Seminar on Analysis, Differential Equations and Mathematical Physics
April 28, 2022

Inverse spectral problem for the matrix Sturm-Liouville operator

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Contents

Scalar Sturm-Liouville IPs

The most complete results in the theory of inverse spectral problems have been obtained for the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda y. \tag{1}$$

1. Marchenko, V.A. Sturm-Liouville Operators and Their Applications, Naukova Dumka, Kiev (1977) (Russian); English transl., Birkhauser (1986).
2. Levitan, B.M. Inverse Sturm-Liouville Problems, Nauka, Moscow (1984) (Russian); English transl., VNU Sci. Press, Utrecht (1987).
3. Freiling, G.; Yurko, V. Inverse Sturm-Liouville Problems and Their Applications, Huntington, NY: Nova Science Publishers (2001).

Scalar Sturm-Liouville IPs

$$-y'' + q(x)y = \lambda y, \quad x \in (0, \pi). \tag{1}$$

The real-valued potential $q \in L_2(0, \pi)$ is uniquely specified by:

1. (Borg, 1946) The eigenvalues $\{\lambda_{n,j}\}_{n=1}^{\infty}$ of the boundary value problems for (1) with BCs $y(0) = y^{(j)}(\pi) = 0$, $j = 0, 1$.

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3. The Weyl function $M(\lambda) = \dfrac{S'(\pi, \lambda)}{S(\pi, \lambda)}$.

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$\{\lambda_{n,0}\}$ — poles, $\{\lambda_{n,1}\}$ — zeros, $\alpha_n^{-1} = \operatorname*{Res}_{\lambda = \lambda_{n,0}} M(\lambda)$.

Scalar inverse Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y, \quad x \in (0, \pi), \tag{1}$$
$$y(0) = y(\pi) = 0. \tag{2}$$

Theorem (Spectral data characterization)

For numbers $\{\lambda_n, \alpha_n\}_{n \geqslant 1}$ to be the spectral data of the problem (1)-(2) with a real-valued potential $q \in L_2(0, \pi)$, the following conditions are necessary and sufficient:

$$\lambda_n \in \mathbb{R}, \quad \lambda_n \ne \lambda_m \text{ if } n \ne m, \quad \alpha_n > 0, \tag{3}$$
$$\sqrt{\lambda_n} = n + \frac{\omega}{\pi n} + \frac{\varkappa_n}{n}, \quad \alpha_n = \frac{\pi}{2n^2} + \frac{\varkappa_{n1}}{n^3}, \tag{4}$$

where

$$\omega = \frac{1}{2} \int_0^\pi q(x)\, dx, \quad \{\varkappa_n\}, \{\varkappa_{n1}\} \in l_2.$$

4 Gelfand, I.M.; Levitan, B.M. On the determination of a differential equation from its spectral function, Izvestiya Akad. Nauk SSSR. Ser. Mat. 15 (1951), 309–360 (Russian).

In this talk, we consider the matrix Sturm-Liouville equation

$$-Y''(x) + Q(x)Y(x) = \lambda Y(x), \quad x \in (0, \pi), \tag{5}$$

with the singular potential $Q \in W_2^{-1}((0,\pi); \mathbb{C}^{m\times m})$, that is, $Q = \sigma'$, $\sigma \in L_2((0,\pi); \mathbb{C}^{m\times m})$, $\sigma(x) = (\sigma(x))^*$ a.e. on $(0,\pi)$.

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Equation (5) can be represented in the equivalent form

$$-(Y^{[1]}(x))' - \sigma(x)Y^{[1]}(x) - \sigma^2(x)Y(x) = \lambda Y(x), \tag{6}$$

where $Y^{[1]}(x) = Y'(x) - \sigma(x)Y(x)$ is the quasi-derivative.

Direct and inverse spectral problems for differential operators with singular coefficients: Savchuk, Shkalikov, Hryniv, Mykytyuk, Djakov, Mityagin, Mirzoev, Korotyaev, ...

Matrix Sturm-Liouville IPs

The majority of studies on matrix Sturm-Liouville IPs on a finite interval deal with the Dirichlet or the Robin BCs:

$$Y(0) = Y(\pi) = 0, \tag{7}$$

$$Y'(0) - H_1 Y(0) = 0, \quad Y'(\pi) + H_2 Y(\pi) = 0, \tag{8}$$

where $H_1, H_2 \in \mathbb{C}^{m \times m}$.

Uniqueness:

5. Carlson, R. An inverse problem for the matrix Schrödinger equation, J. Math. Anal. Appl. 267 (2002), 564–575.
6. Chabanov, V. M. Recovering the M-channel Sturm-Liouville operator from M+1 spectra, J. Math. Phys. 45 (2004), no. 11, 4255–4260.
7. Malamud, M.M. Uniqueness of the matrix Sturm-Liouville equation given a part of the monodromy matrix, and Borg type results, Sturm-Liouville Theory, Birkhäuser, Basel (2005), 237–270.
8. Yurko, V.A. Inverse problems for matrix Sturm-Liouville operators, Russ. J. Math. Phys. 13 (2006), no. 1, 111–118.

Matrix Sturm-Liouville IPs

Constructive solution:

9 Yurko, V. Inverse problems for the matrix Sturm-Liouville equation on a finite interval, Inverse Problems 22 (2006), 1139–1149.

Spectral data characterization:

10 Chelkak, D.; Korotyaev, E. Weyl-Titchmarsh functions of vector-valued Sturm-Liouville operators on the unit interval, J. Func. Anal. 257 (2009), 1546–1588.

11 Mykytyuk, Ya.V.; Trush, N.S. Inverse spectral problems for Sturm-Liouville operators with matrix-valued potentials, Inverse Problems 26 (2009), no. 1, 015009.

12 Bondarenko, N. Spectral analysis for the matrix Sturm-Liouville operator on a finite interval, Tamkang J. Math. 42 (2011), no. 3, 305–327.

13 Chelkak, D.; Matveenko, S. Inverse vector-valued Sturm-Liouville problem. I. Uniqueness theorem, preprint (2013), arXiv:1312.3621 [math.SP].

14 Xu, X.-C. Inverse spectral problem for the matrix Sturm-Liouville operator with the general separated self-adjoint boundary conditions, Tamkang J. Math. 50 (2019), no. 3, 321–336.

In [13, 14], the uniqueness theorems were proved for the matrix Sturm-Liouville equation

$$-Y''(x) + Q(x)Y(x) = \lambda Y(x) \tag{5}$$

with the general self-adjoint separated BCs

$$T_1(Y'(0) - H_1 Y(0)) - T_1^{\perp} Y(0) = 0, \quad T_2(Y'(\pi) - H_2 Y(\pi)) - T_2^{\perp} Y(\pi) = 0, \tag{9}$$

where T_j, $T_j^{\perp}$, $H_j \in \mathbb{C}^{m \times m}$, T_j are orthogonal projectors: $T_j = T_j^* = T_j^2$, $T_j^{\perp} = I - T_j$, I is the unit matrix, $H_j = H_j^* = T_j H_j T_j$, $j = 1, 2$, $Q = Q^* \in L_2((0,\pi); \mathbb{C}^{m \times m})$. BCs (9) were introduced in [15] for the study of differential operators on graphs.

15 Kuchment, P. Quantum graphs. I. Some basic structures, Waves Random Media 14 (2004), no. 1, S107–S128.

Inverse scattering problems for the matrix Sturm-Liouville operators on the half-line:

16 Agranovich, Z.S.; Marchenko, V.A. The inverse problem of scattering theory, Gordon and Breach, New York (1963).

17 Harmer, M. Inverse scattering for the matrix Schrödinger operator and Schrödinger operator on graphs with general self-adjoint boundary conditions, ANZIAM J. 43 (2002), 1–8.

18 Harmer, M. Inverse scattering on matrices with boundary conditions, J. Phys. A. 38 (2005), no. 22, 4875–4885.

19 Aktosun, T.; Weder, R. Direct and Inverse Scattering for the Matrix Schrödinger Equation, Applied Mathematical Sciences, Vol. 203, Springer, Cham, 2021.

[20] Bondarenko, N.P. Direct and inverse problems for the matrix Sturm-Liouville operator with general self-adjoint boundary conditions, Math. Notes 109 (2021), no. 3, 358-378.

[21] Bondarenko, N.P. Inverse problem solution and spectral data characterization for the matrix Sturm-Liouville operator with singular potential, Anal. Math. Phys. 11 (2021), Article number: 145.

IP for boundary value problem L:

$$-(Y^{[1]}(x))' - \sigma(x)Y^{[1]}(x) - \sigma^2(x)Y(x) = \lambda Y(x), \tag{10}$$

$$V_1(Y) = T_1(Y^{[1]}(0) - H_1Y(0)) - T_1^{\perp}Y(0) = 0, \tag{11}$$

$$V_2(Y) = T_2(Y^{[1]}(\pi) - H_2Y(\pi)) - T_2^{\perp}Y(\pi) = 0. \tag{12}$$

Main results:

- uniqueness,
- constructive solution,
- spectral data characterization.

Main Results: Asymptotics

Theorem 1

The boundary value problem L has the countable set of real eigenvalues, which can be numbered as $\{\lambda_{nk}\}_{(n,k)\in J}$ counting with multiplicities in non-decreasing order: $\lambda_{n_1k_1} \leqslant \lambda_{n_2k_2}$, если $(n_1, k_1) < (n_2, k_2)$. The asymptotic relation holds:

$$\sqrt{\lambda_{nk}} = n + r_k + \varkappa_{nk}, \quad (n,k) \in J, \quad \{\varkappa_{nk}\} \in l_2, \tag{13}$$

where

$$J := \{(n,k)\colon n \in \mathbb{N},\ k = \overline{1,m}\} \cup \{(0,k)\colon k = \overline{p^{\perp}+1,m}\},$$
$$p^{\perp} := \dim(Ker T_1 \cap Ker T_2),$$

$\{r_k\}_{k=1}^{m}$ *are the zeros of* $w^0(\rho) := \det(W^0(\rho))$ *on* $[0,1)$,

$$W^0(\rho) := (T_2T_1 + T_2^{\perp}T_1^{\perp})\sin\rho\pi + (T_2^{\perp}T_1 - T_2T_1^{\perp})\cos\rho\pi.$$

Main Results: Definitions

The *Weyl solution* of L is the matrix solution $\Phi(x, \lambda)$ of equation (10) satisfying

$$V_1(\Phi) = I, \quad V_2(\Phi) = 0. \tag{14}$$

The *Weyl matrix* of L is the matrix function

$$M(\lambda) := T_1 \Phi(0, \lambda) + T_1^{\perp} \Phi^{[1]}(0, \lambda). \tag{15}$$

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$$M(\lambda) := T_1 \Phi(0, \lambda) + T_1^{\perp} \Phi^{[1]}(0, \lambda). \tag{15}$$

$M(\lambda)$ is meromorphic in the λ-plane, all its singularities are simple poles which conicide with the eigenvalues of L. Define the *weight matrices*:

$$\alpha_{nk} := \operatorname*{Res}_{\lambda = \lambda_{nk}} M(\lambda), \quad (n, k) \in J. \tag{16}$$

The collection $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J}$ is called the *spectral data* of L.

Main Results: Asymptotics

Let $\lambda_{n_1k_1} = \lambda_{n_2k_2} = \cdots = \lambda_{n_rk_r}$ be a group of multiple eigenvalues maximal by inclusion, $(n_1, k_1) < (n_2, k_2) < \cdots < (n_r, k_r)$. Clearly, $\alpha_{n_1k_1} = \alpha_{n_2k_2} = \cdots = \alpha_{n_rk_r}$. Denote $\alpha'_{n_1k_1} := \alpha_{n_1k_1}$, $\alpha_{n_jk_j} := 0$, $j = \overline{2, r}$. Thus, the sequence of matrices $\{\alpha'_{nk}\}_{(n,k)\in J}$ is defined.

Theorem 2

The asymptotic relation holds:

$$\alpha_n^{(k)} := \sum_{r_s \in J_k} \alpha'_{ns} = \frac{2}{\pi}(T_1 + nT_1^\perp)(A_k + K_{nk})(T_1 + nT_1^\perp), \quad n \geqslant 1, \quad k \in \mathcal{J}, \tag{17}$$

where $\mathcal{J} := \{1\} \cup \{k = \overline{2, m} \colon r_k \ne r_{k-1}\}$, $J_k := \{s = \overline{1, m} \colon r_s = r_k\}$, $\{\|K_{nk}\|\} \in l_2$,

$$A_k := \pi \operatorname*{Res}_{\rho = r_k} (W^0(\rho))^{-1} U^0(\rho),$$

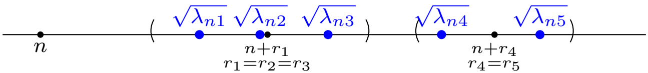
$$U^0(\rho) := (T_2 T_1 + T_2^\perp T_1^\perp) \cos \rho\pi + (T_2 T_1^\perp - T_2^\perp T_1) \sin \rho\pi,$$

$\{A_k\}_{k\in\mathcal{J}}$ *are orthogonal projection matrices such that*

$$rank(A_k) = |J_k|, \qquad A_k A_s = 0, \ k \ne s, \qquad \sum_{k\in\mathcal{J}} A_k = I.$$

Main Results: Asymptotics

$$\sqrt{\lambda_{nk}} = n + r_k + \varkappa_{nk}, \quad k = \overline{1, m}, \quad n \in \mathbb{N} \text{ or } n \in \mathbb{N} \cup \{0\}.$$



Main Results

Without loss of generality we assume that $H_1 = 0$.

$L = L(\sigma, T_1, T_2, H_2)$:

$$-(Y^{[1]}(x))' - \sigma(x)Y^{[1]}(x) - \sigma^2(x)Y(x) = \lambda Y(x),$$
$$V_1(Y) = T_1 Y^{[1]}(0) - T_1^{\perp} Y(0) = 0,$$
$$V_2(Y) = T_2(Y^{[1]}(\pi) - H_2 Y(\pi)) - T_2^{\perp} Y(\pi) = 0.$$

Inverse Problem 1

Given the spectral data $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k)\in J}$, find σ, T_1, T_2, and H_2.

Main Results: Uniqueness

Along with L, consider another boundary value problem $\tilde{L} = L(\tilde{\sigma}, \tilde{T}_1, \tilde{T}_2, \tilde{H}_2)$ of the same form but with different coefficients. We agree that if a symbol γ denotes an object related to L, then the symbol $\tilde{\gamma}$ with tilde denotes the similar object related to $\tilde{L}$. Note that the quasi-derivatives for these two problems are supposed to be different: $Y^{[1]} = Y' - \sigma Y$ for L and $Y^{[1]} = Y' - \tilde{\sigma} Y$ for $\tilde{L}$.

Main Results: Uniqueness

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Theorem 3

If $\lambda_{nk} = \tilde{\lambda}_{nk}$, $\alpha_{nk} = \tilde{\alpha}_{nk}$, $(n,k) \in J$, $J = \tilde{J}$, then

$$\sigma(x) = \tilde{\sigma}(x) + H_1^\diamond \text{ a.e. on } (0,\pi), \quad T_1 = \tilde{T}_1, \quad T_2 = \tilde{T}_2, \quad H_2 = \tilde{H}_2 - T_2 H_1^\diamond T_2, \tag{18}$$

where

$$H_1^\diamond = (H_1^\diamond)^* = T_1^\perp H_1^\diamond T_1^\perp. \tag{19}$$

Thus, the spectral data $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k)\in J}$ uniquely specify the problem L up to a transform (18) given by an arbitrary matrix $H_1^\diamond$ satisfying (19).

Main Results: Uniqueness

Along with L, consider another boundary value problem $\tilde{L} = L(\tilde{\sigma}, \tilde{T}_1, \tilde{T}_2, \tilde{H}_2)$ of the same form but with different coefficients. We agree that if a symbol γ denotes an object related to L, then the symbol $\tilde{\gamma}$ with tilde denotes the similar object related to $\tilde{L}$. Note that the quasi-derivatives for these two problems are supposed to be different: $Y^{[1]} = Y' - \sigma Y$ for L and $Y^{[1]} = Y' - \tilde{\sigma} Y$ for $\tilde{L}$.

Theorem 3

If $\lambda_{nk} = \tilde{\lambda}_{nk}$, $\alpha_{nk} = \tilde{\alpha}_{nk}$, $(n,k) \in J$, $J = \tilde{J}$, *then*

$$\sigma(x) = \tilde{\sigma}(x) + H_1^{\diamond} \text{ a.e. on } (0,\pi), \quad T_1 = \tilde{T}_1, \quad T_2 = \tilde{T}_2, \quad H_2 = \tilde{H}_2 - T_2 H_1^{\diamond} T_2, \tag{18}$$

where

$$H_1^{\diamond} = (H_1^{\diamond})^* = T_1^{\perp} H_1^{\diamond} T_1^{\perp}. \tag{19}$$

Thus, the spectral data $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k)\in J}$ *uniquely specify the problem* L *up to a transform* (18) *given by an arbitrary matrix* $H_1^{\diamond}$ *satisfying* (19).

If (18) holds, then $M(\lambda) \equiv \tilde{M}(\lambda) + H_1^{\diamond}$.

Theorem 4

If $M(\lambda) \equiv \tilde{M}(\lambda)$, *then* $\sigma(x) = \tilde{\sigma}(x)$ *a.e. on* $(0,\pi)$, $T_1 = \tilde{T}_1$, $T_2 = \tilde{T}_2$, $H_2 = \tilde{H}_2$.

Comparison with the Scalar Case

$$-(y^{[1]})' - \sigma(x)y^{[1]} - \sigma^2(x)y = \lambda y, \quad x \in (0,\pi), \tag{20}$$
$$\sigma \in L_2(0,\pi), \quad y^{[1]} = y' - \sigma(x)y.$$

$$\begin{aligned} y(0) = y(\pi) = 0 &\qquad \{\lambda_n, \alpha_n\} \to \sigma(x) + C, \\ y^{[1]}(0) = y(\pi) = 0 &\qquad \{\lambda_n, \alpha_n\} \to \sigma(x), \\ y(0) = y^{[1]}(\pi) + Hy(\pi) = 0 &\qquad \{\lambda_n, \alpha_n\} \to \sigma(x) + C,\ H + C, \\ y^{[1]}(0) = y^{[1]}(\pi) + Hy(\pi) = 0 &\qquad \{\lambda_n, \alpha_n\} \to \sigma(x),\ H. \end{aligned}$$

22 Hryniv, R.O.; Mykytyuk, Y.V. Inverse spectral problems for Sturm-Liouville operators with singular potentials, Inverse Problems 19 (2003), no. 3, 665–684.

23 Guliyev, N.J. Schrödinger operators with distributional potentials and boundary conditions dependent on the eigenvalue parameter, J. Math. Phys. 60 (2019), 063501.

Main Results: Characterization

Consider a group of multiple eigenvalues maximal by inclusion: $\lambda_{n_1 k_1} = \lambda_{n_2 k_2} = \dots = \lambda_{n_r k_r}$. Then $\operatorname{rank}(\alpha_{n_1 k_1}) = r$. Choose a basis $\{\chi_{n_j k_j}\}_{j=1}^{r}$ in $\operatorname{Ran} \alpha_{n_1 k_1}$. Thus, we have defined the vector sequence $\{\chi_{nk}\}_{(n,k) \in J}$.

$$\mathcal{X} := \{X_{nk}\}_{(n,k)\in J}, \quad X_{nk}(x) := \left(\cos(\sqrt{\lambda_{nk}} x) T_1 + \frac{\sin(\sqrt{\lambda_{nk}} x)}{\sqrt{\lambda_{nk}}} T_1^{\perp} \right) \chi_{nk}.$$

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Theorem 5

Let $T_1, T_2 \in \mathbb{C}^{m \times m}$ be arbitrary fixed orthogonal projection matrices. Then, for a collection $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J}$ to be the spectral data of $L = L(\sigma, T_1, T_2, H_2)$, the following conditions are necessary and sufficient:

1. $\lambda_{nk} \in \mathbb{R}$, $\alpha_{nk} \in \mathbb{C}^{m \times m}$, $\alpha_{nk} = \alpha_{nk}^* \geqslant 0$, *$\operatorname{rank}(\alpha_{nk})$ is equal to the multiplicity of the corresponding value λ_{nk}, for all $(n,k) \in J$, and $\alpha_{nk} = \alpha_{ls}$ if $\lambda_{nk} = \lambda_{ls}$.*
2. *The asymptotic relations (13) and (17) hold, where $\{r_k\}_{k=1}^m$ and $\{A_k\}_{k \in \mathcal{J}}$ are defined as in Theorems 1 and 2, respectively, by using the fixed T_1 and T_2.*
3. *$\mathcal{X}$ is complete $L_2((0,\pi); \mathbb{C}^m)$.*

The proof of Theorem 5 and the constructive solution of Inverse Problem 1 are based on the *method of spectral mappings*.

Regular potentials:

[24] Yurko, V.A. Method of Spectral Mappings in the Inverse Problem Theory, Inverse and Ill-Posed Problems Series, Utrecht, VNU Science (2002).

Singular potentials:

[25] Freiling, G.; Ignatiev, M.Y.; Yurko, V.A. An inverse spectral problem for Sturm-Liouville operators with singular potentials on star-type graph, Proc. Symp. Pure Math. 77 (2008), 397–408.

[26] Bondarenko, N.P. Solving an inverse problem for the Sturm-Liouville operator with singular potential by Yurko's method, Tamkang J. Math. 52 (2021), no. 1, 125-154.

Method of Spectral Mappings

- Contour integration in the λ-plane.
- Nonlinear IP is reduced to a linear *main equation* in a Banach space $\mathfrak{B}$:

$$\tilde{\phi}(x) = (\mathcal{I} + \tilde{R}(x))\phi(x), \tag{21}$$

where $\tilde{\phi}(x) \in \mathfrak{B}$ and the linear bounded operator $\tilde{R}(x) \colon \mathfrak{B} \to \mathfrak{B}$ are constructed by the given $\{\lambda_{nk}, \alpha_{nk}\}_{(n,k) \in J}$, and the unknown $\phi(x) \in \mathfrak{B}$ is related to σ and H_2, $\mathcal{I}$ is the unit operator $\mathfrak{B}$.

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- In the scalar case, $\mathfrak{B}$ is the space of infinite bounded sequences $a = [a_{ni}]_{n \geqslant 1,\, i=0,1}$ with the norm $\|a\|_{\mathfrak{B}} = \sup\limits_{n,i} |a_{ni}|$.
- In the matrix case, the special Banach space is constructed by using the grouping the eigenvalues with respect to their asymptotics.

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- In the matrix case, the special Banach space is constructed by using the grouping the eigenvalues with respect to their asymptotics.
- Proof of the main equation solvability by sufficiency.

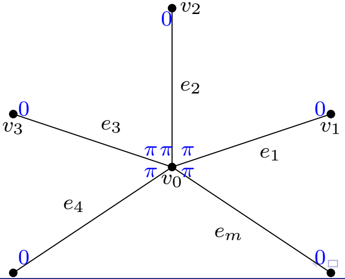
Application to Quantum Graphs

$$-(y_j^{[1]})' - \sigma_j(x_j)y_j^{[1]} - \sigma_j^2(x_j)y_j = \lambda y_j, \quad x_j \in (0,\pi), \quad j = \overline{1,m}, \tag{22}$$

$$y_j(0) = 0, \quad j = \overline{1,m}, \tag{23}$$

$$y_1(\pi) = y_j(\pi), \quad j = \overline{2,m}, \qquad \sum_{j=1}^{m}(y_j^{[1]}(\pi) - hy_j(\pi)) = 0, \tag{24}$$

where $\{\sigma_j\}_{j=1}^m$ are real-valued functions of $L_2(0,\pi)$, $y_j^{[1]} := y_j' - \sigma_j y_j$, $y_j, y_j^{[1]} \in AC[0,\pi]$, $(y_j^{[1]})' \in L_2(0,\pi)$, $j = \overline{1,m}$, $h \in \mathbb{R}$.



Application to Quantum Graphs

The problem (22)-(24) can be represented in the matrix form $L(\sigma, T_1, T_2, H_2)$ with

$$\begin{gathered}\sigma(x) = \operatorname{diag}\{\sigma_j(x)\}_{j=1}^m, \quad T_1 = 0, \quad T_2 = [T_{2,jk}]_{j,k=1}^m, \\ T_{2,jk} = \frac{1}{m},\ j,k = \overline{1,m}, \quad H_2 = hT_2.\end{gathered} \tag{25}$$

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Inverse Problem 2

Given the spectral data $\{\lambda_{nk}, \alpha_{nk}\}_{n \geqslant 1,\, k=\overline{1,m}}$, *find* $\{\sigma_j\}_{j=1}^m$ *and* h.

In the case of regular potentials, this problem statement is equivalent to the problem of [27], which consist in the recovery of the Sturm-Liouville operator on graph by the generalized Dirichlet-to-Neumann map. Inverse Problem 2 problem is overdetermined, its spectral data contains the data of Yurko [28] as a subset.

[27] Brown, B.M.; Weikard, R. A Borg-Levinson theorem for trees, Proc. Royal Soc. A: Math. Phys. Eng. Sci. 461 (2005), 3231–3243.

[28] Yurko, V. Inverse spectral problems for Sturm-Liouville operators on graphs, Inverse Problems 21 (2005), no. 3, 1075–1086.

Application to Quantum Graphs

Theorem 6

Let T_2 be defined by (25). For $\{\lambda_{nk}, \alpha_{nk}\}_{n \geqslant 1,\, k=\overline{1,m}}$ to be the spectral data of (22)-(24), the following conditions are necessary and sufficient:

1. $\lambda_{nk} \in \mathbb{R}$, $\alpha_{nk} \in \mathbb{C}^{m \times m}$, $\alpha_{nk} = \alpha_{nk}^* \geqslant 0$, *rank*$(\alpha_{nk})$ *is equal to the multiplicity of the corresponding value* λ_{nk}, *for all* $(n,k) \in J$, *and* $\alpha_{nk} = \alpha_{ls}$ *if* $\lambda_{nk} = \lambda_{ls}$.
2. *The asymptotic relations hold:*

$$\sqrt{\lambda_{n1}} = n - \frac{1}{2} + \varkappa_{n1}, \qquad \sqrt{\lambda_{nk}} = n + \varkappa_{nk}, \quad k = \overline{2,m},$$

$$\alpha_{n1} = \frac{2n^2}{\pi}(T_2 + K_{n1}), \quad \sum_{k=2}^{m} \alpha'_{nk} = \frac{2n^2}{\pi}(T_2^{\perp} + K_{n2}),$$

   where $\{\varkappa_{nk}\} \in l_2$, $\{\|K_{nk}\|\} \in l_2$.

3. $\mathcal{X}$ *is complete in* $L_2((0,\pi);\mathbb{C}^m)$.
4. *The solution* $\phi(x)$ *of the main equation* $\tilde{\phi}(x) = (\mathcal{I} + \tilde{R}(x))\phi(x)$ *is* ***diagonal*** *for each fixed* $x \in [0,\pi]$.

Thank you for your attention!