Common properties of *Lip* and $\mathcal{H}^\infty$ functions (preliminary report)

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Plan of talk:

1. Basic definitions of *Lip* and $\mathcal{H}^\infty$ and basic background
2. Norm attainment
3. Combining *Lip* and $\mathcal{H}^\infty$

§1 **Basic definitions**

Lip Let (M, d) be a metric space with an arbitrary fixed point $x_0 \in M$ and let Y be a Banach space. Define $Lip_{x_0}(M, Y) :\equiv \{f : M \to Y \mid f(x_0) = 0 \text{ and } \exists C \geq 0$ such that $\|f(x) - f(y)\| \leq Cd(x, y), \forall x, y \in M\}$. $(*)$

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$Lip_{x_0}(M, Y)$ is a Banach space with norm $L(f) :\equiv \inf\{C \mid (*) \text{ holds}\}$. (In other words, $C = \sup_{x \neq y \in M} \frac{||f(x)-f(y)||}{d(x,y)}$.)

$\mathbf{H}^\infty$ For complex Banach spaces X, Y, with B_X being the open unit ball of X, let $\mathcal{H}^\infty(B_X, Y) :\equiv \{f : B_X \to Y \mid f$ is holomorphic and bounded $\}$.

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Notation: When $Y =$ the scalars $\mathbb{K}$, we simply write $Lip_{x_0}(M)$ and $\mathcal{H}^{\infty}(B_X)$.

Basic Background

Three basic results:
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Theorem (Ng, 1971, following Dixmier-1948) For a Banach space Z, suppose that there exists a locally convex topology τ on $\overline{B}_Z$ making it $\tau-$compact. Then there exists a Banach space V whose dual is isometrically isomorphic to Z.

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(2) Let $Z = \mathcal{H}^\infty(B_X)$ and consider $\{\delta_x \mid x \in B_X\}$, which is a subset of $\mathcal{H}^\infty(B_X)^*$. Its predual, the Banach space generated by $\{\delta_x \mid x \in B_X\}$, is denoted $\mathcal{G}(B_X)$.

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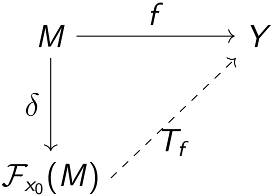
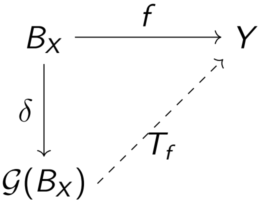


($f : M \to Y$ is Lipschitz)



($f : B_X \to Y$ is bounded & holomorphic)

In each case, the norm of f (as a Lipschitz or an $\mathcal{H}^\infty$ function) equals the norm of the linear operator T_f. In particular, taking $Y = \mathbb{K}$, we see that both $Lip_{x_0}(M)$ and $\mathcal{H}^\infty(B_X)$ are dual spaces.

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Our question: Exactly what are these dense subsets?

Case of $Lip_0[0,1] = (\mathcal{F}_0[0,1])^*$

A good first guess, in general, is that $f \in Lip_{x_0}(M)$ is norm-attaining $\iff$ $L(f)$ is attained. That is, the guess is that $f \in Lip_{x_0}(M)$ is norm-attaining $\iff$ $\exists x, y \in M$ such that $L(f) = \frac{|f(x)-f(y)|}{d(x,y)}$.

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Example *(Kadets, Martin, and Soloviova (2016)) Let $C \subset [0,1]$ be a Cantor set of positive Lebesgue measure. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \int_0^x \chi_C(t)dt$. Then $f \in Lip_0[0,1]$, and $L(f) = ||f'||_\infty = ||\chi_C||_\infty = 1$. For a contradiction, let's suppose that $\exists g \in Lip_0[0,1]$ with $L(f-g) < 1/3$ (and so $L(g) > 2/3$). Suppose further that $\exists x, y \in [0,1]$ such that $L(g) = \frac{|g(x)-g(y)|}{|x-y|}$.*

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Now observe that if $L(g)$ is attained at $\frac{|g(x)-g(y)|}{|x-y|}$, then for every $x', y' \in [x,y]$, $L(g)$ is also equal to $\frac{|g(x')-g(y')|}{|x'-y'|}$. So, let's choose x', y' so that $C \cap (x', y') = \emptyset$. By our definition of f, $f(x') = f(y')$. This yields the following contradiction:

$$1/3 > L(f-g) \geq \frac{|(g-f)(x') - (g-f)(y')|}{|x'-y'|} = \frac{|g(x')-g(y')|}{|x'-y'|} = L(g) > 2/3.$$

There are natural situations of (M, d) where the "good guess" would be correct. For instance, if M is compact and $\mathcal{F}_0(M)$ has the Radon-Nikodym property, then in fact the norm attaining elements of $\mathcal{F}_{x_0}(M)^* = Lip_{x_0}(M)$ are precisely those for which $\exists x, y \in M$ with $L(f) = \frac{|f(x)-f(y)|}{d(x,y)}$.

Case of $\mathcal{H}^\infty(B_X)$

When $X = \mathbb{C}$: As already indicated, a predual of $\mathcal{H}^\infty(B_X)$ is the closed span (in $\mathcal{H}^\infty(B_X)^*$) of the set $\{\delta_x \mid x \in B_X\}$.

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Theorem (see also S. Fisher, 1969) A function $f \in \mathcal{H}^\infty(\mathbb{D})$ is norm attaining relative to the predual if and only if

$$m(\{z \in \mathbb{C} \mid |z| = 1 \text{ and } |f^*(z)| = ||f||_\infty\}) > 0.$$

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(We are using the fact that for $f \in \mathcal{H}^\infty(\mathbb{D})$, the radial limit $f^*(z) = lim_{r \to 1} f(z)$ exists a.e. on the unit circle.)

Without loss, $||f||_\infty = 1$.

$\Leftarrow$ Let $K = \{z \in \mathbb{C} \mid |z| = 1 \text{ and } |f^*(z)| = 1\}$ have positive measure

and define $g \in L_1$ by $g(z) = \begin{cases} 0 & \text{if } z \notin K \\ \frac{\overline{f^*(z)}}{m(K)} & \text{if } z \in K \end{cases}$

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Then $||g||_{L_1} = 1$ and for any $h \in H_0^1$, $\int_{|z|=1}(g+h)f^* = \int_{|z|=1} gf^* + \int_{|z|=1} hf^* = \int_K gf^* = 1$.

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Here we have used Cauchy's integral formula and the fact that $h(0) = 0$.

$\Rightarrow$ (idea)
Suppose that for some $[g] \in L^1/H_0^1, ||[g]|| = 1$, and $f([g]) = 1$. Since $[g] = g + H_0^1$ and $\forall h \in H_0^1,\ \int_{|z|=1} f^*(g+h) = \int_{|z|=1} f^*g$, we have that $||g||_1 = \int_{|z|=1} |g(z)| = 1$.

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For a contradiction, suppose that K $(\equiv \{z \mid |z| = 1 \text{ and } |f^*(z)| = 1\})$ has measure 0. Call $K_n = \{z \mid |z| = 1 \text{ and } |f^*(z)| < \frac{n-1}{n}\}$. So, $\{|z| = 1\} \backslash K = \cup_{n=1}^{\infty} K_n$. Then $1 = \int_{\{\{|z|=1\}\backslash K_n\}} |g| + \int_{K_n} |g| = \int_{|z|=1} f^* g$
$\leq \int_{\{\{|z|=1\}\backslash K_n\}} |f^* g| + \int_{K_n} |f^* g|$
$\leq \int_{\{\{|z|=1\}\backslash K_n\}} |g| + \frac{n-1}{n} \int_{K_n} |g|$. From this it follows that $\forall n$, $\int_{K_n} |g| = 0$.
Therefore, $g = 0$ a.e. on $\cup_n K_n$, and since the measure of K is 0, we conclude that $\int_{|z|=1} |g| = 0$, which is a contradiction to $||g||_1 = 1$. $\square$

Remarks on the higher dimensional case

Much less is known for $\mathbb{C}^n, n \geq 2$. For "important" open unit balls $B \subset \mathbb{C}^n$ (with some "important" norm), it is known that $\mathcal{H}^\infty(B)$ is the dual of a space of type L^1/H^1_0. For instance:

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Proposition Let $f \in \mathcal{H}^\infty(\mathbb{D}^n)$ such that the set $\{w \in \mathbb{T}^n \mid |f^*(w)| = ||f||_\infty\}$ has positive measure. Then f attains its norm as an element of $L^1(\mathbb{T}^n)/\mathcal{H}^1_0(\mathbb{T}^n)$.

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For B^n with unit sphere S^n, one has:

Proposition Let $f \in \mathcal{H}^\infty(B^n)$ such that the set $\{w \in S^n \mid |f^*(w)| = ||f||_\infty\}$ has positive measure. Then attains its norm as an element of $L^1(S^n)/\mathcal{H}_0^1(S^n)$. Conversely, if there is $\varphi \in L^1(S^n), ||\varphi||_{L^1(S^n)} = 1$, such that $\int_{S^n} \varphi \cdot f = ||f||$, then the set $\{w \in S^n \mid |f^*(w)| = ||f||\}$ has positive measure.

Combining Lip and $\mathcal{H}^\infty$

Here, our Lipschitz functions will always be defined on $(M, d) = (B_X, \|\cdot\|)$, where X is a complex Banach space. Also, x_0 will be $0 \in B_X$, meaning that we'll be dealing with $Lip_0(B_X, Y)$.

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Definition Let $\mathcal{H}L_0(B_X, Y) :\equiv Lip_0(B_X, Y) \cap \mathcal{H}^\infty(B_X, Y)$, endowed with the Lipschitz norm

$$f \in \mathcal{H}L_0(B_X, Y) \rightsquigarrow L(f).$$

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This "combination" is reminiscent of work by others, e.g. Bruna & Tugores (and their predecessors), who - among other things - studied $\mathcal{A}^1(\mathbb{D}) :\equiv \{f \in \mathcal{A}(\mathbb{D}) \mid f' \in \mathcal{A}(\mathbb{D})\}$ (where $\mathcal{A}(\mathbb{D})$ is the disc algebra, i.e. $\mathcal{A}(\mathbb{D}) :\equiv \{f : \mathbb{D} \to \mathbb{C} \mid f$ is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}\}$.)

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(c) $\forall f \in \mathcal{H}L_0(B_X, Y), \quad L(f) = ||df||$.

(d) (With $Y = \mathbb{C}$) The mapping

$$f \in \mathcal{H}L_0(B_X) \rightsquigarrow df \in \mathcal{H}^\infty(B_X, X^*)$$

is an isometry <u>into</u>. It is <u>onto</u> $\iff X = \mathbb{C}$.

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Proof idea for (b) $\subseteq$: Suppose that $f \in \mathcal{H}L_0(B_X, Y)$. Then its derivative $df \in \mathcal{H}(B_X, \mathcal{L}(X, Y))$. For any $x, y \in B_X$,

$$||df(x)(y)|| = \lim_{h \to 0} \frac{||f(x + hy) - f(x)||}{||hy||} \leq L(f),$$

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$\supseteq$: Now let $f \in \mathcal{H}^\infty(B_X, Y), f(0) = 0$, such that df is an $H^\infty-$ function. Apply the mean value theorem to $x, y \in B_X$:

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If $X = Y = \mathbb{C}$, then we've shown that $\mathcal{H}L_0(\mathbb{D}) = \{f \in \mathcal{H}^\infty(\mathbb{D}) \mid f(0) = 0 \text{ and } f' \in \mathcal{H}^\infty(\mathbb{D})\}$.