The Landau Hamiltonian with a delta-potential supported on a curve

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Seminar on Analysis, Differential Equations and Mathematical Physics

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PART I

Warm Up: δ-point interactions in 1D

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For $\varepsilon \to 0$ we obtain the boundary condition:

$$-\psi'(y+) + \psi'(y-) + \alpha\psi(y) = 0$$

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is the norm resolvent limit of

$$H_n = -\Delta + V_n, \quad \text{where} \quad \operatorname{supp} V_n \to \{y\},$$

and V_n is suitable locally scaled family of L^1-potentials.

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- Justification as model for localized potentials

PART II

Landau Hamiltonian with δ-potential

2D (unperturbed) Landau Hamiltonian

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Consider magnetic gradient

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where $\Lambda_q = B(2q+1)$, $q \in \mathbb{N}_0$, Landau levels.

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- V. Fock, *Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld*,
  Zeitschrift für Physik 47, 1928, 446–448
- L. Landau, *Diamagnetismus der Metalle*,
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- B. Simon, *Schrödinger operators with singular magnetic vector potentials*,
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- T. Kato, *Remarks on Schrödinger operators with vector potentials*,
  Integral Equations and Operator Theory 1, 1978, 103–113.

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$\mathfrak{a}_\alpha$ is closed semibounded form

2D Landau Hamiltonian with δ-potential

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Notation: $A_\alpha = \nabla_A^2 + \alpha\delta$

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Approach allows to use boundary triples and Weyl functions !

PART III

Spectral theory and approximation

Boundary integral operators to express $(A_\alpha - \lambda)^{-1}$

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G_λ Greens function (explicitely known) of A_0:

$$((A_0 - \lambda)^{-1} f)(x) = \int_{\mathbb{R}^2} G_\lambda(x, y) f(y) dy, \quad \lambda \in \rho(A_0).$$

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The γ**-field** $\rho(A_0) \ni \lambda \mapsto \gamma(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^2)$ is

$$(\gamma(\lambda)\varphi)(s) = \int_\Sigma G_\lambda(s, t)\varphi(t) d\sigma(t), \quad \varphi \in L^2(\Sigma),\ s \in \mathbb{R}^2.$$

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- Related results in 3D: Bony, Bruneau, Raikov, Sambou

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Method: Show for $\lambda_0 \in \mathbb{R}$ and $P_q : L^2(\mathbb{R}^2) \to \ker(A_0 - \Lambda_q)$ that

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is infinite rank operator.

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Proofs inspired by [Pushnitski Rozenblum, 2007].

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In stark contrast to [Klopp Raikov, IMRN 2009] for classical potential $V \geq 0$ (or $V \leq 0$) with $0 < \|V\|_{L^\infty} < 2B$ one has:

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There are ∞-many $r > 0$ with $\ker(A_\alpha - \Lambda_q) \neq \{0\}$ for some q.

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In particular, $H_\varepsilon \to A_\alpha$ in norm resolvent sense as $\varepsilon \to 0$.

References

This talk was based on the papers

- JB, P. Exner, M. Holzmann, V. Lotoreichik
  The Landau Hamiltonian with δ-potentials supported on curves
  Rev. Math. Phys. 32 (2020), 2050010, 51 pp
- JB, M. Holzmann, V. Lotoreichik, G. Raikov
  The fate of Landau levels under δ-interactions
  J. Spectral Theory 12 (2022), 1203-1234

Thank you for your attention