On the discrete eigenvalues of Schrödinger operators with complex potentials

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Based on joint work with Jean-Claude Cuenin (Loughborough), Sukrid Petpradittha (Durham) & František Štampach (Prague)

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Schrödinger operators

Consider $H_V := -\Delta + V$ in Hilbert space $L^2(\mathbb{R}^d)$ where $d \in \mathbb{N}$ arbitrary dimension. Acts as $(H_V\psi)(x) = -\Delta\psi(x) + V(x)\psi(x)$.
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We assume that V is sufficiently regular (e.g. piecewise continuous).

- If $V = 0$ (free Laplacian), then $\sigma(H_0) = \sigma_{\text{ess}}(H_0) = [0, \infty)$.
- We assume that $\lim_{|x|\to\infty} V(x) = 0$. Then $V(H_0 - \mathrm{i})^{-1}$ is compact, hence
$$\sigma_{\text{ess}}(H_V) = \sigma_{\text{ess}}(H_0) = [0, \infty).$$
- Discrete spectrum (isolated eigenvalues of finite algebraic multiplicities) is
$$\sigma_{\text{dis}}(H_V) = \sigma(H_V) \backslash [0, \infty).$$

  Accumulation points of discrete eigenvalues lie in $\sigma_{\text{ess}}(H_V) = [0, \infty)$.

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Accumulation points of discrete eigenvalues lie in $\sigma_{\mathrm{ess}}(H_V) = [0, \infty)$.
- If V real-valued (H_V selfadjoint), then $\sigma_{\mathrm{dis}}(H_V) \subset (-\infty, 0)$. Only possible accumulation point is 0.
- Questions: Where can eigenvalues lie? Modulus of eigenvalues? Distance to essential spectrum? If there are infinitely many eigenvalues, where do they accumulate and how fast?

Lieb-Thirring inequalities (1970s+)

Let

$$\begin{cases} p \geq \frac{d}{2}, & \text{if } d \geq 3, \\ p > 1, & \text{if } d = 2, \\ p \geq 1, & \text{if } d = 1. \end{cases}$$

Lieb-Thirring inequalities: There exists $C_{d,p} > 0$ such that for all for all real-valued $V \in L^p(\mathbb{R}^d)$,

$$\sum_{\lambda \in \sigma_{\mathrm{dis}}(H_V)} |\lambda|^{p-\frac{d}{2}} \leq C_{d,p} \|V\|_{L^p}^p = C_{d,p} \int |V(x)|^p \, \mathrm{d}x.$$

This also gives individual eigenvalue bound $|\lambda|^{p-\frac{d}{2}} \leq C_{d,p} \|V\|_{L^p}^p$.

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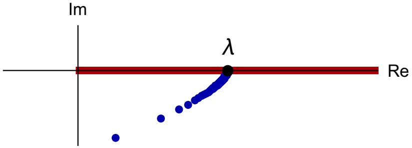
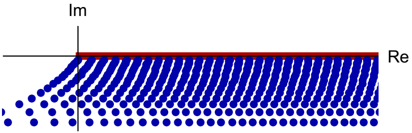
Question: What happens in non-selfadjoint case, i.e. for non-real $V \in L^p(\mathbb{R}^d)$?

Remark: In principle, non-real eigenvalues can accumulate to a non-zero point of the essential spectrum.



Wild behaviour in non-selfadjoint case

Theorem (Bögli 2017, Bögli-Cuenin 2023): Let $p > \frac{d+1}{2}$ and $\varepsilon > 0$. Then $\exists\, V \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $\max\{\|V\|_{L^\infty}, \|V\|_{L^p}\} < \varepsilon$ and $\sigma_{\mathrm{dis}}(H_V)$ accumulates at every point of $[0, \infty)$.



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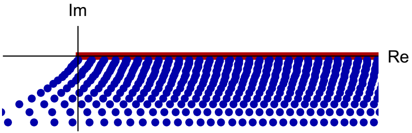

Remark: So for $p > \frac{d+1}{2}$ we can have an arbitrarily small $\|V\|_{L^p}$ but eigenvalues with arbitrarily large modulus.

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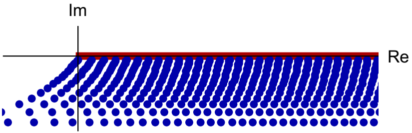

Remark: So for $p > \frac{d+1}{2}$ we can have an arbitrarily small $\|V\|_{L^p}$ but eigenvalues with arbitrarily large modulus.

Remark: If we can find a sequence $(V_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^d)$ with $\lim_{n\to\infty} \|V_n\|_{L^p} = 0$ and $\lambda_n \in \sigma_{\mathrm{dis}}(H_{V_n})$ with $\lim_{n\to\infty} \lambda_n = \lambda \in (0, \infty)$, then take subsequence such that $\sum_n \|V_n\|_{L^p} < \infty$ and consider potential

$$V(x) = \sum_n V_n(x - t_n x_0)$$

where $x_0 \in \mathbb{R}^n \setminus \{0\}$ and $t_n > 0$ sufficiently large (constructed by induction) $\rightsquigarrow$ $\sigma_{\mathrm{dis}}(H_V)$ accumulates to λ. For above wild behaviour, one needs scaling.

Remark on the sharpness of $p > \frac{d+1}{2}$

- $d = 1$: Abramov-Aslanyan-Davies (2001): If $V \in L^1(\mathbb{R})$, then

$$\forall\, \lambda \in \sigma_{\mathrm{dis}}\,(H_V)\,: \quad |\lambda|^{\frac{1}{2}} \leq \frac{1}{2}\|V\|_{L^1}.$$

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- $d \geq 2$: Conjecture by Laptev-Safronov (2009):

  For $p \in \left(\frac{d}{2}, d\right]$ there exists $C_{d,p} > 0$ such that if $V \in L^p(\mathbb{R}^d)$, then

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  * Proved for $p \in \left(\frac{d}{2}, \frac{d+1}{2}\right]$ (Frank, 2011).
  * Disproved for $p > \frac{d+1}{2}$ (Bögli-Cuenin 2023).
  * Embedded eigenvalues $\lambda \in [0, \infty)$: Also satisfy estimate for $p \in \left(\frac{d}{2}, \frac{d+1}{2}\right]$ but not for $p > \frac{d+1}{2}$ (Frank-Simon, 2016). For $p > \frac{d+1}{2}$, they found $(V_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^d)$ with $\lim_{n\to\infty} \|V_n\|_{L^p} = 0$ and $\lambda = 1$ is an embedded eigenvalue of each $-\Delta + V_n$.

Individual eigenvalue bounds and Laptev-Safronov conjecture

In the following always $d \geq 2$.

- Frank (2011): Laptev-Safronov conjecture holds for $p \in \left(\frac{d}{2}, \frac{d+1}{2}\right]$, i.e.

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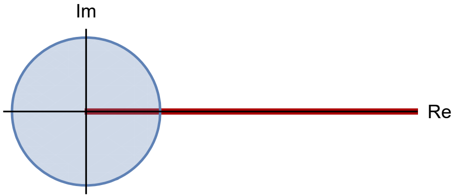

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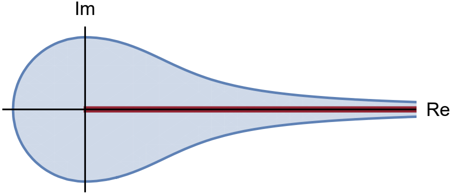
$$\operatorname{dist}(\lambda, [0,\infty))^{p-\frac{d+1}{2}} |\lambda|^{\frac{1}{2}} \leq C_{d,p} \|V\|_{L^p}^p.$$

Note: LHS reduces to $|\lambda|^{p-\frac{d}{2}}$ in the selfadjoint case (when $\lambda \in (-\infty, 0)$).



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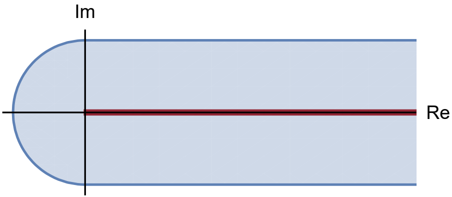
- Remark: For $V \in L^\infty(\mathbb{R}^d)$, since $H_0 = -\Delta$ selfadjoint we know

$$\forall \lambda \in \sigma(H_0 + V) : \quad \operatorname{dist}(\lambda, \sigma(H_0)) \leq \|V\| = \|V\|_{L^\infty} .$$



Proof idea: Birman–Schwinger principle

The Birman–Schwinger principle says that $\lambda \in \mathbb{C}\setminus[0,\infty)$ is an eigenvalue of $-\Delta + V$ if and only if -1 is an eigenvalue of

$$\sqrt{|V|}\,(-\Delta-\lambda)^{-1}\,\frac{V}{\sqrt{|V|}}.$$

The latter implies, for $p, q, q' \in [1,\infty]$ such that $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{q}-\frac{1}{q'}=\frac{1}{p}$:

$$\begin{aligned}
1 &\le \left\| \sqrt{|V|}\,(-\Delta-\lambda)^{-1}\,\frac{V}{\sqrt{|V|}} \right\| \\
&\le \left\| \sqrt{|V|} \right\|_{L^{q'}\to L^2} \left\|(-\Delta-\lambda)^{-1}\right\|_{L^q\to L^{q'}} \left\| \frac{V}{\sqrt{|V|}} \right\|_{L^2\to L^q} \\
&= \|V\|_{L^p} \left\|(-\Delta-\lambda)^{-1}\right\|_{L^q\to L^{q'}}.
\end{aligned}$$

By Kenig–Ruiz–Sogge (1987) (rescaled version by Frank):

$$\left\|(-\Delta-\lambda)^{-1}\right\|_{L^q\to L^{q'}} \le C_{d,q}\,|\lambda|^{-\frac{d+2}{d}+\frac{d}{q}}$$

under assumptions on q, d which translate to $p \in \left(\frac{d}{2}, \frac{d+1}{2}\right]$.

Counterexample for $p > \frac{d+1}{2}$ (Bögli-Cuenin 2023)

Let $H = -\Delta + V_0$ for a real-valued $V_0 \in L^p(\mathbb{R}^d)$.

Theorem: Take $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, and $K \subset \mathbb{R}^d$ compact. Then there exists V such that z is an eigenvalue of $H + V$ and

$$\forall\, x \in \mathbb{R}^d : \quad |V(x)| \leq \frac{\chi_K(x)}{\|\chi_K \, \mathrm{Im}\left((H-z)^{-1}\right) \chi_K\|}.$$

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Proof idea: Assume there exist $\mu \neq 0$ and $f \not\equiv 0$ with

$$\chi(H-z)^{-1}\chi f = \mu f.$$

Let $g := (H-z)^{-1}\chi f$. Then

$$(H-z)g = \chi f = f = \frac{1}{\mu}\chi g.$$

Now take

$$V = -\frac{1}{\mu}\chi.$$

BUT don't know whether $|V(x)|$ small, i.e. $|\mu|$ large!

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Proof: $T := \chi \,\mathrm{Im}\left((H-z)^{-1}\right) \chi$ is compact, positive operator, hence $\mathrm{Specrad}(T) = \|T\|$. Thus there exists real-valued f with $Tf = \|T\| f$. Let $g := (H-z)^{-1}\chi f$. Then $\mathrm{Im}\left((H-z)^{-1}\right)\chi f = \mathrm{Im}(g)$ and hence

$$(H-z)g = \chi f = f = \frac{1}{\|T\|} T f = \frac{1}{\|T\|} \chi \mathrm{Im}(g).$$

Now take

$$V(x) = -\frac{1}{\|T\|}\chi(x)\frac{\mathrm{Im}(g(x))}{g(x)}.$$

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$$\forall\, x \in \mathbb{R}^d : \quad |V(x)| \leq \frac{\chi_K(x)}{\|\chi_K \operatorname{Im}\left((H-z)^{-1}\right)\chi_K\|}.$$

Corollary: Let $\lambda \in \sigma(H)$ and $\psi \in \mathcal{D}(H)$ with $\|\psi\|_{L^2} = 1$ and $H\psi = \lambda\psi$. Assume that $K \subset \mathbb{R}^d$ compact is chosen so large that

$$\left(\int_{\mathbb{R}^d \setminus K} |\psi(x)|^2 \,\mathrm{d}x\right)^{1/2} \leq 1/4.$$

Then, for any $\varepsilon > 0$, there exists $V_\varepsilon \in L^\infty(\mathbb{R}^d)$ such that $z_\varepsilon = \lambda + i\varepsilon$ is an eigenvalue of $H + V_\varepsilon$ and

$$\forall\, x \in \mathbb{R}^d : \quad |V_\varepsilon(x)| \leq \frac{\chi_K(x)}{\|\chi_K \operatorname{Im}\left((H-z_\varepsilon)^{-1}\right)\chi_K\|} \leq 4\varepsilon\chi_K(x).$$

Note: $\|V_\varepsilon\|_{L^p} \leq 4\varepsilon\|\chi_K\|_{L^p}$ can be made arbitrarily small.

$\rightsquigarrow$ One can perturb the embedded eigenvalue $\lambda = 1$ of the Frank–Simon example, thus disproving the Laptev-Safronov conjecture $|z|^{p-d/2} \leq C\|V\|_{L^p}^p$.

Generalised Lieb–Thirring inequalities for non-selfadjoint case

Theorem (Frank-Laptev-Lieb-Seiringer 2006): For $p \geq \frac{d}{2} + 1$ there exists $C_{d,p} > 0$ such that, for all $V \in L^p(\mathbb{R}^d)$, the eigenvalues of H_V satisfy

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(eigenvalues in left half-plane) and, for all $t > 0$,

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(eigenvalues outside sector with semi-angle $\arctan t$).

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Theorem (Demuth-Hansmann-Katriel 2009): For $p \geq \frac{d}{2} + 1$ and $\varepsilon > 0$ there exists $C_{d,p,\varepsilon} > 0$ such that for all $V \in L^p(\mathbb{R}^d)$

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Note: LHS reduces to $\sum_{\lambda \in \sigma_{\mathrm{dis}}(H_V)} |\lambda|^{p-\frac{d}{2}}$ in selfadjoint case, as in LT ineq.

Open problem

Open problem (Demuth-Hansmann-Katriel 2013):
Prove the existence of $C_{d,p} > 0$ such that for all $V \in L^p(\mathbb{R}^d)$

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or find a counterexample.

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Theorem (Bögli-Štampach 2021): Let $d = 1$ and $p \geq 1$. Then

$$\lim_{t \to +\infty} \left(\frac{1}{\|\mathrm{i}t\chi_{[-1,1]}\|_{L^p}^p} \sum_{\lambda \in \sigma_{\mathrm{dis}}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}t\chi_{[-1,1]}\right)} \frac{\operatorname{dist}(\lambda, [0,\infty))^p}{|\lambda|^{\frac{1}{2}}} \right) = \infty.$$

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Theorem (Bögli-Petpradittha-Štampach 2024+): This works also in higher dimensions, with potential $\mathrm{i}t\chi_{B_1(0)}$.

New Lieb–Thirring type inequalities

Theorem (Bögli 2023): Let $d \in \mathbb{N}$ and $p \geq \frac{d}{2} + 1$. Let $f : [0,\infty) \to (0,\infty)$ be a continuous, non-increasing function. If $\int_0^\infty f(s)\,\mathrm{d}s < \infty$, then there exists $C_{d,p,f} > 0$ such that, for any $V \in L^p(\mathbb{R}^d)$,

$$\sum_{\lambda \in \sigma_{\mathrm{dis}}(-\Delta+V)} \frac{\mathrm{dist}(\lambda,[0,\infty))^p}{|\lambda|^{d/2}} f\left(-\log\left(\frac{\mathrm{dist}(\lambda,[0,\infty))}{|\lambda|}\right)\right) \leq C_{d,p,f} \|V\|_{L^p}^p$$

where $C_{d,p,f} = C_{d,p} \cdot \left(\int_0^\infty f(s)\,\mathrm{d}s + f(0)\right)$ with $C_{d,p} > 0$ independent of f.

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Theorem (Bögli 2023): Let $d \in \mathbb{N}$ and $p \geq \frac{d}{2} + 1$. Let $f : [0, \infty) \to (0, \infty)$ be a continuous, non-increasing function. If $\int_0^\infty f(s)\,\mathrm{d}s < \infty$, then there exists $C_{d,p,f} > 0$ such that, for any $V \in L^p(\mathbb{R}^d)$,

$$\sum_{\lambda \in \sigma_{\mathrm{dis}}(-\Delta+V)} \frac{\mathrm{dist}(\lambda, [0,\infty))^p}{|\lambda|^{d/2}} f\left(-\log\left(\frac{\mathrm{dist}(\lambda, [0,\infty))}{|\lambda|}\right)\right) \leq C_{d,p,f} \|V\|_{L^p}^p$$

where $C_{d,p,f} = C_{d,p} \cdot \left(\int_0^\infty f(s)\,\mathrm{d}s + f(0)\right)$ with $C_{d,p} > 0$ independent of f.

Remark: Demuth-Hansmann-Katriel result corresponds to $f(s) = \exp(-\varepsilon s)$. There are L^1-functions with a slower decay at infinity, such as

$$f(s) \sim \frac{1}{s^{1+\varepsilon}}, \quad f(s) \sim \frac{1}{s \log(s)^{1+\varepsilon}}, \quad f(s) \sim \frac{1}{s \log(s) \log(\log(s))^{1+\varepsilon}}, \quad \ldots .$$

New Lieb–Thirring type inequalities

Theorem (Bögli 2023): Let $d \in \mathbb{N}$ and $p \geq \frac{d}{2} + 1$. Let $f : [0,\infty) \to (0,\infty)$ be a continuous, non-increasing function. If $\int_0^\infty f(s)\,\mathrm{d}s < \infty$, then there exists $C_{d,p,f} > 0$ such that, for any $V \in L^p(\mathbb{R}^d)$,

$$\sum_{\lambda \in \sigma_{\mathrm{dis}}(-\Delta+V)} \frac{\mathrm{dist}(\lambda,[0,\infty))^p}{|\lambda|^{d/2}} f\left(-\log\left(\frac{\mathrm{dist}(\lambda,[0,\infty))}{|\lambda|}\right)\right) \leq C_{d,p,f}\|V\|_{L^p}^p$$

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Theorem (Bögli 2023, Bögli-Petpradittha-Štampach 2024+): Let $p \geq 1$. Let $f : [0,\infty) \to (0,\infty)$ be a continuous, non-increasing function with $\int_0^\infty f(s)\,\mathrm{d}s = \infty$. Then

$$\sup_{V \in L^p(\mathbb{R})} \frac{\sum_{\lambda \in \sigma_{\mathrm{dis}}(-\Delta+V)} \frac{\mathrm{dist}(\lambda,[0,\infty))^p}{|\lambda|^{1/2}} f\left(-\log\left(\frac{\mathrm{dist}(\lambda,[0,\infty))}{|\lambda|}\right)\right)}{\|V\|_{L^p}^p} = \infty.$$

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Thank you for your attention!