Gradient estimates for harmonic and generalized harmonic functions

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The class of (α, β)-harmonic functions in $\mathbb{D}$

Let, for $\alpha, \beta \in \mathbb{C}$

$$L_{\alpha,\beta} = (1 - |z|^2)\left((1 - |z|^2)\frac{\partial^2}{\partial z \partial \overline{z}} + \alpha z \frac{\partial}{\partial z} + \beta \overline{z} \frac{\partial}{\partial \overline{z}} - \alpha\beta\right)$$

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1. $L_{0,0} = 4^{-1}(1 - |z|^2)^2 \Delta$,
2. A function u in $C^2(\mathbb{D})$ is (α, β)-harmonic if $L_{\alpha,\beta} u = 0$.
3. The vector space of all such functions is denoted by $h_{\alpha,\beta}(\mathbb{D})$.

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3. The vector space of all such functions is denoted by $h_{\alpha,\beta}(\mathbb{D})$.
4. If $\alpha > -1$ the $(0, \alpha)$-harmonic functions are called α-harmonic
5. if $\alpha \in \mathbb{R}$ the $(\frac{\alpha}{2}, \frac{\alpha}{2})$-harmonic functions are called T_α-harmonic functions.

Early studies of (α, β)-harmonic functions

The spaces we deal with appeared in a more general contest of the unit ball in $\mathbb{C}^n$ in connection with H^p theory for the Heisenberg group, there $z\partial_z$ and $\overline{z}\partial_{\overline{z}}$ are replaced by operators

$$R = \sum_j z_j \partial_{z_j}, \qquad \overline{R} = \sum_j \overline{z_j} \partial_{\overline{z_j}}.$$

[Gel] D. Geller, Some results in H^p theory for the Heisenberg group, Duke Math. J., 47 (1980) 365–390.
Later these spaces were investigated in the paper
[ABC] P. Ahern, J. Bruna, C. Cascante, H^p-theory for generalized $\mathcal{M}$-harmonic functions in the unit ball, Indiana Univ. Math. J., 45(2) (1996) 103–135.
The case of the unit disc was treated in
[KO] M. Klintborg, A. Olofsson, A series expansion for generalized harmonic functions, Analysis and Math. Physics, vol 11, (2021)

The function $u_{\alpha,\beta}$: pointwise and L^1 estimate

Let $\alpha, \beta \in \mathbb{C}$. The following function is used to construct integral representations of (α, β)-harmonic functions:

$$u_{\alpha,\beta}(z) = \frac{(1-|z|^2)^{\alpha+\beta+1}}{(1-z)^{\alpha+1}(1-\overline{z})^{\beta+1}}, \qquad |z| < 1 \tag{1}$$

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Let $\alpha, \beta \in \mathbb{C}$. Then $u_{\alpha,\beta} \in h_{\alpha,\beta}(\mathbb{D})$ and

$$|u_{\alpha,\beta}(z)| \le e^{\frac{\pi}{2}|\Im\alpha - \Im\beta|} \frac{(1-|z|^2)^{\Re\alpha+\Re\beta+1}}{|1-z|^{\Re\alpha+\Re\beta+2}}, \quad |z| < 1. \tag{2}$$

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Moreover, if $\Re\alpha + \Re\beta > -1$, the following estimate holds:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u_{\alpha,\beta}(re^{i\theta})| d\theta \le e^{\frac{\pi}{2}|\Im\alpha - \Im\beta|} \frac{\Gamma(\Re\alpha + \Re\beta + 1)}{\Gamma^2\left(\frac{\Re\alpha+\Re\beta}{2} + 1\right)}, \quad 0 \le r < 1. \tag{3}$$

The derivatives of $u_{\alpha,\beta}$, KO

$$\frac{\partial u_{\alpha,\beta}}{\partial z}(z) = \left(-(\alpha+\beta+1)\frac{\overline{z}}{1-|z|^2} + \frac{\alpha+1}{1-z}\right) u_{\alpha,\beta}(z),$$

$$\frac{\partial u_{\alpha,\beta}}{\partial \overline{z}}(z) = \left(-(\alpha+\beta+1)\frac{z}{1-|z|^2} + \frac{\beta+1}{1-\overline{z}}\right) u_{\alpha,\beta}(z).$$

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$$\partial^k u_{\alpha,\beta}(z) = \frac{(\alpha+1)_k}{(1-z)^k} u_{\alpha,\beta}(z) + \overline{z} g_k(z), \quad |z|<1, \quad k \in \mathbb{N}$$

where, for $z \in \mathbb{D}$, $g_1(z) = -(\alpha+\beta+1)(1-|z|^2)^{-1} u_{\alpha,\beta}(z)$ and

$$g_k(z) = \frac{(\alpha+1)_{k-1}}{(1-z)^{k-1}} g_1(z) + \partial^{k-1} g_{k-1}(z) \in C^\infty(\mathbb{D}), \qquad k \geq 2.$$

The (α, β)-Poisson kernel

In the following we assume that

$$\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}^- = \mathbb{C} \setminus \{-1, -2, \dots\}, \Re\alpha + \Re\beta > -1.$$

The (α, β)-Poisson kernel is defined by

$$P_{\alpha,\beta}(z, \zeta) = c_{\alpha,\beta} u_{\alpha,\beta}(z\bar{\zeta}), \qquad z \in \mathbb{D}, \quad \zeta \in \mathbb{T} \tag{4}$$

where a normalizing constant $c_{\alpha,\beta}$ is given by

$$c_{\alpha,\beta} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}. \tag{5}$$

This choice of $c_{\alpha,\beta}$ ensures that

$$\lim_{|z|\to 1} \int_{\mathbb{T}} P_{\alpha,\beta}(z, \zeta) dm(\zeta) = 1.$$

The (α, β)-Poisson integral

Definition

For $f \in L^1(\mathbb{T})$ the (α, β)-Poisson integral of f is defined by

$$P_{\alpha,\beta}[f](z) = \int_{\mathbb{T}} P_{\alpha,\beta}(z, \zeta) f(\zeta) dm(\zeta), \qquad z \in \mathbb{D}. \tag{6}$$

Since $P_{\alpha,\beta}(z, \zeta)$ is (α, β) - harmonic with respect to $z \in \mathbb{D}$ for each fixed $\zeta \in \mathbb{T}$ we have

$$P_{\alpha,\beta}[f] \in h_{\alpha,\beta}(\mathbb{D}), \qquad f \in L^1(\mathbb{T}).$$

Definition of $h^p_{\alpha,\beta}(\mathbb{D})$ spaces

Definition

Let $1 \le p \le \infty$. The space $h^p_{\alpha,\beta}(\mathbb{D})$ consists of all $u \in h_{\alpha,\beta}(\mathbb{D})$ such that

$$\|u\|_{\alpha,\beta;p} = \sup_{0\le r<1} \left(\int_{\mathbb{T}} |u(r\zeta)|^p dm(\zeta) \right)^{\frac{1}{p}} < \infty, \qquad 1 \le p < \infty,$$

$$\|u\|_{\alpha,\beta;p} = \sup_{z\in\mathbb{D}} |u(z)| < +\infty, \qquad p = +\infty.$$

Theorem

Let $u \in h^p_{\alpha,\beta}(\mathbb{D})$, $1 < p \le \infty$. Then there is a unique $\psi \in L^p(\mathbb{T})$ such that $u = P_{\alpha,\beta}[\psi]$.

Jelena Gajić, Miloš Arsenović and Miodrag Mateljević, *H^p* theory of separately (α, β)-harmonic functions in the unit polydisc, arXiv math.CV 2305.10858 (2023)

A review of related results: T_α-harmonic case

$$\|Du(z)\| = \sup\{|Du(z)\zeta| : |\zeta| = 1\} = |u_z(z)| + |u_{\overline{z}}(z)|. \qquad (7)$$

1. Colonna, 1989: If $u : \mathbb{D} \to \mathbb{D}$ is harmonic, then for $z \in \mathbb{D}$

$$\|Du(z)\| \le \frac{4}{\pi}\frac{1}{1-|z|^2}.$$

2. Chen, Vuorinen, 2015: If $\alpha > -1$ and $u \in h^\infty_{\alpha/2,\alpha/2}(\mathbb{D})$, then

$$\|Du(z)\| \le \frac{2+\alpha+|\alpha z|}{1-|z|^2}\|u\|_{\alpha/2,\alpha/2,+\infty}, \qquad z \in \mathbb{D}.$$

3. Khalfallah, Mateljević, 2021: Under the same assumptions:

$$\|Du(0)\| \le \frac{2(\alpha+2)}{\pi}\frac{\Gamma^2\left(\frac{\alpha}{2}+1\right)}{\Gamma(\alpha+1)}\|u\|_{\alpha/2,\alpha/2,+\infty}.$$

A review of related results: α-harmonic case

1. Li, Wang, Xiao, 2017:
   Suppose $u \in C(\overline{\mathbb{D}}) \cap h_{0,\alpha}(\mathbb{D})$ where $\alpha > -1$. Then

$$\|Du(z)\| \le 2(\alpha+2)\frac{\Gamma(\alpha+1)}{\Gamma^2\left(\frac{\alpha}{2}+1\right)}\frac{1}{1-|z|^2}\|u\|_{0,\alpha,+\infty}, \quad z \in \mathbb{D}.$$

2. P. Li, Rasila, Wang,2020; M. Li, X. Chen, 2022:
   For φ in $C(\mathbb{T})$ we have:

$$\|DP_{0,\alpha}[\varphi](z)\| \le \begin{cases} 2^{1-\alpha}(1-|z|^2)^{\alpha-1}\|\varphi\|_\infty, & -1<\alpha<0, \\ (1+\alpha)2^{1+\alpha}\frac{1}{1-|z|^2}\|\varphi\|_\infty, & \alpha \ge 0. \end{cases}$$

A case of two real parameters α and β

Note that in all of the above results we had one parameter case and moreover all the estimates are obtained in the supremum norm. The next result deals with two parameters case, in the supremum norm.

1. Khalfallah, Mhamdi,2024: Let $\alpha, \beta \in (-1, +\infty)$ where $\alpha + \beta > -1$. Then, for $\varphi \in C(\mathbb{T})$:

$$\|DP_{\alpha,\beta}[\varphi](z)\| \leq \left| \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \right| \frac{\Gamma(\alpha+\beta+1)}{\Gamma^2\left(\frac{\alpha+\beta+2}{2}\right)} \cdot \frac{|\alpha+1| + |\beta+1| + |\alpha| + |\beta|}{1-|z|^2} \|\varphi\|_\infty, \quad z \in \mathbb{D}.$$

General case: overview

We generalize all of the above results, except Colonna's on harmonic functions. The results, obtained with J. Gajic, will apear in JMAA.

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5. We do not equire continuity up to the boundary when the estimate is in the supremum norm
6. We obtain asymptoticaly sharp estimates, as $|z| \to 1$, of derivatives of arbitrary order.

A formula for $\|DP_{\alpha,\beta}[\varphi](z)\|$

For φ in $L^1(\mathbb{T})$ and z in $\mathbb{D}$ we have

$$\begin{aligned}\|DP_{\alpha,\beta}[\varphi](z)\| =&|c_{\alpha,\beta}| \left[\left| \int_{\mathbb{T}} \left(-\frac{(\alpha+\beta+1)\overline{z}}{1-|z|^2} + \frac{\alpha+1}{1-z\overline{\zeta}}\overline{\zeta} \right) u_{\alpha,\beta}(z\overline{\zeta})\varphi(\zeta)dm(\zeta) \right| \right. \\ &\left. + \left| \int_{\mathbb{T}} \left(-\frac{(\alpha+\beta+1)z}{1-|z|^2} + \frac{\beta+1}{1-\overline{z}\zeta}\zeta \right) u_{\alpha,\beta}(z\overline{\zeta})\varphi(\zeta)dm(\zeta) \right| \right],\end{aligned}$$

$$\|DP_{\alpha,\beta}[\varphi](0)\| = |c_{\alpha,\beta}| \left(\left| \int_{\mathbb{T}} (\alpha+1)\overline{\zeta}\varphi(\zeta)dm(\zeta) \right| + \left| \int_{\mathbb{T}} (\beta+1)\zeta\varphi(\zeta)dm(\zeta) \right| \right).$$

Setting $\lambda_1 = |\alpha+1|$, $\lambda_2 = |\beta+1|$ and

$$I(\varphi) = \lambda_1 \left| \int_{\mathbb{T}} \overline{\zeta}\varphi(\zeta)dm(\zeta) \right| + \lambda_2 \left| \int_{\mathbb{T}} \zeta\varphi(\zeta)dm(\zeta) \right|. \qquad (8)$$

we have $\quad \|DP_{\alpha,\beta}[\varphi](0)\| = |c_{\alpha,\beta}| I(\varphi), \qquad \varphi \in L^1(\mathbb{T}). \qquad (9)$

Sharp estimate for $\|DP_{\alpha,\beta}[\varphi](0)\|$ in terms of $\|\varphi\|_p$

Lemma

For every $\varphi \in L^1(\mathbb{T})$ we have

$$I(\varphi) = \frac{1}{2\pi} \max_{|\eta|=1} \left| \int_{-\pi}^{\pi} e^{-it} \varphi(e^{it}) \left(\lambda_1 + \lambda_2 \frac{\overline{\eta}}{\eta} e^{2it} \right) dt \right| \tag{10}$$

Theorem

Let $1 \leq p \leq +\infty$ and let q be the exponent conjugate to p. Then

$$\sup_{\|\varphi\|_p \leq 1} \|DP_{\alpha,\beta}[\varphi](0)\| = \frac{|c_{\alpha,\beta}|}{2\pi} \left(\int_{-\pi}^{\pi} \left| |\alpha+1| + |\beta+1| e^{2it} \right|^q dt \right)^{\frac{1}{q}}, \tag{11}$$

with obvious modification in the case $p = 1$, $q = +\infty$.

Sharp estimate of $\|DP_{\alpha,\beta}[\varphi](0)\|$ in terms of $\|\varphi\|_\infty$

This is, of course, a special case of the previous Theorem:
For all $\varphi \in L^\infty(\mathbb{T})$ we have

$$\|DP_{\alpha,\beta}[\varphi](0)\| \leq D(\alpha,\beta)\|\varphi\|_\infty,$$

where the constant

$$D(\alpha,\beta) = \frac{2|c_{\alpha,\beta}|(|\alpha+1|+|\beta+1|)E\left(2\frac{\sqrt{|(\alpha+1)(\beta+1)|}}{|\alpha+1|+|\beta+1|}\right)}{\pi}$$

on the right hand side is the best possible. Here $E(k)$ is complete elliptic integral of the second kind.

1. T_α-harmonic functions where $\alpha > -1$ is real. ($(\alpha/2, \alpha/2)$-harmonic functions).

$$\|Du(0)\| \leq \frac{2(\alpha+2)}{\pi} \frac{\Gamma^2\left(\frac{\alpha}{2}+1\right)}{\Gamma(\alpha+1)}, \qquad u = P_{\alpha/2,\alpha/2}[\varphi], \quad \|\varphi\|_\infty \leq 1.$$

2. Colonna's estimate at the origin with constant $4/\pi$.

An estimate of $\|DP_{\alpha,\beta}[\varphi](z)\|$ in terms of $\|\varphi\|_p$

Lemma

Let $u_{\alpha,\beta}$ be as in (1) and $1 \le p < \infty$. Then for $z \in \mathbb{D}$

$$\int_{\mathbb{T}} |u_{\alpha,\beta}(z\overline{\zeta})|^p dm(\zeta) \le e^{\frac{p\pi}{2}|\Im\alpha - \Im\beta|} \frac{\Gamma(p(\Re\alpha + \Re\beta + 2) - 1)}{\Gamma^2\left(\frac{p(\Re\alpha+\Re\beta+2)}{2}\right)} (1-|z|^2)^{1-p}. \tag{12}$$

Theorem

Let $\varphi \in L^p(\mathbb{T}), 1 \le p \le \infty$. Then

$$\|DP_{\alpha,\beta}[\varphi](z)\| \le C_{\alpha,\beta,p} \frac{|\alpha+1| + |\beta+1| + |\alpha z| + |\beta z|}{(1-|z|^2)^{1+\frac{1}{p}}} \|\varphi\|_{L^p(\mathbb{T})},$$

where $C_{\alpha,\beta,p}$ is a constant that depends only on α, β and p.

Special cases of the previous thteorem

1. For T_α-harmonic function $u = P_{\alpha/2,\alpha/2}[\varphi]$ where $\alpha > -1$

$$\|Du(z)\| \leq \frac{2+\alpha+|\alpha z|}{1-|z|^2}\|\varphi\|_\infty, \qquad f = P_{\alpha/2,\alpha/2}[\varphi], .$$

2. The case of α-harmonic funcgtions:

$$\|DP_{0,\alpha}[\varphi](z)\| \leq (|\alpha|+\alpha+2)\frac{\Gamma(\alpha+1)}{\Gamma^2\left(\frac{\alpha}{2}+1\right)} \cdot \frac{\|\varphi\|_\infty}{1-|z|^2}, \quad \alpha > -1. \tag{13}$$

SHARPER, $-1 < \alpha < +\infty$.

Lemma

Let $u_{\alpha,\beta}$ be as in (1) for some $\alpha, \beta \in \mathbb{C}$. For all $k \geq 0, l \geq 0$ the equality holds

$$\partial^k \overline{\partial}^l u_{\alpha,\beta}(z) = P_{k,l}\left(\frac{1}{1-|z|^2}, \frac{1}{1-z}, \frac{1}{1-\overline{z}}, f_1(z), \ldots, f_N(z)\right) u_{\alpha,\beta}(z),$$

where $N = N_{k,l}$, the functions $f_1, \ldots, f_{N_{k,l}}$ are C^∞ on $\mathbb{C}^\star = \mathbb{C}\setminus\{0\}$ and $P_{k,l}$ is a polynomial whose total degree in $\frac{1}{1-|z|^2}, \frac{1}{1-z}, \frac{1}{1-\overline{z}}$ is equal to $k + l$. In other words,

$$P_{k,l} = P_{k,l}\left(\frac{1}{1-|z|^2}, \frac{1}{1-z}, \frac{1}{1-\overline{z}}\right)$$

is polynomial with coefficients from $C^\infty(\mathbb{C}^\star)$ and $\deg P_{k,l} = k + l$.

Proposition

Let $u_{\alpha,\beta}$ be as in (1) for some $\alpha, \beta \in \mathbb{C}$. Then

$$\left|\partial^k \overline{\partial}^l u_{\alpha,\beta}(z)\right| \leq C_{\alpha,\beta,k,l} \frac{|u_{\alpha,\beta}(z)|}{(1-|z|^2)^{k+l}} \quad z \in \mathbb{D}. \tag{14}$$

Theorem

Let $\varphi \in L^p(\mathbb{T}), 1 \leq p \leq +\infty$.

$$\left|\partial^k \overline{\partial}^l u(z)\right| \leq C_{\alpha,\beta,k,l,p} \frac{\|\varphi\|_p}{(1-|z|^2)^{k+l+\frac{1}{p}}} \tag{15}$$

for $u = P_{\alpha,\beta}[\varphi]$.

Inhomogeneous case

J. Gajic investigated further inhomogeneous equation $L_{\alpha,\beta} = g$, a solution operator is given by Green's operator $G_{\alpha,\beta}$. Let us state a couple of results she obtained:

$$|G_{\alpha,\beta}g(z)| \leq C(1-|z|^2)^{\Re\alpha+\Re\beta+1/q}\|g\|_{L^p(\mathbb{D})}.$$

$$|DG_{\alpha,\beta}g(z)| \leq C(1-|z|^2)^{\Re\alpha+\Re\beta}\|g\|_\infty, \qquad g \in C(\overline{\mathbb{D}}).$$

Harmonc functions in the upper half space

1. $\mathbb{H}^{n+1} = \{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$,
2. The surface measure of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is $n\omega_n$,

$$P(x,y) = P_y(x) = c_n \frac{y}{(y^2 + |x|^2)^{(n+1)/2}}, \qquad x \in \mathbb{R}^n, \quad y > 0.$$

where

$$c_n = \Gamma\left(\frac{n+1}{2}\right) \pi^{-\frac{n+1}{2}}.$$

The harmonic extension of a function φ on $\mathbb{R}^n$ to $\mathbb{H}^{n+1}$ is

$$P[\varphi](x,y) = c_n \int_{\mathbb{R}^n} \varphi(t) \frac{y}{(y^2 + |x-t|^2)^{(n+1)/2}} \, dt$$

Modulii of continuity

Definition

We say that a function $\omega : [0, +\infty) \to \mathbb{R}$ is a modulus of continuity if it is continuous, concave and increasing on $[0, +\infty)$ and strictly positive on $(0, +\infty)$.

Definition

We say that a modulus of continuity satisfies (A) condition if there is a constant M such that

$$\int_1^\infty \omega(ys)s^{-3}\,ds \le M\omega(y) \quad y > 0, \qquad (A)$$

for some constant $M = M_\omega$.

The classical modulii of continuity $\omega(t) = t^\alpha$ where $0 < \alpha \le 1$ satisfy the (A) condition.

A key lemma

Lemma

Let $\varphi \in L^p(\mathbb{R}^n)$ *for some* $1 \le p \le \infty$. *Assume*

$$|\varphi(t) - \varphi(x^0)| \le \omega(|t - x^0|), \qquad t \in \mathbb{R}^n \tag{16}$$

for some $x^0 \in \mathbb{R}^n$ *and some modulus of continuity* ω *which satisfies condition* (A). *Then the harmonic extension* $g = P[\varphi]$ *of* φ *satisfies the following estimate:*

$$\left|\frac{\partial g}{\partial x_j}(x^0, y)\right| \le C(n, \omega)\frac{\omega(y)}{y}, \qquad 0 < y < +\infty, \quad 1 \le j \le n. \tag{17}$$

Proof of the lemma

Proof.

For all $(x,y) \in \mathbb{H}^{n+1}$ and all $j = 1, \ldots, n$ we have

$$\begin{aligned}
\frac{\partial g}{\partial x_j}(x,y) &= c_n \int_{\mathbb{R}^n} \varphi(t) \frac{\partial}{\partial x_j} \frac{y}{(y^2 + |x-t|^2)^{(n+1)/2}} dt \\
&= -(n+1)c_n \int_{\mathbb{R}^n} \varphi(t) y \frac{x_j - t_j}{(y^2 + |x-t|^2)^{\frac{n+3}{2}}} dt \\
&= -(n+1)c_n \int_{\mathbb{R}^n} [\varphi(t) - \varphi(x)] y \frac{x_j - t_j}{(y^2 + |x-t|^2)^{\frac{n+3}{2}}} dt.
\end{aligned}$$

The last equality follows from the observation that $x_j - t_j$ is an odd function of the variable $x - t$. □

Proof continued

Proof.

Therefore, using spherical coordinates centered at x^0, we obtain

$$\left|\frac{\partial g}{\partial x_j}(x^0, y)\right| \leq (n+1)c_n y \int_{\mathbb{R}^n} \omega(|x^0 - t|)\frac{|x^0 - t|}{(y^2 + |x^0 - t|^2)^{\frac{n+3}{2}}}\, dt$$

$$= n(n+1)c_n\omega_n y \int_0^\infty \omega(r)\frac{r^n\, dr}{(y^2 + r^2)^{\frac{n+3}{2}}}$$

$$= n(n+1)c_n\omega_n \frac{1}{y}\int_0^\infty \frac{\omega(ys)s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds$$

Let us denote the above integral by $I(y)$. □

Proof continued

Proof.

Then we have

$$\begin{aligned} I(y) &= \int_0^1 \frac{\omega(ys)s^n}{(1+s^2)^{\frac{n+3}{2}}}ds + \int_1^\infty \frac{\omega(ys)s^n}{(1+s^2)^{\frac{n+3}{2}}}ds \\ &\le \int_0^1 \frac{\omega(y)s^n}{(1+s^2)^{\frac{n+3}{2}}}ds + \int_1^\infty \frac{\omega(ys)}{s^3}ds \\ &\le (1+M)\omega(y). \end{aligned}$$

and this proves desired estimate (17). □

The proof shows that one can take
$C(n,\omega) = n(n+1)c_n\omega_n(1+M_\omega)$.

Systems of conjugate harmonic functions

We recall that a system of harmonic functions u_j, $0 \le j \le n$, on $\mathbb{H}^{n+1}$ is called a conjugate system if it satisfies the following system of equations

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \tag{18}$$

where $x_0 = y$.
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When $n = 2$ this reduces to the classical CR equations.
Given a function $f = f(x_0, x_1, \ldots, x_n)$ harmonic in $\mathbb{H}^{n+1}$ one gets a conjugate system by setting $u_j = \partial f / \partial x_j$, $0 \leq j \leq n$.
The above system allows one to infer estimates of $\partial u_0 / \partial x_0 = \partial u_0 / \partial y$ from the estimates of $\partial u_j / \partial x_j$ for $1 \leq j \leq n$, this is how one proves the following proposition.

Conjugate systems and vertical derivatives

Proposition

Let ω be a modulus of continuity, $E \subset \mathbb{R}^n \cong \partial\mathbb{H}^{n+1}$ and let f_j, $0 \leq j \leq n$, be a system of conjugate functions on $\mathbb{H}^{n+1}$. Assume

$$\left|\frac{\partial f_j}{\partial x_j}(x, y)\right| \leq \frac{\omega(y)}{y} \qquad x \in E, \quad y > 0, \quad 1 \leq j \leq n.$$

Then we have

$$\left|\frac{\partial f_0}{\partial y}(x, y)\right| \leq n\frac{\omega(y)}{y}, \qquad x \in E, \quad y > 0.$$

Main result

Let R_j, $1 \leq j \leq n$, be the Riesz operators. These operators are bounded linear operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Theorem

Let $f \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$ and set $f_j = R_j f$ for $1 \leq j \leq n$. Let $u = P[f]$ be the harmonic extension of f to the upper half space $\mathbb{H}^{n+1}$. Let ω be a modulus of continuity satisfying condition (A) and let E be a subset of $\mathbb{R}^n \cong \partial\mathbb{H}^{n+1}$. Assume

$$|f_j(t) - f_j(x)| \leq \omega(|t-x|), \qquad x \in E, \quad t \in \mathbb{R}^n, \quad 1 \leq j \leq n. \tag{19}$$

Then there is a constant $C = C(n)$ such that

$$\left|\frac{\partial u}{\partial y}(x,y)\right| \leq C(n)\frac{\omega(y)}{y}, \qquad y > 0, \quad x \in E. \tag{20}$$

Thank you for your attention!