Refinements of Berry-Esseen inequality in terms of Lyapunov coefficients

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Setting

$X_1, \dots, X_n$ independent random variables, $\mathbb{E}X_k = 0$, $\mathrm{Var}(X_k) = \sigma_k^2$,

$$S_n = X_1 + \dots + X_n.$$

Normalization: $\mathbb{E}S_n^2 = \sigma_1^2 + \dots + \sigma_n^2 = 1$.

Distribution functions: $F_n(x) = \mathbb{P}\{S_n \le x\}$.

Kolmogorov distance from the standard normal law:

$$\Delta_n = \sup_x |F_n(x) - \Phi(x)|,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy, \quad x \in \mathbb{R}.$$

IID model: $X_k = \xi_k/\sqrt{n}$ with iid ξ_k such that $\mathbb{E}\xi_1 = 0$, $\mathbb{E}\xi_1^2 = 1$.

Weighted sums iid: $X_k = a_k \xi_k$ $(a_1^2 + \dots + a_n^2 = 1,\ a_k \in \mathbb{R})$.

CLT and Lyapunov coefficients

Lyapunov coefficients:

$$L_p = \sum_{k=1}^{n} \mathbb{E}\,|X|^p, \qquad p > 2.$$

IID model:

$$L_p = n^{-\frac{p-2}{2}}\,\beta_p, \qquad \beta_p = \mathbb{E}\,|\xi_1|^p.$$

Theorem (Berry-Esseen inequality). Up to some absolute constant $c > 0$,

$$\Delta_n \leq cL_3.$$

In particular, in the iid model,

$$\Delta_n \leq \frac{c\beta_3}{\sqrt{n}}.$$

Note: $L_3 \geq \frac{1}{\sqrt{n}}$.

Truncated Lyapunov coefficients

Definition:

$$R_p = \sum_{k=1}^{n} \mathbb{E} \min\{1, |X_k|^{p-2}\} X_k^2, \qquad p > 2.$$

Note: $\frac{c}{\sqrt{n}} \leq R_3 \leq \min\{1, L_3\}$.

Theorem (improved Berry-Esseen inequality): $\Delta_n \leq cR_3$.

History in the typical model of increasing sums: $S_n = \frac{1}{B_n} \sum_{k=1}^{n} \xi_k$

M.L. Katz (1963), V.V. Petrov (1965), Ju.P. Studnev (1958, 1965), L.V. Osipov (1965), W. Feller (1968)

Constants: $c = 2.02$ (V.Yu. Korolev, S.V. Popov 2011), $c = 1.87$ (A. Dorofeeva, V.Yu. Korolev 2017)

Reproved (as investigated by R.A. Gabdullin, V.A. Makarenko, I.G. Shevtsova 2018):

H. Paditz (1986), A.D. Barbour and P. Hall (2001 Stein method), L.H.Y. Chen and Q.M. Shao (2001)

Connection with theorems by Lindeberg and Osipov

Consider the sums

$$S_n = \frac{1}{B_n} \sum_{k=1}^{n} \xi_k, \qquad \mathbb{E}\xi_k = 0, \quad B_n^2 = \sum_{k=1}^{n} \mathbb{E}\xi_k^2, \quad V_k(x) = \mathbb{P}\{X_k \le x\}.$$

For $\varepsilon > 0$, define

$$\Lambda_n(\varepsilon) = \frac{1}{B_n} \sum_{k=1}^{n} \int_{|x|>\varepsilon B_n} x^2 \, dV_k(x), \qquad L_n(\varepsilon) = \frac{1}{B_n} \sum_{k=1}^{n} \int_{|x|\le\varepsilon B_n} |x|^3 \, dV_k(x) \le \varepsilon.$$

Theorem (Lindeberg): If $\Lambda_n(\varepsilon) \to 0$ for any $\varepsilon > 0$, then $\Delta_n \to 0$ as $n \to \infty$.

Theorem (Osipov):

$$|\Delta_n| \le C\,(\Lambda_n(\varepsilon) + L_n(\varepsilon)).$$

Notes: Let $X_k = \frac{1}{B_n}\xi_k$. Gabdullin-Makarenko-Shevtsova 2018: For any $\varepsilon > 0$,

$$R_3 \;=\; \Lambda_n(1) + L_n(1) \;\le\; \Lambda_n(\varepsilon) + L_n(\varepsilon).$$

Moments smaller than 3

Moments $p = 2 + \delta$, $0 < \delta < 1$.

Since $\min(1, |x|)\, x^2 \leq |x|^{2+\delta}$ for all $x \in \mathbb{R}$, we have

$$R_3 \leq L_{2+\delta}.$$

Corollary:

$$\Delta_n \leq c L_{2+\delta}.$$

In particular, in the iid model,

$$\Delta_n \leq \frac{c\beta_{2+\delta}}{n^{\delta/2}}.$$

Notes: The case $0 < \delta < 1$: Lyapunov (according to Osipov)
Necessary and sufficient conditions for $\Delta_n = O(n^{-\delta/2})$: I. A. Ibragimov (1966).

Optimality of Berry-Esseen bounds

Weighted sums:

$$S_n = a_1\xi_1 + \dots + a_n\xi_n, \qquad a_1^2 + \dots + a_n^2 = 1 \ \ (a_k \in \mathbb{R})$$

with iid ξ_k such that $\mathbb{E}\xi_1 = 0$, $\mathbb{E}\xi_1^2 = 1$. Put $\alpha_3 = \mathbb{E}\xi_1^3$, $\beta_p = \mathbb{E}\,|\xi_1|^p$.

Theorem 1. $a)$ Let $\alpha_3 \neq 0$ and $\beta_4 < \infty$. If all $a_k \geq 0$, then with $c' = c'(\alpha_3, \beta_4)$

$$c' L_3 \leq \Delta_n \leq c L_3.$$

$b)$ If $\beta_4 \neq 3$ and $\beta_5 < \infty$, then with $c = c'(\beta_4, \beta_5)$

$$c' L_4 \leq \Delta_n \leq c L_3.$$

Note:

$$L_3 = \beta_3 \sum_{k=1}^n |a_k|^3 \geq \beta_3 \, (\max_k |a_k|)^3,$$

implying

$$L_4 \leq c L_3^{4/3}, \qquad c = \beta_4 \beta_3^{-1/3}.$$

New Berry-Esseen-type bounds

As before, let $X_1, \dots, X_n$ be independent r.v. with mean zero, and let $\mathbb{E} S_n^2 = 1$.

Theorem 2. Suppose that X_k have finite 4-th moments with $\mathbb{E} X_k^3 = 0$. For any $\delta \in (0, 1]$,

$$\Delta_n \leq c \left(\frac{1}{\delta} L_4 + L_{2+\delta}^{1/\delta} \right).$$

Moreover, if the distributions of X_k are symmetric and have finite absolute moments of order $2 + \delta$, then

$$\Delta_n \leq c \left(\frac{1}{\delta} R_4 + L_{2+\delta}^{1/\delta} \right).$$

Notes. If $\delta = 1$, we return to the Berry-Esseen theorem (symmetric case).
In general, $\delta \to L_{2+\delta}^{1/\delta}$ is non-decreasing. In particular, $L_3 \leq \sqrt{L_4}$.

Examples

Let the i.i.d. random variables ξ_k have $\mathbb{E}\xi_1 = \mathbb{E}\xi_1^3 = 0$, $\mathbb{E}\xi_1^2 = 1$, $\mathbb{E}\xi_1^4 < \infty$. For $0 < \alpha < \frac{1}{2}$, consider

$$S_n = \frac{1}{b_n} \sum_{k=1}^{n} \frac{1}{k^\alpha} \xi_k, \qquad b_n \sim n^{\frac{1}{2}-\alpha}.$$

Main case: $\frac{1}{3} < \alpha < \frac{1}{2}$. Choose $\delta < \frac{1}{\alpha} - 2)$. Then

$$L_3 \sim n^{-3\,(\frac{1}{2}-\alpha)}, \qquad L_4 \sim n^{-4\,(\frac{1}{2}-\alpha)} \sim L_3^{4/3}, \qquad L_{2+\delta}^{1/\delta} \sim \frac{1}{\sqrt{n}} = o(L_3).$$

Therefore, Th2 improves the usual B-E bound and yields

$$\Delta_n \le c\left(L_4 + \frac{1}{\sqrt{n}}\right).$$

Moreover, by Th1,

$$\Delta_n \sim L_4, \qquad \frac{3}{8} \le \alpha < \frac{1}{2}.$$

Note: For $\frac{1}{4} < \alpha < \frac{1}{3}$,

$$L_3 \sim L_{2+\delta}^{1/\delta} \sim \frac{1}{\sqrt{n}}, \qquad L_4 = o(L_3).$$

Chebyshev–Edgeworth correction

Let X_k have finite absolute moments of an integer order $p \geq 4$, and $\Phi_{p-1}(x)$ denote the Chebyshev–Edgeworth correction for $F_n(x)$ based on moments of X_k up to order $p-1$.

First corrections.

$$\Phi_3(x) \;=\; \Phi(x) - \frac{\gamma_3}{3!}\,(x^2-1)\varphi(x), \qquad \gamma_3 = \sum_{k=1}^{n} \mathbb{E}X_k^3,$$

$$\Phi_4(x) \;=\; \Phi(x) - \frac{\gamma_4}{4!}\,(x^3-3x)\varphi(x), \qquad \gamma_4 = \sum_{k=1}^{n} (\mathbb{E}X_k^4 - 3) \quad (\text{if } \mathbb{E}X_k^3 = 0).$$

Theorem 3.

$$\sup_x |F_n(x) - \Phi_{p-1}(x)| \leq c_p \left(\delta^{-\frac{p-2}{2}}\, L_p + L_{2+\delta}^{1/\delta}\right), \qquad 0 < \delta \leq 1.$$

Previous example:

$$\sup_x |F_n(x) - \Phi_{p-1}(x)| \leq c_{p,\alpha}\, \frac{\beta_p}{\sqrt{n}}, \qquad p > \frac{1}{1-2\alpha}.$$

Normal approximation for characteristic functions

Let $X_1, \dots, X_n$ be independent r.v. with mean zero, and let $\mathbb{E} S_n^2 = 1$. Put $f_n(t) = \mathbb{E}\, e^{itS_n}$.

Esseen's smoothing:

$$c\Delta_n \le \int_0^T \left| \frac{f_n(t) - e^{-t^2/2}}{t} \right| dt + \frac{1}{T}, \qquad T > 0.$$

It is well-known that

$$|f_n(t) - e^{-t^2/2}| \le cL_3 \min(1, |t|^3)\, e^{-t^2/6}, \qquad |t| \le \frac{1}{L_3}.$$

Lemma 1 (Osipov). We have

$$|f_n(t) - e^{-t^2/2}| \le cR_3 \min(1, |t|^3)\, e^{-t^2/6}, \qquad |t| \le \frac{1}{32\, R_3}.$$

Lemma 2. If the distributions of X_k are symmetric about the origin, then

$$|f_n(t) - e^{-t^2/2}| \le cR_4 \min(1, |t|^3)\, e^{-t^2/6}, \qquad |t| \le \frac{1}{32\, R_3}.$$

Characteristic functions for single random variables

Let X be a random variable with $\mathbb{E}X = 0$, $\mathrm{Var}(X) = \sigma^2$, and let

$$f(t) = \mathbb{E}\, e^{itX}, \quad t \in \mathbb{R}.$$

Lemma 3. For all $t \in \mathbb{R}$,

$$f(t) = 1 - \frac{\sigma^2 t^2}{2} + \theta t^2\, \mathbb{E}\, \min\{1, |tX|\} X^2, \qquad |\theta| \le 2.$$

Moreover, if the distribution of X is symmetric, then

$$f(t) = 1 - \frac{\sigma^2 t^2}{2} + \theta t^2\, \mathbb{E}\, \min\{1, (tX)^2\} X^2, \qquad |\theta| \le 4.$$

Cox-Kemperman inequality

Lemma 4 (C-K 1983). Given independent random variables X and Y with mean zero and finite absolute moments of order $p \geq 2$,

$$\mathbb{E}\,|X+Y|^p \leq 2^{p-2}\left(\mathbb{E}\,|X|^p + \mathbb{E}\,|Y|^p\right).$$

Cases of equality: X and Y have a symmetric Bernoulli distribution.

Proof: For all $x, y \in \mathbb{R}$,

$$|x+y|^p \leq 2^{p-2}\left(|x|^p + |y|^p + x\,\mathrm{sign}(y)\,|y|^{p-1} + y\,\mathrm{sign}(x)\,|x|^{p-1}\right).$$

Upper bounds for characteristic functions

Lemma 5. We have

$$|f_n(t)| \leq 2e^{-t^2/4}, \qquad |t| \leq \frac{1}{32\,R_3}.$$

Proof: Apply Lemma 3.

Lemma 6. For all $\delta \in (0, 2]$,

$$|f_n(t)| \leq e^{-\delta t^2/6}, \qquad |t| \leq \frac{1}{L_{2+\delta}^{1/\delta}}.$$

In particular,

$$|f_n(t)| \leq e^{-t^2/6}, \qquad |t| \leq \frac{1}{L_3}.$$

Proof of Lemma 6

Let $X_1, \dots, X_n$ be independent r.v. with mean zero, variances σ_k^2 such that $\mathbb{E} S_n^2 = 1$. Put $v_k(t) = \mathbb{E}\, e^{itX_k}$, so that

$$f_n(t) = \mathbb{E}\, e^{itS_n} = v_1(t) \dots v_n(t).$$

We use

$$\cos x \le 1 - \frac{1}{2}\, x^2 + \frac{1}{2} \cdot 12^{-\delta/2}\, |x|^{2+\delta}, \quad x \in \mathbb{R}.$$

Applying C-K inequality, this gives

$$\begin{aligned} |v_k(t)|^2 &= \mathbb{E}\, \cos(t(X_k - Y_k)) \\ &\le 1 - \sigma_k^2 t^2 + \frac{1}{2} \cdot 12^{-\delta/2}\, |t|^{2+\delta}\, \mathbb{E}\, |X_k - Y_k|^{2+\delta} \\ &\le 1 - \sigma_k^2 t^2 + 3^{-\delta/2}\, |t|^{2+\delta}\, \mathbb{E}\, |X_k|^{2+\delta} \\ &\le \exp\big\{ - \sigma_k^2 t^2 + 3^{-\delta/2}\, |t|^{2+\delta}\, \mathbb{E}\, |X_k|^{2+\delta} \big\}. \end{aligned}$$

Proof of Lemma 6 (continuation)

Thus,

$$|v_k(t)| \le \exp\Big\{ -\frac{\sigma_k^2 t^2}{2} + \frac{1}{2}\cdot 3^{-\delta/2}\,|t|^{2+\delta}\,\mathbb{E}\,|X_k|^{2+\delta}\Big\}.$$

Multiplying over all $k \le n$,

$$|f_n(t)| \le \exp\Big\{ -\frac{t^2}{2} + \frac{1}{2}\cdot 3^{-\delta/2}\,|t|^{2+\delta}\,L_{2+\delta}\Big\}.$$

As a result, if $|t|^{\delta} L_{2+\delta} \le 1$,

$$|f_n(t)| \le \exp\Big\{ -\frac{1}{2}\,(1 - 3^{-\delta/2})\,t^2\Big\}.$$

To simplify, use $1 - 3^{-\delta/2} \ge \frac{1}{3}\,\delta$ for $0 < \delta \le 2$.