23)$$

where $\Re(p_2) > \Re(p_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0,\ r \geq 0$ *and* $\mid z \mid < 1.$

Properties of Extended Hypergeometric functions

Theorem

Consider $\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z)$ function. The following integral representation hold:
(2)

$$\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z) = \frac{1}{B(p_1, p_2 - p_1)} \int_0^1 t^{p_1-1}(1-t)^{p_2-p_1-1} e^{zt} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt, \tag{24}$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$ *and* $r \geq 0$.

Properties of Extended Hypergeometric functions

Theorem

The Mellin Transformations for $F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z)$ *are given by :*
(1)

$$\mathbf{M}\{F^{(r)}_{(r_1,r_2)}(p_0, p_1, p_2; z); s\} = \frac{\Gamma_0^{(r_1,r_2)}(s)B(p_1+s, p_2-p_1+s)}{B(p_1, p_2-p_1)} F(p_0, p_1+s, p_2+2s; z), \quad (25)$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$, $s \geq 0$, $r \geq 0$ *and* $\mid z \mid < 1$.

Properties of Extended Hypergeometric functions

Theorem

The Mellin Transformations for $\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z)$ *are given by : (2)*

$$\mathbf{M}\{\Phi^{(r)}_{(r_1,r_2)}(p_1, p_2; z); s\} = \frac{\Gamma_0^{(r_1,r_2)}(s) B(p_1 + s, p_2 - p_1 + s)}{B(p_1, p_2 - p_1)} \Phi(p_1 + s, p_2 + 2s; z), \quad (26)$$

where $\Re(p_2) > \Re(p_1) > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$, $s \geq 0$ *and* $r \geq 0$.

G. W. Leibniz (1695–1697)

In 1697, Leibniz mentioned in his letter to J. Wallis and J. Bernulli about the possible approach to fractional-order differentiation in that sense, that for non-integer values of n the definition could be the following:

$$\frac{d^n e^{mx}}{dx^n} = m^n\, e^{mx}$$

L. Euler (1730)

$$\frac{d^n x^m}{dx^n} = m(m-1)\cdots(m-n+1)x^{m-n}$$
$$\Gamma(m+n) = m(m-1)\cdots(m-n+1)\Gamma(m-n+1)$$
$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

Euler suggested to use this relationship also for negative or non-integer (rational) values of n. Taking $m = 1$ and $n = \frac{1}{2}$, Euler obtained:

$$\frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}} = \sqrt{\frac{4x}{\pi}} \quad \left(\frac{2}{\sqrt{\pi}} x^{\frac{1}{2}}\right)$$

Introduction to Fractional Calculus

J. B. J. Fourier.(1820–1822)

The first step to generalization of the notion of differentiation for arbitrary functions was done by J. B. J. Fourier.

J. Liouvilli (1832-1855)

Liouville was the first to point out the existence of the right-sided and left-sided differentials and integrals through his three approaches.

Introduction to Fractional Calculus

G.F.B. Riemann (1826-1866)

The initialized fractional calculus was born lately in the later half of the twentieth century; Riemanns notation is as follows with the complimentary function.

$$D^{-\nu}\left\{f(t)\right\} = \frac{1}{\Gamma(\nu)}\int_c^t (t-\tau)^{\nu-1} f(\tau)\,d\tau + \Psi(\tau)$$

Cauchy formula for *nth* **derivative in complex variables is**

$$f^n(z) = \frac{n!}{j2\pi}\oint \frac{f(\tau)}{(\tau-z)^{n+1}}\,d\tau,$$

and for non-integer $n=\nu$**, a branch point of the function**

- **Since that time the factional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent.**
- **But it was not until 1884 that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematics.**

Introduction to Fractional Calculus

- **By then the theory had been extended to include operators D^{ν}, where ν could be rational or irrational, positive or negative, real or complex.**

- **Thus the name fractional calculus became somewhat of a misnomer. A better description might be differentiation and integration to an arbitrary order.**

Introduction to Fractional Calculus

(1900-2022)

It is practically impossible to name all made important contribution in construction of the early stages of the building of fractional calculus. New era in the development of this branch of science began 40–50 years ago due to numerous application of fractional-type models and is continued up to now. One can mention a large list of areas of application as follows

Fractional operators appear at studying various problems of the Analysis of Differential Equations

Diethelm, K.

The Analysis of Differential Equations of Fractional Order: An Application-Oriented Exposition Using Differential Operators of Caputo Type, Lecture Notes in Mathematics 2004, Springer Verlag, Berlin - Heidelberg (2010)

Introduction to Fractional Calculus

Kilbas, A. A., Srivastava, H. M., Trujillo, J. J.

Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies. 204. Elsevier, Amsterdam, etc. (2006)

Kiryakova, V.

- **Generalized Fractional Calculus and Applications. Harlow, Longman (1994)**
- **On two Saigo's fractional integral operators in the class of univalent functions, Fractional Calculus & Applied**

Introduction to Fractional Calculus

Miller, K. S., Ross, B.

An Introduction to the Fractional Calculus and Fractional Differential Equations, New York, John Wiley and Sons, 1993

Oldham, K. B., Spanier, J

The Fractional Calculus, Academic Press, New York (1974)

Podlubny, I.

Fractional Differential Equations, Academic Press, New York (1999)

Introduction to Fractional Calculus

Samko, S. G., Kilbas, A. A., Marichev, O. I

Fractional Integrals and Derivatives: Theory and Applications. New York and London, Gordon and Breach Science Publishers (1993)

Tenreiro Machado J.A., Kiryakova V., Mainardi F.

History of fractional calculus, Commun. Nonlinear Sci. Numer. Simulat., 16, 1140–1153 (2011)

Introduction to Fractional Calculus

Applications in physics

Hilfer, R.

Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)

In modeling and control

Caponetto, R., Dongola, G., Fortuna, L., and Petráš, I.

Fractional Order Systems: Modeling and Control Applications, World Scientific, Singapore (2010)

Introduction to Fractional Calculus

nano-and cosmic-physics

Uchaikin, V. V.:

Fractional derivatives for physicists and engineers. Vol. I. & II Background and theory. Springer, Berlin - Higher Education Press, Beijing (2013)

Mamikon Gulian on Fractional Calculus & Hidden Physics

The theory of fractional calculus (FC) has successfully been utilized to describe the fractal problems, real life problems in engineering mathematics. In 2021, Jain et al. [9], introduce a new extension of classical Riemann-Liouville fractional derivative operator by the application of extended beta function defined in [8] and also established some good results for this extended Riemann-Liouville fractional derivative operator.

Applications in Fractional Calculus

Definition

$$D^{u,(r)}_{z,(r_1,r_2)}[f(z)] = \begin{cases} \frac{1}{\Gamma(-u)} \int_0^z (z-t)^{-u-1} E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right) f(t)dt, \\ \quad (\Re(u) < 0) \\ \\ \frac{d^m}{dz^m}\{D^{u-m,(r)}_{z,(r_1,r_2)} f(z)\}, \\ \quad (m-1 \leq \Re(u) < m, \;\; m \in \mathbb{N}) \end{cases} \tag{27}$$

here $(\min\{\Re(r_1), \Re(r_2)\} > 0$, $\Re(r) > 0$ and $E_{r_1,r_2}(z)$ is 2-parameter Mittag-Leffler function).

Applications in Fractional Calculus

Theorem

Assume $\Re(u) < 0$, $\Re(k) < -1$ *then*

$$D_{z,(r_1,r_2)}^{u,(r)}[z^k] = \frac{B_{(r_1,r_2)}^{(r)}(k+1,-u)}{\Gamma(-u)} z^{k-u}, \tag{28}$$

where $B_{(r_1,r_2)}^{(r)}(x,y)$ is extended beta function.

Applications in Fractional Calculus

Theorem

Consider $\Re(u) < 0$, $\Re(k) > 0$, $\Re(l) > 0$ *and* $\mid z \mid < 1$ *then*

$$D_{z,(r_1,r_2)}^{k-u,(r)}[z^{k-1}(1-z)^{-l}] = \frac{\Gamma(k)}{\Gamma(u)} z^{u-1} F_{(r_1,r_2)}^{(r)}(l,k,u;z), \qquad (29)$$

where, $F_{(r_1,r_2)}^{(r)}(a,b,c;z)$ **is extended Gauss Hypergeometric function.**

Applications in Fractional Calculus

Theorem

Consider $\Re(u) < 0$,$\Re(\beta) > 0$ and $s > 0$ then the Mellin transform for the extended Riemann-Liouville fractional derivative operator defined as (27), is given by the following:

$$\mathbf{M}\left[D^{u,(r)}_{z,(r_1,r_2)}[z^{\beta}];s\right] = \frac{z^{\beta-u}}{\Gamma(-u)}\Gamma_0^{r_1,r_2}(s)B(\beta+s+1,-u+s). \quad (30)$$

Applications in Fractional Calculus

Later in [10], we define a new extension of the classical Caputo fractional derivative operator by the application of extended beta function.

Definition

$$D_{z,(r_1,r_2)}^{u,(r)}[f(z)] = \frac{1}{\Gamma(m-u)} \int_0^z (z-t)^{m-u-1} E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right) \frac{d^m}{dt^m} f(t)dt. \tag{31}$$

Here, $\min\{\Re(r_1), \Re(r_2)\} > 0$, $\Re(r) > 0$, $m-1 < \Re(u) < m$, $m \in N$; and $E_{r_1,r_2}(z)$ is the 2-parameter Mittag-Leffler function.

Applications in Fractional Calculus

Theorem

Consider $m-1<\Re(u)<m$, $\Re(u)<\Re(k)$. *Then*

$$D^{u,(r)}_{z,(r_1,r_2)}[z^k]=\frac{\Gamma(k+1)}{\Gamma(k-u+1)}\frac{B^{(r)}_{(r_1,r_2)}(k-m+1,m-u)}{B(k-m+1,m-u)}z^{k-u}. \tag{32}$$

where, $B^{(r)}_{(r_1,r_2)}(x,y)$ **is extended beta function.**

Applications in Fractional Calculus

Theorem

Consider $\Re(\lambda) > m - 1$ *and* $s > 0$*. Then, the Mellin transform for the extended Caputo fractional derivative operator defined as* (31) *is given by the following expression:*

$$\mathbf{M}\left[D^{u,(r)}_{z,(r_1,r_2)}[z^{\lambda}]; s\right] = \frac{\Gamma(\lambda+1)\Gamma^{(r_1,r_2)}_{0}(s)}{\Gamma(\lambda-m+1)\Gamma(m-u)} B(m-u+s, \lambda-m+s+1)z^{\lambda-u}. \tag{33}$$

Conclusion Remark

We conclude our investigation by remarking that the results presented in this presentation are easily converted by using the interesting known extension of the beta function and other special functions such as trigonometric functions, exponential function, beta function, and Gamma function, it will increase the rate to find out the solution of differential equations symmetrically. The results presented here are important for the extension of other special functions in the theory of Special functions and Fractional calculus.

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References I

[1] Chaudhry, M.A.; Zubair, S.M. Generalized incomplete gamma functions with Applications. J. Comput. Appl. Math. 1994, 55, 99124.

[2] Chaudhry, M.A.; Qadir, A.; Rafique, M.; Zubair, S.M. Extension of Eulers Beta function. J. Comput. Appl. Math. 1997, 78, 1932.

[3] Özergin, E.; Özarslan, M.A.; Altn, A. Extension Gamma, Beta and Hypergeometric Functions. J. Comput. Appl. Math. 2011, 235, 46014610.

References II

[4] Shadab, M.S.; Jabee, S.J.; Choi, J.C. An extended Beta function and its applications. Far East J. Math. Sci. (FJMS) 2018, 103, 235251.

[5] Goyal, R.; Momani, S.; Agarwal, P.; Rassias, M.T. An Extension of Beta Function by Using Wimans Function. Axioms 2021, 1, 0.

[6] Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. Mittage Leffler Functions Related Topics and Application, 2nd ed.; Springer: New York, NY, USA, 2020.

References III

[7] Jain, S.; Goyal, R.; Agarwal, P.; Lupica, A.; Cesarano, C. Some results of extended beta function and hypergeometric functions by using Wiman's function, Mathemtics 2021, 9(22), 2944.

[8] Goyal, R.; Agarwal, P.; Momami, S.; Rassias, M.T. An Extension of Beta Function by Using wiman's function. Axioms, 2021,10, 187.

References IV

[9] Jain, S.; Goyal, R.; Agarwal, R.P.; Agarwal, P., Certain Extension of Riemann Liouville Fractional Derivative Operator by Using Wiman's Function, book chapter edited by Prof. Panos. M. Pardolas and Prof. Themistocles M. Rassias, Mathemtical Analysis, Optimization, Approximation and Applications, 2021.

[10] Goyal, R.; Argawal, P.; Parmentier, A.; Cesarano, C. An extension of Caputo fractional derivative operator by use of Wimans function. Symmetry 2021,

[11] Wiman, A. U ber der Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$. Acta Math. 1905, 29, 191-201.r

References V

[12] Shadab, M.S.; Jabee, S.J.; Choi, J.C. An extended Beta function and its applications, Far East J. Math. Sci. (FJMS), 2018, 103, 235–251.

[13] Chaudhry, M.A.; Qadir, A.; Srivastava, H.M.; Paris, R.B. Extended hypergeometric and confluent hypergeometric functions, Appl. Math. Comput., 2004, 159, 589–602.

[14] Olver, W.J.F.; Lozier, W.D.; Boisvert, F.R.; Clark, W.C.,NIST Handbook of Mathematical Functions ; Cambridge University Press, New York, NY, USA, 2010.

References VI

[15] Rainville, E.D., Special Functions; Macmillan: New York, NY, USA, 1960.


Thank you!