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On a new formulation of the inverse problem of determining the order of fractional derivatives in partial differential equations

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Introduction.

Let us consider the following initial-boundary value problem:

$$\partial_t^\rho u(x,t) = \Delta u(x,t), \quad x \in \Omega \subset R^N,\ 0 < \rho < 1,\ t > 0, \tag{1}$$

$$Bu(x,t) \equiv \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{2}$$

$$\lim_{t\to 0} J^{1-\rho} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}, \tag{3}$$

where n is the unit outward normal vector of $\partial\Omega$. The fractional Riemann-Liouville integral and the Riemann-Liouville derivative correspondingly have the form

$$J^\rho f(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-\tau)^{\rho-1} f(\tau) d\tau,\ \partial_t^\rho f(t) = D J^{1-\rho} f(t),\ D = \frac{d}{dt}.$$

The fractional derivative in the sense of Caputo has the form:

$$D_t^\rho f(t) = J^{1-\rho} D f(t).$$

In the paper [1] it is shown that initial conditions in the form (3) have physical meaning, and that the corresponding quantities can be obtained from measurements.

[1] Heymans N., Podlubny I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. Rheol. acta. 2006, 45, 765-771 p.

We will call problem (1)-(3) a forward problem. It often happens that we know that $\rho \in (0, 1)$, but we do not know its exact value. Unfortunately, there is no device that could accurately measure this parameter ρ. The only way to find this parameter is analytically, and we arrive at the inverse problem of determining the order of the fractional derivative ρ. In the last decade, this type of inverse problem has been actively studied by many specialists. A number of interesting results have been obtained that have a certain applied significance (see e.g., the survey paper [2]).

A few words about how we came to these problems.

In 2020, when the pandemic began, S. Umarov (New Haven, USA) and YangQuan Chen (University of California) created a mathematical model of the spread of coronavirus. It turned out that the differential equations modeling this process is a nonlinear fractional differential equation. And the data presented by Johns Hopkins University on COVID-19 outbreaks in different countries showed that the order of the fractional derivative ρ in this equation lies in the interval $(0, 1)$. To find the exact value, Umarov proposed to search for this parameter analytically and we began to study the corresponding inverse problem.

[2] Z. Li, Y. Liu, M. Yamamoto, Inverse problems of determining parameters of the fractional partial differential equations. Handbook of fractional calculus with applications. 2, DeGruyter (2019), 431-443.

I have given a similar report at seminars in Europe (this is Part 1 of this talk). Since then our new results [3,4] have appeared and I have noticed very interesting works [5,6]. I will dwell on these works at the end of the report (this will be Part 2).

[3] Sh. Alimov, R. Ashurov, On determining the fractional exponent of the subdiffusion equation, arXive:2411.19852v1[math.AP]29Nov2024.
[4] R. Ashurov, I. Sulaymonov, Monotonicity of the Mittag-Leffler function and determining the fractional exponent of the subdiffusion equation, arXive:2501.01724v1[math.AP]3Jan2025.

[5] G. Li, Z. Wang, X. Jia, Y. Zhang, An inverse problem of determining the fractional order in the TFDE using the measurement at one space-time point, Fractional Calculus and Applied Analysis, 26:1770–1785, (2023).
[6] A. N. Artyushin, An inverse problem of recovering the variable order of the derivative in a fractional diffusion equation, Siberian Mathematical Journal, 2023, Vol. 64, No. 4, pp. 796–806

Part 1

Let us make some general comments on the remaining works published to date.

1) all the publications assumed the fractional derivative in the sense of Caputo;

2) fractional derivative of order $0 < \rho < 1$ (multi-term, distributed order);

3) only homogeneous equations were considered;

4) the spectrum of the elliptic part is discrete;

5) the authors proved only (with the exception of the work [5], see below) the uniqueness of the solution (this is very important from application point of view);

6) the authors used the following condition as an over-determination condition:

$$u(x_0, t) = h(t), \ \ 0 < t < T, \tag{4}$$

at a monitoring point $x_0 \in \Omega$. Since the authors used the asyptotics of the Mittag-Leffler function in their proof, T must be large enough.

We will give another formulation of the inverse problem and methods for its solution. In this formulation, previously unconsidered cases can be covered (existence).

Let us recall the formulation in which the inverse problem was considered (let us call it Inverse Problem I; $u(x,t)$ and ρ unknown)

$$\begin{cases} D_t^\rho u(x,t) = \Delta u(x,t),\ 0<\rho<1, \quad x\in\Omega\subset R^N,\ t>0, \\ Bu(x,t) \equiv \frac{\partial u(x,t)}{\partial n} = 0, \quad x\in\partial\Omega,\ t\geq 0, \\ u(x,0)=\varphi(x), \quad x\in\overline{\Omega}, \\ u(x_0,t)=h(t),\ 0<t<T,\ x_0\in\Omega. \end{cases}$$

Does a solution exist for an arbitrarily given function $h(t)$? The answer is no.

Since $x_0\in\Omega$, then the function $h(t)$ must be the trace of the solution of the subdiffusion equation, and since any solution of it is infinitely differentiable with respect to $t>0$, then the function $h(t)$ must also be infinitely differentiable. It turns out that this is still not enough: see Theorem 7.2 [5] on the existence of a solution, which is formulated on more than one journal page, i.e., it requires the fulfillment of a large number of conditions on the data of the problem. This means that it is not always possible to prove the existence of a solution with the condition $u(x_0,t)=h(t)$.

[5] Janno, J. Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time-fractional diffusion equation. Electronic J. Differential Equations V. 216(2016), pp. 1-28,

In this regard, in the survey paper [2] the following problem is formulated:
Open Problem. Is it possible to investigate the inverse problem using the value of the solution at a fixed time t_0 as observation data?

For our part, we would add that it would be mathematically correct to indicate not a function, as in the condition (4), for finding one unknown number ρ, but to indicate one number.

Based on this, we propose the following over-determination condition

$$F(u(x,t_0)) = d_0, \tag{5}$$

with some functional F. Since $u(x,t_0)$ depends on ρ, then by introducing the notation $f(\rho) = F(u(x,t_0))$, condition (5) can be rewritten as

$$f(\rho) = d_0. \tag{6}$$

Now our task is to find such a functional F that the function $f(\rho)$ is strictly monotone. If this succeeds, then the equation (6) has a unique solution ρ if and only if

$$\inf_{0<\rho<1} f(\rho) < d_0 < \sup_{0<\rho<1} f(\rho).$$

Now let us consider the inverse problem in the following formulation (let us call it Inverse Problem II; $u(x,t)$ and ρ unknown)

$$\begin{cases} \partial_t^\rho u(x,t) = \Delta u(x,t), \quad x \in \Omega \subset R^N, \ t > 0, \\ Bu(x,t) \equiv \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, \ t \geq 0, \\ \lim\limits_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}, \\ f(\rho) \equiv (u(\cdot,t_0), v_1) = |\Omega|^{-1/2} \int\limits_\Omega u(x,t_0)dx = d_0, \quad t_0 \geq 1, \end{cases}$$

where $v_1(x) = |\Omega|^{-1/2}$ is the first egenfunction of the spectral problem:

$$-\Delta v(x) = \lambda v(x), \, x \in \Omega, \quad Bv(x) = 0, \ x \in \partial\Omega.$$

Note an important fact for our reasoning: $\lambda_1 = 0$.

Recall, the Mittag-Leffler function $E_{\rho,\mu}(z)$ has the form:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}.$$

The unique solution of the corresponding forward problem has the form:

$$u(x,t) = \sum_{j=1}^{\infty} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^{\rho}) v_j(x), \quad \varphi_j = (\varphi, v_j).$$

Then, since $\lambda_1 = 0$,

$$f(\rho) = (u(\cdot, t_0), v_1) = \varphi_1 t_0^{\rho-1} E_{\rho,\rho}(0) = \frac{\varphi_1 t_0^{\rho-1}}{\Gamma(\rho)}.$$

Lemma 1

Let $\varphi_1 \neq 0$ and $t_0 \geq 1$. Then $f(\rho)$ is strictly monotone and

$$\lim_{\rho \to +0} f(\rho) = 0, \quad f(1) = \varphi_1.$$

From equation $f(\rho) = d_0$ and the condition $\inf_{0<\rho<1} f(\rho) < d_0 < \sup_{0<\rho<1} f(\rho)$ one has

Theorem 2

Let $\varphi_1 \neq 0$. Then Inverse Problem II has a unique solution $\{u(x,t), \rho\}$ if and only if

$$0 < \frac{d_0}{\varphi_1} < 1.$$

This result is published in the paper [8]. In Inverse Problem II one can consider a non-homogeneous subdifussion equation (see [9]) and fractional wave equation (see [10]) Note in these papers: Riemann-Liouville derivative and $\lambda_1 = 0$. The idea of solving the inverse problem is the same as above, but technical difficulties arise. For example, in case $f(x,t) \equiv 0$ and $\rho \in (1,2)$ the over-determination condition has the form:

$$\varphi_1 t_0^{\rho-1} E_{\rho,\rho}(0) + \psi_1 t_0^{\rho-2} E_{\rho,\rho-1}(0) = d_0.$$

Of course, in Inverse Problem II, instead of the Laplace operator with the Neumann condition, one can take an arbitrary positive self-adjoint abstract operator with a single condition: $\lambda_1 = 0$ (see [10]).

[8] R.R. Ashurov, S.R. Umarov, Determination of the order of farctional derivative for subdiffusion equations. // Fractional Calculus and Applied Analysis, 2020, V. 23, № 6, pp. 1647-1662.
[9] R. R.Ashurov, Y.Fayziev. Determination of fractional order and source term in a fractional subdiffusion equation, // Lobachevski Journal of Mathematics, 2021, Vol.42, No3, pp. 508-516.
[10] Ashurov, R.; Sitnik, S. Identification of the Order of the Fractional Derivative for the Fractional Wave Equation. // Fractal Fract. 2023, 7, 67.

The disadvantage of Inverse Problem II is the restriction $\lambda_1 = 0$ and in this formulation it is impossible to consider equations with the Caputo derivative.

Indeed, if in Inverse Problem II we take the Caputo derivative, then the solution to the forward problem has the form

$$u(x,t) = \sum_{j=1}^{\infty} \varphi_j E_{\rho,1}(-\lambda_j t^{\rho}) v_j(x).$$

Then (remember, $\lambda_1 = 0$)

$$f(\rho) \equiv \int_{\Omega} u(x,t_0) v_1(x) dx = \varphi_1 E_{\rho,1}(0) = \varphi_1,$$

and we can not use the above formulation of the inverse problem, since the overdetermination condition $f(\rho) = d_0$ is not an equation for finding ρ.

Now we will give another method for solving the inverse problem, in which condition $\lambda_1 = 0$ is removed, and, moreover, in the new formulation one can also consider equations with a fractional Caputo derivative.

Let us call the following problem Inverse Problem III ($u(x,t)$ and ρ unknown)

$$\begin{cases} \partial_t^\rho u(x,t) = \Delta u(x,t), \quad x \in \Omega \subset R^N, \ t > 0, \\ Bu(x,t) \equiv \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, \ t \geq 0, \\ \lim\limits_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}, \\ F(\rho; t_0) \equiv \int\limits_\Omega |u(x,t_0)|^2 dx = ||u(x,t_0)||^2 = d_0, \quad t_0 \geq T_0. \end{cases}$$

It is easy to see that the following lemma solves Inverse Problem III.

Lemma 3

Given ρ_0 from interval $0 < \rho_0 < 1$, there exists a number $T_0 = T_0(\lambda_1, \rho_0)$, such that for all $t_0 \geq T_0$ and for arbitrary $\varphi \in L_2(\Omega)$ function $F(\rho; t_0)$ is monotonously decreasing with respect to $\rho \in [\rho_0, 1]$.

Theorem 4

Let $t_0 \geq T_0$. Then the inverse problem has a unique solution $\{u(t), \rho\}$ if and only if

$$F(1; t_0) \leq d_0 \leq F(\rho_0; t_0).$$

Theorem 4 (proved in [11]) gives a positive answer to the Open problem posed in Z. Li et al. [2]: one may recover the order ρ by the value of $F(\rho;t)$ at a fixed time instant t_0.

Sketch of the proof of Lemma 3. Solution of the forward problem has the form:

$$u(x,t_0)=\sum_{k=1}^{\infty}(\varphi,v_k)t_0^{\rho-1}E_{\rho,\rho}(-\lambda_k t_0^{\rho})v_k(x).$$

Therefore, by Parseval's equality

$$F(\rho;t_0)=||u(x,t_0)||^2=\sum_{k=1}^{\infty}|(\varphi,v_k)|^2|t_0^{\rho-1}E_{\rho,\rho}(-\lambda_k t_0^{\rho})|^2,$$

and Lemma 3 from the following statement.

Lemma 5

Let $0<\rho_0<1$. Then there is a positive number $T_0=T_0(\rho_0,\lambda_1)$ such that functions $e_\lambda(\rho)=t_0^{\rho-1}E_{\rho,\rho}(-\lambda t_0^{\rho})$ are positive and monotonically decrease in $\rho\in[\rho_0,1]$ for all $t_0\geq T_0$ and $\lambda\geq\lambda_1$.

[11] Sh. Alimov and R. Ashurov, Inverse problem of determining an order of the Riemann-Liouville time-fractional derivative, Progr. Fract. Differ. Appl. 8 (2022), 1-8.

The sketch of the proof of Lemma 5. Let $0 < \beta < \pi$ and $\delta(\beta)$ stand for a contour oriented by non-decreasing $\arg \zeta$ and consisting of the following parts:
1) the ray $\arg \zeta = -\beta$, $|\zeta| \geq 1$,
2) the arc $-\beta \leq \arg \zeta \leq \beta$, $|\zeta| = 1$,
3) the ray $\arg \zeta = \beta$, $|\zeta| \geq 1$.
The contour $\delta(\beta)$ is called the Hankel path.

If $\frac{\pi}{2}\rho < \beta < \pi\rho$, then we arrive at (see [12], p. 135)

$$t_0^{\rho-1} E_{\rho,\rho}(-\lambda t_0^\rho) = -\frac{1}{\lambda^2 t_0^{\rho+1}\Gamma(-\rho)} + \frac{1}{2\pi i \rho \lambda^2 t_0^{\rho+1}} \int\limits_{\delta(\beta)} \frac{e^{\zeta^{1/\rho}} \zeta^{\frac{1}{\rho}+1}}{\zeta + \lambda t_0^\rho} d\zeta.$$

We choose $\beta = \frac{3\pi}{4}\rho$, $\rho \in [\rho_0, 1]$. Recall $e_\lambda(\rho) = t_0^{\rho-1} E_{\rho,\rho}(-\lambda t_0^\rho)$.

To show the validity of the lemma, it suffices to prove that $\frac{d}{d\rho} e_\lambda(\rho) < 0$ for all $\rho \in [\rho_0, 1)$ and sufficiently large t_0. The positiveness of $e_\lambda(\rho)$ is a consequence of the inequality $e_\lambda(1) = e^{-\lambda t} > 0$.

[12] M. M. Dzherbashian [=Djrbashian], Integral Transforms and Representation of Functions in the Complex Domain (in Russian), M. NAUKA (1966).

If we consider the Caputo derivative in Inverse Problem III, then the solution of the forward problem will have the form:

$$u(x,t_0)=\sum_{k=1}^{\infty}(\varphi,v_k)E_{\rho,1}(-\lambda_k t_0^{\rho})v_k(x).$$

Therefore

$$F(\rho;t_0)=||u(x,t_0)||^2=\sum_{k=1}^{\infty}|(\varphi,v_k)|^2|E_{\rho,1}(-\lambda_k t_0^{\rho})|^2.$$

For this case we have the following [13]

Lemma 6

Given ρ_0 from the interval $0<\rho_0<1$, there exists a number $T_0=T_0(\lambda_0,\rho_0)$, such that for all $t_0\geq T_0$ and $\lambda\geq\lambda_0$ the function $\ell_\lambda(\rho)=E_\rho(-\lambda t_0^{\rho})$ is positive and monotonically decreasing with respect to $\rho\in[\rho_0,1]$ and

$$\ell_\lambda(1)\leq\ell_\lambda(\rho)\leq\ell_\lambda(\rho_0).$$

[13] Sh. Alimov, R. Ashurov, Inverse problem of determining an order of the Caputo time- fractional derivative for a subdiffusion equation.// J. Inverse Ill-Posed Probl. V. 28, Issue 5, pp. 651-658, 2020;

Let $\delta(\beta)$ be the Hankel path and $\beta = \frac{3\pi}{4}\rho$, $\rho \in [\rho_0, 1)$. Then ([12], p. 135)

$$E_\rho(-\lambda t_0^\rho) = \frac{1}{\lambda t_0^\rho \Gamma(1-\rho)} - \frac{1}{2\pi i \rho \lambda t_0^\rho} \int\limits_{\delta(\beta)} \frac{e^{\zeta^{1/\rho}} \zeta}{\zeta + \lambda t_0^\rho} d\zeta.$$

To prove the lemma it suffices to show that the derivative $\frac{d}{d\rho}\ell_\lambda(\rho)$ is negative for all $\rho \in [\rho_0, 1)$ and for suficiently large t_0. The positivity of $\ell_\lambda(\rho)$ follows from the inequality $\ell_\lambda(1) = e^{-\lambda t} > 0$.

After this lemma is proven, the inverse problem for the equation with the Caputo derivative is solved in the same way as above (see [13]).

Operators with continuous spectrum.

Problem: find a function $u(x,t) \in L_2^m(R^N)$, $t>0$, such that $(0<\rho\leq 1)$

$$D_t^\rho u(x,t) + A(D)u(x,t) = 0, \quad x \in R^N, \quad t>0,$$
$$u(x,0) = \varphi(x), \quad x \in R^N,$$

where $\varphi(x)$ is a given continuous function. Here $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ - a positive elliptic differential operator with constant coefficients and continuous spectrum $\sigma = [0,\infty)$.

If $\tau > \frac{N}{2}$ and $\varphi \in L_2^\tau(R^N)$ (by the way, this condition can not be improved), then the forward problem has a unique solution [14]

$$u(x,t) = \int\limits_{R^N} E_\rho(-A(\xi)t^\rho)\,\hat{\varphi}(\xi)e^{ix\xi}d\xi.$$

Let us fix a vector $\xi_0 \neq 0$, such that $\hat{\varphi}(\xi_0) \neq 0$ and put $\lambda_0 = A(\xi_0) > 0$. In [14] the uniqueness and existance of the papametr ρ is proved with the following over-determination condition:

$$F(\rho; t_0) \equiv |\hat{u}(\xi_0,t_0)| = d_0\Big(= E_{\rho,1}(-\lambda_0 t_0^\rho)\,|\hat{\varphi}(\xi_0)|\Big).$$

[14] R. Ashurov, R. Zunnunov, Initial-boundary value and inverse problems for subdiffusion equation in R^N, Fractional Differential Calculus, 2020, V. 10, N 2, pp. 291-306.

Two-parameter inverse problem.

Tatar and Ulusoy (2013) considered the initial-boundary value problem for the differential equation ($\rho =?$, $\sigma =?$)

$$D_t^\rho u(t,x) = -(-\triangle)^\sigma u(t,x), \quad t>0,\ x\in(-1,1),$$

where $\triangle^\sigma$ is the one-dimensional fractional Laplace operator, $\rho\in(0,1)$ and $\sigma\in(0,1)$. Let the initial function φ satisfy the conditions

$$\varphi(x)\in C^\infty(0,1),\ \varphi_k\geq 0, \text{for all} \quad k\geq 1.$$

The authors proved only the uniqueness theorem for the two-parameter inverse problem with the over-determination condition $u(0,t)=h(t)$, $0<t<T$.

M. Yamamoto (2020) proved the uniqueness theorem for the above two-parameter inverse problem in an N-dimensional bounded domain Ω with a smooth boundary $\partial\Omega$ and over-determination condition $u(x_0,t)=h(t)$, $0<t<T$, $x_0\in\Omega$.

In work [14] a two-parameter inverse problem for the equation is also considered:

$$D_t^\rho u(x,t) + A^\sigma(D)u(x,t) = 0, \quad x\in R^N, \quad t>0.$$

[14] R. Ashurov, R. Zunnunov, Initial-boundary value and inverse problems for subdiffusion equation in R^N, Fractional Differential Calculus, 2020, V. 10, N 2, pp. 291-306.

The Rayleigh-Stokes equation

Consider the inverse problem for the Rayleigh-Stokes equation (u and α unknown)

$$\begin{cases} \partial_t u(x,t) - (1+\gamma\,\partial_t^\alpha)\Delta u(x,t) = 0, \quad x\in\Omega, \quad 0<t\le T; \\ u(x,t)=0, \quad x\in\partial\Omega, \quad 0<t\le T; \\ u(x,0)=\varphi(x), \quad x\in\Omega; \\ ||u(x,t_0)||^2_{L_2(\Omega)} = d_0, \end{cases} \tag{7}$$

where $\gamma>0$ is a fixed constant, $\partial_t = \partial/\partial t$, $\alpha\in(0,1)$, d_0 and t_0 are given numbers.

The forward problem (7) plays an important role in describing the behavior of some non-Newtonian fluids (see e.g. [15] and [16]). In recent decades this problem has received much attention due to its importance for applications (see, e.g. [17]). The fractional derivative ∂_t^α in the model is used to capture the viscoelastic behavior of the flow (see, e.g. [15-16]).

[15] Tan, W.C.; Masuoka, T. Stokes' first problem for a second grade fluid in a porous half-space with heated boundary, Int. J. Non-Linear Mech. 40 (2005) 515-522.
[16] Fetecau, C.; Jamil, M.; Fetecau, C.; Vieru, D. The Rayleigh-Stokes problem for an edge in a generalized Oldroyd-B fluid. Z. Angew. Math. Phys., 60, 921-933, 2009.
[17] Bazhlekova, E., Jin, B., Lazarov, R., Zhou, Z.: An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid. Numer. Math. 131, 1-31 (2015)

The unique solution of the forward problem has the form (see [17]):

$$u(x,t) = \sum_{k=1}^{\infty} B_\alpha(\lambda_k, t)\varphi_k v_k(x),$$

where

$$B_\alpha(\lambda, t) = \int_0^\infty e^{-rt} b_\alpha(\lambda, r) dr,$$

where

$$b_\alpha(\lambda, r) = \frac{\gamma}{\pi} \frac{\lambda r^\alpha \sin \alpha\pi}{(-r + \lambda\gamma r^\alpha \cos \alpha\pi + \lambda)^2 + (\lambda\gamma r^\alpha \sin \alpha\pi)^2}.$$

Therefore, the over-determination condition can be rewritten as

$$F(\alpha; t_0) = ||u(x, t_0)||^2 = \sum_{k=1}^{\infty} B_\alpha^2(\lambda_k, t_0)|\varphi_k|^2 = d_0.$$

[17] Bazhlekova, E., Jin, B., Lazarov, R., Zhou, Z.: An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid. Numer. Math. 131, 1-31 (2015)

Lemma 7

Let $\gamma > 0$, $\lambda \geq \lambda_1$ and $\alpha \in (0,1)$ be given numbers. There exists a positive number $T_0 = T_0(\gamma, \alpha, \lambda_1) \geq 1$ such that for any t_0, $t_0 \geq T_0$, one has

$$\partial_\alpha B_\alpha(\lambda, t_0) < 0.$$

Various properties of the function $B_\alpha(\lambda, t)$ (in particular, monotonicity with respect to t) have been studied by many authors (see [17]). But monotonicity with respect to the parameter α has not been considered by anyone.

Theorem 8

There exists a number $T_0 = T_0(\gamma, \alpha, \lambda_1) \geq 1$ such that for every $t_0 \geq T_0$ and every $\varphi \in L_2(\Omega)$ the function $F(\alpha; t_0) \equiv ||u(x, t_0)||^2$ strictly decreasing in $\alpha \in (0,1)$.

This theorem implies the unique solvability of the inverse problem (see [18]). On the other hand, this statement has independent interest.

[18] R. Ashurov, O. Mukhiddinova, Inverse problem of determining the order of the fractional derivative in the Rayleigh-Stokes equation, Fractional Calculus and Applied Analysis, 2023, 26:1691–1708

Equations of mixed-type.

Let $0 < \alpha < 1$ and $1 < \beta < 2$. Consider initial-boundary value problem for mixed-type equation (see [19]) ($u(x,t) =?$, $\alpha =?$, $\beta =?$)

$$\begin{cases} D_{0t}^{\alpha} u(x,t) - \Delta u(x,t) = 0, & t > 0,\ \Omega \subset R^N; \\ D_{t0}^{\beta} u(x,t) - \Delta u(x,t) = 0, & -T < t < 0. \end{cases}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq -T,$$

$$u(x,-T) = \varphi(x), \quad x \in \overline{\Omega},$$

(gluing condition)

$$u(x,+0) = u(x,-0), \quad \lim_{t \to +0} D_{0t}^{\alpha}(x,t) = u_t(x,-0).$$

M. Ruzhansky, E. T. Karimov et al. (2015, 2016, 2020). $N = 1$, $u(x,t) =?$, $f(x) =?$.

[19] R. Ashurov, R. Zunnunov, The inverse problem of determining the order of the fractional derivative in equations of mixed-type, Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 12, pp. 2714-2729.

The unique solution of the forward problem has the form [19]:

$$u^+(x,t) = \sum_{k=1}^{\infty} \frac{E_{\alpha,1}\big(-\lambda_k t^\alpha\big)\varphi_k v_k(x)}{\Delta_k}, \quad 0 \le t < \infty, \tag{8}$$

$$u^-(x,t) = \sum_{k=1}^{\infty} \frac{\Big[|t|\lambda_k E_{\beta,2}\big(-\lambda_k|t|^\beta\big) - E_{\beta,1}\big(-\lambda_k|t|^\beta\big)\Big]\varphi_k v_k(x)}{\Delta_k}, \quad -T \le t \le 0, \tag{9}$$

where

$$\Delta_k \equiv \Delta_k(\beta,T) = \lambda_k T E_{\beta,2}\big(-\lambda_k T^\beta\big) - E_{\beta,1}\big(-\lambda_k T^\beta\big) > \delta_0 > 0, \quad k \ge 1, \; T >> 1.$$

Since there are two unknowns, we set the following two additional conditions:

$$F_1(\alpha,\beta;t_1) \equiv \int\limits_\Omega |u(x,t_1)|^2 dx = d_1, \; F_2(\alpha,\beta;t_2) \equiv \int\limits_\Omega u(x,-t_2)v_{k_0}(x)dx = d_2,$$

where d_1, d_2 are the given numbers, t_1, t_2 are positive numbers, defined later, and $k_0 \ge 1$ is an integer such that $\varphi_{k_0} \neq 0$.

[19] R. Ashurov, R. Zunnunov, The inverse problem of determining the order of the fractional derivative in equations of mixed-type, Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 12, pp. 2714-2729.

Using the explicit form of the solution (8) and (9) we will have

$$F_1(\alpha,\beta;t_2)=\int\limits_{\Omega}|u(x,t_1)|^2dx=\sum_{k=1}^{\infty}\left|E_{\alpha,1}(-\lambda_k t^{\alpha})\frac{\varphi_k}{\Delta_k(T,\beta)}\right|^2=d_1,$$

$$F_2(\alpha,\beta;t_2)\equiv F_2(\beta;t_2)=\frac{\Delta_{k_0}(t_2,\beta)}{\Delta_{k_0}(T,\beta)}=d_2.$$

Remember

$$\Delta_k(\beta,T)=\lambda_k T E_{\beta,2}\big(-\lambda_k T^{\beta}\big)-E_{\beta,1}\big(-\lambda_k T^{\beta}\big).$$

First, we will prove that F_2 is a monotone function of β and show the unique solvability of the second equation and by putting β into the first equation we will prove the monotonicity of the function F_1 with respect to the variable α. And from here the unique solvability of the inverse problem follows.

For the best of our knowledge, this is the first work, where the inverse problem is investigated for an equations of mixed-type (see also [20, 21]).

[20] R. Ashurov, R. Zunnunov, An analog of the Tricomi problem for a mixed-type equation with fractional derivative. Inverse problems. Lobachevskii Journal of Mathematics. 44(8), 3224–3238 (2023)
[21] R. Ashurov, M. Murzambetova, Inverse problem of determining the fractional derivative order in the mixed-type equation. Uzbek Mathematical Journal. 67(3), 45-52 (2023)

Inverse problems related to the problem under consideration

The authors of [22] studied an inverse problem for simultaneously determining the order ρ and a source function $(u(x,t) =?,\ f(x) =?,\ \rho =?)$:

$$\begin{cases} D_t^\rho u(x,t) - \triangle u(x,t) = f(x),\ x \in \Omega,\ 0 < t < \infty; \\ u(x,t) = 0, \quad x \in \partial\Omega,\ 0 < t < \infty; \\ u(x,0) = \varphi(x), \quad x \in \overline{\Omega}; \\ \dfrac{\partial u}{\partial n}(x_i, t) = h_i(t),\ t \in (0,\infty),\ x_i \in \Omega,\ i = 1, 2. \end{cases}$$

Here Ω is the unit disc in R^2. If $1/2 < \rho < 1$, then the authors proved the uniqueness theorem. In [23] we considered the same type of problem in $\Omega \subset R^N$, N arbitrary and $\rho \in (0,1)$ and also proved the existence of all unknowns.

[22] Zhiyuan Li and Zhidong Zhang. Unique determination of fractional order and source term in a fractional diffusion equation from sparse boundary data. Inverse Problems, Volume 36, Number 11 (2020).
[23] R. Ashurov, Y. Fayziev. Determination of fractional order and source term in a fractional subdiffusion equation. Eurasian Mathematical Journal, European Mathematical Journal, Volume 13, Number 1 (2022), 19- 31.

Fractional derivatives can be taken both in the Riemann-Liouville sense and in the Caputo sense. Using two over-determination conditions

$$\int_{\Omega} u(x, t_0) v_1(x) dx = d_0$$

and

$$u(x, T) = \psi(x),$$

the existence and uniqueness of three unknowns $u(x, t)$, $f(x)$ and ρ are proved in work [23].

Theorem 9

There exists a number $T_0(\varphi_1, \psi_1) > 0$ such that if $t_0 \geq T_0(\varphi_1, \psi_1)$ and $\varphi_1^2 + \psi_1^2 \neq 0$, then $\exists$ unique $\{u(x, t), f(x), \rho\}$ if and only if

$$\min\left\{|\psi_1|,\ \left|\varphi_1\left[1 - \frac{t_0}{T}\right] + \frac{t_0\psi_1}{T}\right|\right\} < d_0 <$$

$$< \max\left\{|\psi_1|,\ \left|\varphi_1\left[1 - \frac{t_0}{T}\right] + \frac{t_0\psi_1}{T}\right|\right\}.$$

Part 2.

We start with the work [3].

Motivation. In the paper of Hatano et al. [25] the equation $D_t^\rho u = \Delta u$ is considered in $\Omega \subset \mathbb{R}^N$ with the Robin boundary condition and the initial function $\varphi(x)$. They defined the order ρ from observation data $u(x_0, t)$ and $u_t(x_0, t)$: if $\varphi \in C_0^\infty(\Omega)$, $\varphi(x) \geq 0$ or ≤ 0 and is not identically zero on $\overline{\Omega}$, then

$$\rho = -\lim_{t\to\infty} \left[t u_t(x_0, t) [u(x_0, t)]^{-1} \right],$$

with an arbitrary $x_0 \in \Omega$. In our opinion, this formula has one inconvenience when using it in real-life problems. Firstly, $u_t(x, t)$ does not participate in the equation and secondly when calculating the derivative of an approximately known solution, we allow new inaccuracies. In this regard, the question arises: is it possible to represent this formula without using $u_t(x, t)$?

In work [3] exactly this formula is established without the condition of preserving the sign of the initial function $\varphi(x)$.

[3] Sh. Alimov, R. Ashurov, On determining the fractional exponent of the subdiffusion equation, arXive:2411.19852v1[math.AP]29Nov2024.
[25] Y. Hatano, J. Nakagawa, S. Wang and M. Yamamoto, Determination of order in fractional diffusion equation, J. Math-for-Ind. 5A (2013), 51–57.

Let us consider following the work [25] the initial-boundary value problem with the Robin boundary condition (see [3]):

$$\begin{cases} \partial_t^\rho u(x,t) - \Delta u(x,t) = 0, \quad t > 0, \\ \lim\limits_{t\to 0} J_t^{\rho-1} u(x,t) = \phi, \\ \dfrac{\partial u}{\partial n} + \sigma(x) v(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{10}$$

where $\phi \in C(\overline{\Omega})$ is the given initial function. Here $\Omega \subset \mathbb{R}^N$ is an arbitrary domain with a smooth boundary.

The spectrum of the Laplace operator with the corresponding boundary condition consists of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < ..., \quad \lambda_k \to +\infty, \quad (k \to +\infty),$$

each of which has finite multiplicity (see [26], Chapter III, Theorem 3.5). We will first show that the function $\sigma(x)$ can be chosen so that the first eigenvalue λ_1 is negative.

[3] Sh. Alimov, R. Ashurov, On determining the fractional exponent of the subdiffusion equation, arXive:2411.19852v1[math.AP]29Nov2024.
[26] Y. Berezanskii, Expansions in eigenfunctions of selfadjoint operators, Translations of Mathematical Monographs 17 (1968), pp. 809.

Indeed, for any smooth function $v(x)$ the identity

$$\int\limits_{\Omega} [-\Delta v(x)] v(x)\, dx \;=\; \int\limits_{\Omega} \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 dx - \int\limits_{\partial\Omega} v(x) \frac{\partial v}{\partial n}\, ds(x)$$

holds, obtained by integration by parts (see, e.g., [26], p. 90). If $v(x)$ is an eigenfunction then

$$\lambda \;=\int\limits_{\Omega} \sum_{i=1}^{N} \left(\frac{\partial v}{\partial x_i} \right)^2 dx + \int\limits_{\partial\Omega} |v(x)|^2 \sigma(x)\, ds(x).$$

Let us divide the boundary $\partial\Omega$ of the domain Ω into the following three parts:

$$\begin{aligned} E^+(\sigma) \;&=\; \{x \in \partial\Omega \mid \sigma(x) > 0\}, \\ E^-(\sigma) \;&=\; \{x \in \partial\Omega \mid \sigma(x) < 0\}, \\ E^0(\sigma) \;&=\; \{x \in \partial\Omega \mid \sigma(x) = 0\}. \end{aligned}$$

The flow exits through the surface $E^+(\sigma)$, the flow enters through $E^-(\sigma)$, and at the points $E^0(\sigma)$ the flow is zero. Note that in the case when $\mathrm{mes}_{N-1} E^-(\sigma) > 0$, the eigenvalue may turn out to be negative. At the same time, exponentially growing solutions appear. We examine exactly this case.

Now it is clear how one can construct a specific boundary value problem (10) with some function $\sigma(x)$ (which has a specific physical meaning) so that the first eigenvalue of the corresponding spectral problem will be negative.

So our basic assumption is this:

$$\lambda_1 < 0, \quad \lambda_1 \neq -1. \tag{11}$$

Let p be the multiplicity of the eigenvalue λ_1 and $v_{1,k}(x)$ be the eigenfunctions corresponding to λ_1. Let

$$P(\phi) \;=\; \{x \in \Omega \mid \sum_{k=1}^{p} (\phi, v_{1,k}) v_{1,k}(x) = 0\},$$

The following statement is true (without involvement of $u_t(x,t)$).

Theorem 10

Let condition (11) be satisfied and let $0 < \rho < 1$. Then for any function $\phi \in C_0^\infty(\Omega)$ at each point $x \in \Omega \setminus P(\phi)$ the following equality

$$\frac{1}{\ln|\lambda_1|} \lim_{t \to +\infty} \left(\ln \ln |u(x,t)| - \ln t\right) \;=\; \frac{1}{\rho}$$

holds.

Main result of the work [4].

Motivation. The authors of [6] considered the following inverse problem:

$$\begin{cases} D_t^\rho u(x,t) - c\,\Delta u(x,t) = 0, \quad x \in \Omega, \quad t > 0, \\ u(x,0) = \phi, \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ U(\rho; x_0, t_0) := u(x_0, t_0) = d_0, \end{cases} \tag{12}$$

(note, over-determination condition given at a single space-time point (x_0, t_0)) and proved the following very interesting result:

Theorem 11

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$ and the diffusion coefficient $c > 0$ be sufficiently small and the measured time $t_0 > 0$ be sufficiently large. Moreover, suppose that there exists a positive integer $K > 1$ and a measured point $x_0 \in \Omega$ such that $v_k(x_0)$ and ϕ_k satisfy the condition:
(A1) $\phi_k \geq 0$ and $v_k(x_0) \geq 0, k = 1, 2, \ldots, K$; and for some $k_0 \in \{1, 2, \ldots, K\}$, $\phi_{k_0} > 0$ and $v_{k_0}(x_0) > 0$.

Then the function $U(\rho; x_0, t_0) = \sum\limits_{k=1}^{\infty} \phi_k v_k(x_0) E_\rho(-ct_0^\rho)$ strictly monotonically decreases on $\rho \in (0,1)$, and the inverse problem (12) has a unique solution.

But unfortunately, condition (A1) is not sufficient for function $U(\rho; x_0, t_0)$ to be strictly monotone on $\rho \in (0, 1)$ (one can construct a counterexample showing that this is not the case).

Nevertheless, the inverse problem considered in this paper and the proposed method are very interesting. At the same time, the authors proved several auxiliary lemmas, which are undoubtedly of independent interest. Having slightly modified this method, we were able (see [4]) to prove the following results:

1) the requirement of a sufficiently small coefficient c was removed from the considered equation;
2) the restriction on the dimension of the domain $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$ was removed;
3) it is shown that the conditions of the above theorem of [6] is not complete and an updated version of this theorem is proved.

[4] R. Ashurov, I. Sulaymonov, Monotonicity of the Mittag-Leffler function and determining the fractional exponent of the subdiffusion equation, arXive:2501.01724v1[math.AP]3Jan2025.
[6] G. Li, Z. Wang, X. Jia, Y. Zhang, An inverse problem of determining the fractional order in the TFDE using the measurement at one space-time point, Fractional Calculus and Applied Analysis, 26:1770–1785, (2023).

The transition to the N-dimensional domain is based on the following lemma of V.A. Ilyin [27]:

Lemma 12

Let the function $g(x)$ satisfy the conditions

1. $g(x) \in C^p(\overline{\Omega}),\ \frac{\partial^{p+1} g(x)}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}} \in L_2(\Omega),\ p+1 = p_1 + p_2 + \ldots p_n, p \geq 1,$
2. $g(x)|_{\partial\Omega} = \Delta g(x)|_{\partial\Omega} = \cdots = \Delta^{\left[\frac{p}{2}\right]} g(x)|_{\partial\Omega} = 0.$

Then the number series $\sum\limits_{k=1}^{\infty} g_k^2 \lambda_k^{p+1}$ converges, where $g_k = (g, v_k)$.

Lemma 13

Let $\rho_0 \in (0,1)$ and $t_1 > 0$. If the function $\varphi(x)$ satisfy conditions of Lemma 12 with the exponent $p = \left[\frac{N}{2}\right]$. Then for any number $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon, \rho_0, t_1)$ such that the following inequality holds for all $x \in \Omega,\ t \in [t_1, T], \rho \in [\rho_0, 1]$:

$$\sum_{k=n_0+1}^{\infty} \left| \varphi_k v_k(x) \frac{d}{d\rho} E_\rho(-\lambda_k t^\rho) \right| < \varepsilon.$$

[27] V. Ilyin, "On the solvability of mixed problems for hyperbolic and parabolic equations", Russian Math. Surveys. 15, No. 2, 97-154 (1960).

Now we present the theorem about the solution of Inverse Problem (12).

Theorem 14

Let $\varepsilon > 0$ and $t_0 \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho_0}}}, \frac{1}{e^{\frac{7}{2}}}\right)\right]$. Suppose that function $\varphi(x)$ satisfies the conditions of Lemma 13 and for some $x_0 \in \Omega$ the quantities $v_k(x_0)$ and φ_k satisfy the following conditions:

1. $\varphi_k \geq 0$ and $v_k(x_0) \geq 0$ for $k = 1, 2, \ldots, n_0$, where $n_0 = n_0(\varepsilon, \rho_0, t_0)$ from Lemma 13;
2. φ_{k_0} and $v_{k_0}(x_0)$ satisfying following inequality for some $k_0 \in \{1, 2, \ldots, n_0\}$

$$\varphi_{k_0} v_{k_0}(x_0) > \frac{\varepsilon}{M_{k_0}},$$

   where $M_{k_0} = \frac{d}{d\rho} E_\rho(-\lambda_{k_0} t_0^\rho)$.

Then the nonlinear function $U(\rho; x_0, t_0)$ is strictly monotonically increasing on $\rho \in [\rho_0, 1]$. Moreover, Inverse Problem (12) has a unique solution if and only if

$$d_0 \in [U(\rho_0; x_0, t_0), U(1; x_0, t_0)]. \tag{13}$$

Remark 15

Let $M_{k_0} = \min_{1 \le k \le n_0} M_k$. Then from the given proof it follows that condition (2) of Theorem 14 can be replaced by the condition

$$\sum_{k=1}^{n_0} \varphi_k v_k(x_0) > \frac{\varepsilon}{M_{k_0}}. \tag{14}$$

Of course, the conditions of Theorem 14 are quite strict, especially the condition of non-negativity of several eigenfunctions. The question naturally arises: is it possible to formulate an initial-boundary value problem for which a similar inverse problem is uniquely solvable under less restrictive conditions? A positive answer to this question is given in the work [4].

[4] R. Ashurov, I. Sulaymonov, Monotonicity of the Mittag-Leffler function and determining the fractional exponent of the subdiffusion equation, arXive:2501.01724v1[math.AP]3Jan2025.

Consider the inverse problem:

$$\begin{cases} \partial_t^\rho u(x,t) - \Delta u(x,t) = 0, \quad t > 0, \quad x \in \Omega, \\ \lim\limits_{t\to 0} J_t^{\rho-1} u(x,t) = \phi, \\ \dfrac{\partial u}{\partial n} + \sigma(x) v(x) = 0, \quad x \in \partial\Omega, \\ U(\rho; x_0, t_0) \equiv u(x_0, t_0) = d_0, \end{cases} \tag{15}$$

with the assumption:

$$\lambda_1 < 0, \quad \lambda_1 \neq -1.$$

The solution of the inverse problem (15) exists and is unique, which follows directly from the following statement (see [4]):

Lemma 16

Let $x_0 \in \Omega \setminus P(\phi)$. Then there is a number $T_0 = T_0(\rho, \lambda^*)$, $\lambda^* = \min\limits_j |\lambda_j|$, such that for all $t_0 \geq T_0$ the function $U(\rho; x_0, t_0) \equiv u(x_0, t_0)$ of ρ is strictly monotone.

Remember, the set $P(\phi)$ has the form:

$$P(\phi) = \{x \in \Omega \mid \sum_{k=1}^{p} (\phi, v_{1,k}) v_{1,k}(x) = 0\},$$

where p is the multiplicity of the eigenvalue λ_1 and $v_{1,k}(x)$ be the eigenfunctions corresponding to λ_1.

In conclusion, I would like to note the recent interesting work of A. N. Artyushin [6], where the inverse problem with a variable order of the derivative is considered:

$$\begin{cases} D_t^{\rho(x)} u(x,t) - \Delta u(x,t) = f(x,t),\ 0 < \rho(x) \leq \lambda < 1, \quad x \in \Omega \subset R^N,\ 0 < t < T, \\ Bu(x,t) \equiv \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = 0, \quad x \in \overline{\Omega}, \\ u(x,T) = \varphi(x),\ x \in \overline{\Omega}. \end{cases} \tag{16}$$

The author has proved the existence of some operator A such that a solution of the inverse problem $\rho(x)$ is a fixed point of A: $A\rho(x) = \rho(x)$. Without having an explicit form of the operator A, the author was able to apply the Birkhoff–Tarski theorem to this operator and prove the following statement: When certain conditions are met on the functions $f(x,t)$ and $f_t(x,t)$, the solution to the inverse problem $\rho(x)$ exists and is unique if function $\varphi(x) \in W_2^2(\Omega)$ is positive and satisfies the following conditions:

$$\frac{\partial \varphi(x)}{\partial n} = 0,\ x \in \partial\Omega,\ \text{and } \Gamma(1-\lambda)T^\lambda G(x) \leq \varphi(x) \leq G(x),$$

where $G(x) = f(x,T) + \Delta\varphi(x)$.

[6] A. N. Artyushin, An inverse problem of recovering the variable order of the derivative in a fractional diffusion equation, Siberian Mathematical Journal, 2023, Vol. 64, No. 4, pp. 796–806.

Motivated by this work, we (with I. Sulaymonov) have proved the following results:

Theorem 17

Let $\rho_0 \in (0,1)$. Then, for any $t \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho_0}}}, \frac{1}{e^{\frac{7}{2}}}\right)\right]$, the Mittag-Leffler function $E_\rho(-t^\rho)$ is monotonically increasing in $\rho \in [\rho_0, 1]$.

Theorem 18

Let $\rho_0 \in (0,1)$. Then, for any $t \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho_0}}}, \frac{1}{e^{\frac{13}{6}}}\right)\right]$, the function $t^{\rho-1}E_{\rho,\rho}(-t^\rho)$ is monotonically decreasing in $\rho \in [\rho_0, 1]$.

In proving these statements we used some ideas from the recent paper [5]. In that paper it was proved that the function $g(\rho) = E_\rho(-ct^\rho)$ has a negative derivative $g(\rho) < 0$ if t is large enough and c is small enough.

[4] R. Ashurov, I. Sulaymonov, Monotonicity of the Mittag-Leffler function and determining the fractional exponent of the subdiffusion equation, arXive:2501.01724v1[math.AP]3Jan2025.
[5] G. Li, Z. Wang, X. Jia, Y. Zhang, An inverse problem of determining the fractional order in the TFDE using the measurement at one space-time point, Fractional Calculus and Applied Analysis, 26:1770–1785, (2023).

Thank you for attention!