ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2016, Vol. 293, pp. 107–126. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © V.I. Burenkov, T.V. Tararykova, 2016, published in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2016, Vol. 293, pp. 113–132.

An Analog of Young's Inequality for Convolutions of Functions for General Morrey-Type Spaces

V. I. Burenkov[a] **and T. V. Tararykova**[b]

Received January 15, 2015

Abstract—An analog of the classical Young's inequality for convolutions of functions is proved in the case of general global Morrey-type spaces. The form of this analog is different from Young's inequality for convolutions in the case of Lebesgue spaces. A separate analysis is performed for the case of periodic functions.

DOI: 10.1134/S0081543816040088

1. INTRODUCTION. GENERAL MORREY-TYPE SPACES

Over the last three decades, the general local and global Morrey-type spaces have been in the focus of many studies (see, e.g., [12, 16, 17, 19, 21, 24, 25, 29, 30, 37]).

In particular, for a certain range of the numerical parameters $0 < p_1, p_2, \theta_1, \theta_2 \le \infty$ of the general local Morrey-type spaces $LM_{p_1\theta_1,w_1(\cdot)}$ and $LM_{p_2\theta_2,w_2(\cdot)}$, necessary and sufficient conditions on the functional parameters w_1 and w_2 have been obtained under which the maximal operator [11–13], the fractional maximal operator [7, 14, 16], the Riesz potential [8, 15, 17], genuine singular integral operators [18, 19], and the Hardy operator [20, 22] are bounded as operators acting from the space $LM_{p_1\theta_1,w_1(\cdot)}$ to the space $LM_{p_2\theta_2,w_2(\cdot)}$. In those studies, only natural assumptions—ensuring that the spaces $LM_{p_1\theta_1,w_1(\cdot)}$ and $LM_{p_2\theta_2,w_2(\cdot)}$ are nontrivial—have been initially made about the functions w_1 and w_2.

The recent survey papers [3, 23, 28, 34–36] describe in detail the present state of the operator theory in general Morrey-type spaces and various applications of this theory.

One of the most common definitions of general Morrey-type spaces is as follows. Let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ of radius $r > 0$ with center at a point $x \in \mathbb{R}^n$.

Definition 1. Let $0 < p, \theta \le \infty$, and let w be a nonnegative Lebesgue measurable function on the half-axis $(0,\infty)$ that is not equivalent to zero. The *local Morrey-type space* $LM_{p\theta,w(\cdot)} \equiv LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is the space of all Lebesgue measurable functions f on $\mathbb{R}^n$ with finite quasinorm

$$\|f\|_{LM_{p\theta,w(\cdot)}} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)}.$$

The *global Morrey-type space* $GM_{p\theta,w(\cdot)} \equiv GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is the space of all Lebesgue measurable functions f on $\mathbb{R}^n$ with finite quasinorm

$$\|f\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \big\|w(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,\infty)}.$$

Remark 1. If the function w is equivalent to zero (in short, $w \sim 0$) on (t,∞) for some $t > 0$, then we set

$$b = \inf\{t > 0\colon\ w \sim 0 \ \text{ on } \ (t,\infty)\}.$$

[a] Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.
[b] School of Mathematics, Cardiff University, Senghennydd Road, CF24 4AG Cardiff, Wales, UK.
E-mail addresses: burenkov@cardiff.ac.uk (V.I. Burenkov), tararykovat@cardiff.ac.uk (T.V. Tararykova).

If $w(r) = 0$ and $\|f\|_{L_p(B(x,r))} = \infty$, then we assume that $w(r)\|f\|_{L_p(B(x,r))} = 0$. Under this agreement,

$$\|f\|_{LM_{p\theta,w(\cdot)}} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,b)}, \qquad \|f\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n}\big\|w(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,b)}.$$

In the case of local Morrey-type spaces (in contrast to global Morrey-type spaces), the finiteness of $\|f\|_{LM_{p\theta,w(\cdot)}}$ does not impose any constraints on the behavior of the function f for $|x| \ge b$. For definiteness, we assume that $f(x) = 0$ for $|x| \ge b$.

If $\theta = \infty$ and $w(\cdot) \equiv 1$, then $LM_{p\infty,1} = GM_{p\infty,1} = L_p(\mathbb{R}^n)$, while if $\theta = \infty$ and $w(r) = r^{-\lambda}$, $0 \le \lambda \le n/p$, then

$$GM_{p\infty,r^{-\lambda}} \equiv M_p^\lambda$$

is the classical Morrey space and

$$LM_{p\infty,r^{-\lambda}} \equiv LM_p^\lambda$$

is a local version of the Morrey space.

The spaces M_p^λ are nontrivial (i.e., they consist not only of functions equivalent to zero on $\mathbb{R}^n$) if and only if $0 \le \lambda \le n/p$. The spaces LM_p^λ are nontrivial if and only if $\lambda \ge 0$. For $\lambda = 0$, we have $LM_p^0 = M_p^0 = L_p$. For $\lambda = n/p$, we have $M_p^{n/p} = L_\infty$.

Let us discuss the relationship between the Morrey spaces M_p^λ and the Nikol'skii spaces H_p^λ $(1 \le p \le \infty,\ \lambda > 0)$ consisting of all Lebesgue measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{H_p^\lambda} = \|f\|_{L_p} + \|f\|_{\dot H_p^\lambda} < \infty,$$

where

$$\|f\|_{\dot H_p^\lambda} = \sup_{h\in\mathbb{R}^n,\ h\ne 0} |h|^{-\lambda}\|\Delta_h^\sigma f\|_{L_p};$$

here $\Delta_h^\sigma f$ is the difference of the function f of order $\sigma \in \mathbb{N}$ with step $h \in \mathbb{R}^n$ and $\sigma > \lambda$. (These definitions for different $\sigma > \lambda$ are equivalent.)

Note that for $1 \le p \le \infty$ and $0 < \lambda < n/p$,

$$H_p^\lambda \subset M_p^\lambda$$

(the inclusion is strict); moreover, for any $\varepsilon > 0$,

$$H_p^\lambda \not\subset M_p^{\lambda+\varepsilon}.$$

See Nikol'skii's paper [31] as well as his survey paper [32] for details. (For $n = 1$, see [26]; regarding generalizations, see [2, 4–6, 9, 10, 27].) A detailed account of the theory of the spaces H_p^λ can be found in the books [1, 33, 38].

We will say that $f \in (H_p^\lambda)^{\mathrm{loc}}$ if $f\eta \in H_p^\lambda$ for any infinitely continuously differentiable function η with compact support.

In some cases, the parameter λ of the spaces M_p^λ and LM_p^λ behaves like the smoothness parameter λ of the spaces H_p^λ; for example,

$$\|f(\varepsilon x)\|_{LM_p^\lambda} = \varepsilon^{\lambda-n/p}\|f(x)\|_{LM_p^\lambda}, \qquad \|f(\varepsilon x)\|_{M_p^\lambda} = \varepsilon^{\lambda-n/p}\|f(x)\|_{M_p^\lambda},$$
$$\|f(\varepsilon x)\|_{\dot H_p^\lambda} = \varepsilon^{\lambda-n/p}\|f(x)\|_{\dot H_p^\lambda}$$

for all $\varepsilon > 0$, and for $\alpha \in \mathbb{R}$

$$|x|^\alpha \in LM_p^\lambda \quad \Leftrightarrow \quad |x|^\alpha \in M_p^\lambda \quad \Leftrightarrow \quad |x|^\alpha \in (H_p^\lambda)^{\mathrm{loc}} \quad \Leftrightarrow \quad \alpha \ge \lambda - \frac{n}{p}$$

under appropriate assumptions on the parameters p and λ.

The first natural question concerning general Morrey-type spaces is to find out for what functions w the spaces $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$ are nontrivial. To answer this question, we need the following definition.

Definition 2. Let $0 < p,\theta \le \infty$. Then Ω_θ is the set of all functions w that are nonnegative, Lebesgue measurable on $(0,\infty)$, not equivalent to zero, and are such that

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty \tag{1.1}$$

for some $t > 0$. Further, $\Omega_{p\theta}$ is the set of all functions w that are nonnegative, Lebesgue measurable on $(0,\infty)$, not equivalent to zero, and are such that

$$\|w(r)r^{n/p}\|_{L_\theta(0,t)} < \infty, \qquad \|w(r)\|_{L_\theta(t,\infty)} < \infty \tag{1.2}$$

for some $t > 0$, or, which is equivalent,

$$\Big\|w(r)\Big(\frac{r}{t+r}\Big)^{n/p}\Big\|_{L_\theta(0,\infty)} < \infty \tag{1.3}$$

for some $t > 0$.

Note that if condition (1.2) (and, hence, condition (1.3)) holds for some $t > 0$, then it holds for all $t > 0$.

Let

$$a = \inf\big\{t > 0\colon\ \|w\|_{L_\theta(t,\infty)} < \infty\big\}.$$

Note that if $w \in \Omega_{p\theta}$, then $a = 0$.

Lemma 1 [13, 20]**.** *Let $0 < p,\theta \le \infty$, and let w be a nonnegative Lebesgue measurable function on $(0,\infty)$ that is not equivalent to zero.*

The space $LM_{p\theta,w(\cdot)}$ is nontrivial if and only if $w \in \Omega_\theta$, and the space $GM_{p\theta,w(\cdot)}$ is nontrivial if and only if $w \in \Omega_{p\theta}$.

Moreover, if $w \in \Omega_\theta$, then the space $LM_{p\theta,w(\cdot)}$ contains all functions $f \in L_p(\mathbb{R}^n)$ that vanish on $B(0,t)$ for some $t > a$.

If $w \in \Omega_{p\theta}$, then $L_p(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n) \subset GM_{p\theta,w(\cdot)}$.

Let $w \in \Omega_\theta$ or $w \in \Omega_{p\theta}$; suppose that the function w is not equivalent to zero on the interval (t,∞) for any $t > 0$. It may happen that the function w is equivalent to zero on some subintervals of the interval (a,∞), which is inconvenient for some applications. One can overcome this disadvantage by replacing the function w with another function $\widetilde{w}$ that is positive on (a,∞) and is such that the quasinorms $\|f\|_{LM_{p\theta,w(\cdot)}}$ and $\|f\|_{LM_{p\theta,\widetilde{w}(\cdot)}}$, as well as $\|f\|_{GM_{p\theta,w(\cdot)}}$ and $\|f\|_{GM_{p\theta,\widetilde{w}(\cdot)}}$, differ little from each other. More precisely, the following statement is valid.

Let Ω_θ^+ and $\Omega_{p\theta}^+$ be the sets of all functions $w \in \Omega_\theta$ and $w \in \Omega_{p\theta}$, respectively, that are positive on $(0,\infty)$.

Theorem 1 [37]**.** *Let $0 < p,\theta \le \infty$ and $w \in \Omega_\theta$; suppose that the function w is not equivalent to zero on the interval (t,∞) for any $t > 0$.*

If $\theta < \infty$, then, for any $\varepsilon > 0$, there exists a function $w_\varepsilon \in \Omega_\theta^+$ such that $w_\varepsilon \ge w$ on $(0,\infty)$, $LM_{p\theta,w_\varepsilon(\cdot)} = LM_{p\theta,w(\cdot)}$, and

$$\|f\|_{LM_{p\theta,w(\cdot)}} \le \|f\|_{LM_{p\theta,w_\varepsilon(\cdot)}} \le (1+\varepsilon)\|f\|_{LM_{p\theta,w(\cdot)}}$$

for all $f \in LM_{p\theta,w(\cdot)}$.

If $\theta = \infty$, then there exists a function $\widetilde{w} \in \Omega_\infty^+$ such that $\widetilde{w} \geq w$ on $(0,\infty)$, $LM_{p\infty,\widetilde{w}(\cdot)} = LM_{p\infty,w(\cdot)}$, and

$$\|f\|_{LM_{p\infty,\widetilde{w}(\cdot)}} = \|f\|_{LM_{p\infty,w(\cdot)}} \tag{1.4}$$

for all $f \in LM_{p\infty,w(\cdot)}$. Moreover, there exists a function $\overline{w} \in \Omega_\infty^+$ such that $\overline{w} \geq w$ almost everywhere on $(0,\infty)$, $\overline{w}$ does not increase and is continuous on the right on (a,∞), $LM_{p\infty,\overline{w}(\cdot)} = LM_{p\infty,w(\cdot)}$, and equality (1.4) with $\overline{w}$ instead of $\widetilde{w}$ is satisfied.

A similar statement is valid if the classes Ω_θ and Ω_θ^+ are everywhere replaced with $\Omega_{p\theta}$ and $\Omega_{p\theta}^+$ and the local Morrey-type spaces $LM_{p\infty,w(\cdot)}$ are replaced with the global Morrey-type spaces $GM_{p\infty,w(\cdot)}$.

In the present study we prove an analog of the classical Young's inequality for convolutions in the case of general global Morrey-type spaces. The form of this analog is different from Young's inequality for convolutions in the case of Lebesgue spaces. Section 2 is of auxiliary character. The main result is contained in Section 3. In Section 4, we present inequalities for truncated convolutions. The case of periodic functions is considered in Section 5.

2. ANALOG OF A MULTIPLICATIVE INEQUALITY FOR GENERAL LOCAL AND GLOBAL MORREY-TYPE SPACES

In the case of Lebesgue spaces, the following well-known multiplicative inequality is valid:

$$\|f\|_{L_p(\Omega)} \leq \|f\|_{L_{p_1}(\Omega)}^{\alpha_1} \|f\|_{L_{p_2}(\Omega)}^{\alpha_2} \tag{2.1}$$

for all $f \in L_{p_1}(\Omega) \cap L_{p_2}(\Omega)$, where Ω is a measurable set in $\mathbb{R}^n$ and

$$0 \leq \alpha_1, \alpha_2 \leq 1, \qquad \alpha_1 + \alpha_2 = 1, \qquad 0 < p_1, p_2, p \leq \infty, \qquad \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = \frac{1}{p}. \tag{2.2}$$

The following inequalities are analogs of inequality (2.1) for local and global Morrey-type spaces.

Lemma 2. *Let condition (2.2) hold and let, in addition,*

$$0 < \theta_1, \theta_2, \theta \leq \infty, \qquad \frac{\alpha_1}{\theta_1} + \frac{\alpha_2}{\theta_2} = \frac{1}{\theta}, \tag{2.3}$$

and

$$w(r) = w_1^{\alpha_1}(r) w_2^{\alpha_2}(r), \qquad r > 0. \tag{2.4}$$

Then

(1) *$w \in \Omega_\theta$ for all $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, and the inequality*

$$\|f\|_{LM_{p\theta,w(\cdot)}} \leq \|f\|_{LM_{p_1\theta_1,w_1(\cdot)}}^{\alpha_1} \|f\|_{LM_{p_2\theta_2,w_2(\cdot)}}^{\alpha_2} \tag{2.5}$$

is valid for all $f \in LM_{p_1\theta_1,w_1(\cdot)} \cap LM_{p_2\theta_2,w_2(\cdot)}$;

(2) *$w \in \Omega_{p\theta}$ for all $w_1 \in \Omega_{p_1\theta_1}$ and $w_2 \in \Omega_{p_2\theta_2}$, and the inequality*

$$\|f\|_{GM_{p\theta,w(\cdot)}} \leq \|f\|_{GM_{p_1\theta_1,w_1(\cdot)}}^{\alpha_1} \|f\|_{GM_{p_2\theta_2,w_2(\cdot)}}^{\alpha_2} \tag{2.6}$$

is valid for all $f \in GM_{p_1\theta_1,w_1(\cdot)} \cap GM_{p_2\theta_2,w_2(\cdot)}$; in particular, for $0 \leq \lambda_1 \leq n/p_1$ and $0 \leq \lambda_2 \leq n/p_2$,

$$\|f\|_{M_p^{\alpha_1\lambda_1+\alpha_2\lambda_2}} \leq \|f\|_{M_{p_1}^{\lambda_1}}^{\alpha_1} \|f\|_{M_{p_2}^{\lambda_2}}^{\alpha_2}.$$

Proof. 1. Let $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$. According to Definition 2, for some $t_1, t_2 > 0$ we have $\|w_1\|_{L_{\theta_1}(t_1,\infty)} < \infty$ and $\|w_2\|_{L_{\theta_2}(t_2,\infty)} < \infty$. Let $t = \max\{t_1, t_2\}$. Using conditions (2.2) and (2.3), we apply inequality (2.1) in which p, p_1, and p_2 are replaced by θ, θ_1, and θ_2. Then

$$\|w\|_{L_\theta(t,\infty)} \le \|w_1\|^{\alpha_1}_{L_{\theta_1}(t_1,\infty)} \|w_2\|^{\alpha_2}_{L_{\theta_2}(t_2,\infty)} < \infty.$$

Thus, $w \in \Omega_\theta$.

Next, let $f \in LM_{p_1\theta_1,w_1(\cdot)} \cap LM_{p_2\theta_2,w_2(\cdot)}$. Applying inequality (2.1) with $\Omega = B(0,r)$, equality (2.4), and then taking into account condition (2.3) and Hölder's inequality with exponents θ, θ_1/α_1, and θ_2/α_2, we obtain

$$\begin{aligned}
\|f\|_{LM_{p\theta,w(\cdot)}} &= \big\| w(r) \|f\|_{L_p(B(0,r))} \big\|_{L_\theta(0,\infty)} \le \big\| w(r) \|f\|^{\alpha_1}_{L_{p_1}(B(0,r))} \|f\|^{\alpha_2}_{L_{p_2}(B(0,r))} \big\|_{L_\theta(0,\infty)} \\
&= \big\| \big(w_1(r)\|f\|_{L_{p_1}(B(0,r))}\big)^{\alpha_1} \big(w_2(r)\|f\|_{L_{p_2}(B(0,r))}\big)^{\alpha_2} \big\|_{L_\theta(0,\infty)} \\
&\le \big\| \big(w_1(r)\|f\|_{L_{p_1}(B(0,r))}\big)^{\alpha_1} \big\|_{L_{\theta_1/\alpha_1}(0,\infty)} \big\| \big(w_2(r)\|f\|_{L_{p_2}(B(0,r))}\big)^{\alpha_2} \big\|_{L_{\theta_2/\alpha_2}(0,\infty)} \\
&= \big\| w_1(r)\|f\|_{L_{p_1}(B(0,r))} \big\|^{\alpha_1}_{L_{\theta_1}(0,\infty)} \big\| w_2(r)\|f\|_{L_{p_2}(B(0,r))} \big\|^{\alpha_2}_{L_{\theta_2}(0,\infty)} \\
&= \|f\|^{\alpha_1}_{LM_{p_1\theta_1,w_1(\cdot)}} \|f\|^{\alpha_2}_{LM_{p_2\theta_2,w_2(\cdot)}},
\end{aligned}$$

which implies inequality (2.5).

2. Now, let $w_1 \in \Omega_{p_1\theta_1}$ and $w_2 \in \Omega_{p_2\theta_2}$. Let us verify condition (1.3) for the function w. Using condition (2.3), we apply Hölder's inequality with exponents θ, θ_1/α_1, and θ_2/α_2. Then

$$\begin{aligned}
\Big\| w(r)\Big(\frac{r}{t+r}\Big)^{n/p} \Big\|_{L_\theta(0,\infty)} &= \Big\| \Big(w_1(r)\Big(\frac{r}{t+r}\Big)^{n/p_1}\Big)^{\alpha_1} \Big(w_2(r)\Big(\frac{r}{t+r}\Big)^{n/p_2}\Big)^{\alpha_2} \Big\|_{L_\theta(0,\infty)} \\
&\le \Big\| w_1(r)\Big(\frac{r}{t+r}\Big)^{n/p_1} \Big\|^{\alpha_1}_{L_{\theta_1}(0,\infty)} \Big\| w_2(r)\Big(\frac{r}{t+r}\Big)^{n/p_2} \Big\|^{\alpha_2}_{L_{\theta_2}(0,\infty)} < \infty.
\end{aligned}$$

Thus, $w \in \Omega_{p\theta}$.

Next, let $f \in GM_{p_1\theta_1,w_1(\cdot)} \cap GM_{p_2\theta_2,w_2(\cdot)}$. According to inequality (2.5),

$$\begin{aligned}
\|f\|_{GM_{p\theta,w(\cdot)}} &= \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w(\cdot)}} \le \sup_{x\in\mathbb{R}^n} \Big(\|f(x+\cdot)\|^{\alpha_1}_{LM_{p_1\theta_1,w_1(\cdot)}} \|f(x+\cdot)\|^{\alpha_2}_{LM_{p_2\theta_2,w_2(\cdot)}} \Big) \\
&\le \Big(\sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p_1\theta_1,w_1(\cdot)}} \Big)^{\alpha_1} \Big(\sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p_2\theta_2,w_2(\cdot)}} \Big)^{\alpha_2} \\
&= \|f\|^{\alpha_1}_{GM_{p_1\theta_1,w_1(\cdot)}} \|f\|^{\alpha_2}_{GM_{p_2\theta_2,w_2(\cdot)}}. \qquad \square
\end{aligned}$$

Remark 2. In Section 3, we will make use of inequality (2.6) in the case when $w_1 = w_2 = w \in \Omega_{\theta_1} \cap \Omega_{\theta_2}$.

3. ANALOG OF YOUNG'S INEQUALITY FOR CONVOLUTIONS OF FUNCTIONS IN THE CASE OF GENERAL GLOBAL MORREY-TYPE SPACES

Let f_1 and f_2 be measurable functions and

$$(f_1 * f_2)(x) = \int\limits_{\mathbb{R}^n} f_1(x-y) f_2(y)\, dy, \qquad x \in \mathbb{R}^n,$$

be the convolution of these functions.

In this section, we formulate and prove an analog of Young's inequality for convolutions in Lebesgue spaces:

$$\|f_1 * f_2\|_{L_p} \le \|f_1\|_{L_{p_1}} \|f_2\|_{L_{p_2}} \tag{3.1}$$

for all $f_k \in L_{p_k}$, $k = 1, 2$, where

$$1 \le p_1, p_2 \le p \le \infty, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1. \tag{3.2}$$

If $1 \le p_2 = p \le \infty$, then the inequality takes the form

$$\|f_1 * f_2\|_{L_p} \le \|f_1\|_{L_1} \|f_2\|_{L_p}. \tag{3.3}$$

Applying the generalized Minkowski inequality for integrals twice, we can easily prove that if $1 \le p, \theta \le \infty$ and $w \in \Omega_{p\theta}$, then

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1\|_{L_1} \|f_2\|_{GM_{p\theta,w(\cdot)}} \tag{3.4}$$

for all $f_1 \in L_1$ and $f_2 \in GM_{p\theta,w(\cdot)}$; in particular, for any $0 \le \lambda \le n/p$ and all $f_1 \in L_1$ and $f_2 \in M_p^\lambda$,

$$\|f_1 * f_2\|_{M_p^\lambda} \le \|f_1\|_{L_1} \|f_2\|_{M_p^\lambda}. \tag{3.5}$$

These are direct analogs of Young's inequality (3.3) (L_p is replaced by $GM_{p\theta,w(\cdot)}$ and M_p^λ, respectively).

Indeed, for all $x \in \mathbb{R}^n$,

$$\|f_1 * f_2\|_{L_p(B(x,r))} = \Bigg\| \int\limits_{\mathbb{R}^n} f_2(x-y) f_1(y)\, dy \Bigg\|_{L_p(B(x,r))} \le \int\limits_{\mathbb{R}^n} \|f_2(x-y)\|_{L_p(B(x,r))} |f_1(y)|\, dy$$
$$= \int\limits_{\mathbb{R}^n} \|f_2\|_{L_p(B(x-y,r))} |f_1(y)|\, dy$$

and

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \big\| w(r) \|f_1 * f_2\|_{L_p(B(x,r))} \big\|_{L_\theta(0,\infty)}$$
$$\le \sup_{x\in\mathbb{R}^n} \Bigg\| \int\limits_{\mathbb{R}^n} w(r) \|f_2\|_{L_p(B(x-y,r))} |f_1(y)|\, dy \Bigg\|_{L_\theta(0,\infty)}$$
$$\le \sup_{x\in\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \big\| w(r) \|f_2\|_{L_p(B(x-y,r))} \big\|_{L_\theta(0,\infty)} |f_1(y)|\, dy \le \|f_1\|_{L_1} \|f_2\|_{GM_{p\theta,w(\cdot)}}.$$

Remark 3. In inequality (3.4), one cannot replace the global space $GM_{p\theta,w(\cdot)}$ by the local space $LM_{p\theta,w(\cdot)}$ even if one adds a constant factor independent of f_1 and f_2 to the right-hand side. In particular, for any $0 < p \le \infty$, $\lambda > 0$, and any $A > 0$, the inequality

$$\|f_1 * f_2\|_{LM_p^\lambda} \le A \|f_1\|_{L_1} \|f_2\|_{LM_p^\lambda} \tag{3.6}$$

with arbitrary $f_1 \in L_p$ and $f_2 \in LM_p^\lambda$ fails, as is shown in the following example.[1]

[1]This example was proposed by E.D. Nursultanov.

Let $n = 1$ and

$$f_{1k} = \chi_{[-k-1,-k]}, \qquad f_{2k} = \chi_{[k,k+1]}, \qquad k \in \mathbb{N}.$$

Then

$$\|f_{1k}\|_{L_1} = 1, \qquad \|f_{2k}\|_{L_p} = 1, \qquad \|f_{2k}\|_{LM_p^\lambda} \le \sup_{r\ge k} r^{-\lambda}\|f_{2k}\|_{L_p(-r,r)} \le k^{-\lambda}.$$

On the other hand, for $0 \le x \le 1/2$,

$$(f_{1k} * f_{2k})(x) = \int\limits_{\mathbb{R}^n} f_{1k}(x-y)f_{2k}(y)\,dy = \int\limits_{[k,k+1]\cap(x-[-k-1,-k])} dy = \int\limits_{[k+x,k+1]} dy \ge \frac{1}{2}$$

and, for all $k \in \mathbb{N}$,

$$\|f_{1k} * f_{2k}\|_{LM_p^\lambda} \ge \sup_{r\ge 1} r^{-\lambda}\|f_{1k} * f_{2k}\|_{L_p(0,r)} \ge \|f_{1k} * f_{2k}\|_{L_p(0,1/2)} \ge 2^{-1-1/p}.$$

Replacing f_1 with f_{1k} and f_2 with f_{2k} in (3.6), we see that this inequality is impossible.

Remark 4. Note that in the case of the spaces H_p^λ, for any $\lambda > 0$ and any p_1, p_2, and p satisfying condition (3.2), the following direct analog of Young's inequality holds:

$$\begin{aligned}\|f_1 * f_2\|_{\dot H_p^\lambda} &= \sup_{h\in\mathbb{R}^n,\, h\ne 0} |h|^{-\lambda}\|\Delta_h^\sigma(f_1 * f_2)\|_{L_p} = \sup_{h\in\mathbb{R}^n,\, h\ne 0} |h|^{-\lambda}\|f_1 * \Delta_h^\sigma f_2\|_{L_p}\\ &\le \Bigl(\sup_{h\in\mathbb{R}^n,\, h\ne 0} |h|^{-\lambda}\|\Delta_h^\sigma f_2\|_{L_{p_2}}\Bigr)\|f_1\|_{L_{p_1}} = \|f_1\|_{L_{p_1}}\|f_2\|_{\dot H_{p_2}^\lambda}\end{aligned}$$

(it is assumed here that $\sigma \in \mathbb{N}$ and $\sigma > \lambda$). However, in the case of the global spaces $GM_{p\theta,w(\cdot)}$, for any p_1, p_2, and p satisfying condition (3.2), the direct analog of Young's inequality

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1\|_{L_{p_1}}\|f_2\|_{GM_{p_2\theta,w(\cdot)}}$$

fails for $p_1 > 1$ even if one adds a constant factor independent of f_1 and f_2 to the right-hand side. In particular, for any $A > 0$, the inequality

$$\|f_1 * f_2\|_{M_p^\lambda} \le A\|f_1\|_{L_{p_1}}\|f_2\|_{M_{p_2}^\lambda} \tag{3.7}$$

with arbitrary $f_1 \in L_{p_1}$ and $f_2 \in M_{p_2}^\lambda$ fails to hold.

This is obvious if $n/p < \lambda \le n/p_2$. Indeed, it follows from (3.7) that $f_1 * f_2 \in M_p^\lambda$ with $\lambda > n/p$, which implies that the convolution $f_1 * f_2$ is equivalent to zero on $\mathbb{R}^n$ for all $f_1 \in L_{p_1}$ and $f_2 \in M_{p_2}^\lambda$, but this is impossible.

For $n = 1$ and any $0 < \lambda \le 1/p$, this is confirmed by the following example.[2] Let $\alpha = 1/(\lambda p_2)$ and

$$f_1 = \sum_{k=2}^\infty k^{-1/p_1}(\ln k)^{-2/p_1}\chi_{[-k^\alpha-1,-k^\alpha+1]}, \qquad f_2 = \sum_{k=2}^\infty \chi_{[k^\alpha,k^\alpha+1]}.$$

It is obvious that $f_1 \in L_{p_1}$ and $f_2 \notin L_{p_2}$. Let us prove that $f_2 \in M_{p_2}^\lambda$. Indeed, if $x < -2r$, then $\|f_2\|_{L_{p_2}(x-r,x+r)} = 0$. If $r < 1$, then, for all $x \in \mathbb{R}$,

$$r^{-\lambda}\|f_2\|_{L_{p_2}(x-r,x+r)} \le r^{-\lambda}(2r)^{1/p_2} \le 2.$$

[2]This example was also proposed by E.D. Nursultanov.

If $r > 1$ and $x > 2r$, then $x - r - 1 > 0$ and, since $a^{\lambda p_2} - b^{\lambda p_2} \le (a-b)^{\lambda p_2}$ for $a \ge b \ge 0$ because of the inequality $\lambda p_2 \le 1$, we have

$$r^{-\lambda}\|f_2\|_{L_{p_2}(x-r,x+r)} \le r^{-\lambda}\Bigg(\sum_{k^\alpha+1>x-r,\ k^\alpha<x+r} 1\Bigg)^{1/p_2} = r^{-\lambda}\Bigg(\sum_{(x-r-1)^{\lambda p_2}<k<(x+r)^{\lambda p_2}} 1\Bigg)^{1/p_2}$$

$$= r^{-\lambda}\big((x+r)^{\lambda p_2} - (x-r-1)^{\lambda p_2} + 1\big)^{1/p_2} \le r^{-\lambda}\big((2r+1)^{\lambda p_2}+1\big)^{1/p_2} \le (3^{\lambda p_2}+1)^{1/p_2} \le 4.$$

Finally, if $r > 1$ and $-r \le x \le 2r$, then

$$r^{-\lambda}\|f_2\|_{L_{p_2}(x-r,x+r)} \le r^{-\lambda}\|f_2\|_{L_{p_2}(0,3r)} \le r^{-\lambda}\Bigg(\sum_{k^\alpha<3r} 1\Bigg)^{1/p_2}$$

$$\le r^{-\lambda}\big((3r)^{\lambda p_2}+1\big)^{1/p_2} \le (3^{\lambda p_2}+1)^{1/p_2} \le 4.$$

Thus,

$$\|f_2\|_{M^\lambda_{p_2}} = \sup_{x\in\mathbb{R}}\sup_{r>0} r^{-\lambda}\|f_2\|_{L_{p_2}(x-r,x+r)} < \infty.$$

At the same time, for $0 \le x \le 1$ and $k^\alpha \le y \le k^\alpha + 1$, we have $-k^\alpha - 1 \le x - y \le -k^\alpha + 1$; therefore,

$$(f_1 * f_2)(x) = \sum_{k=2}^{\infty} \int\limits_{k^\alpha}^{k^\alpha+1} f_1(x-y) = \sum_{k=2}^{\infty} k^{-1/p_1}(\ln k)^{-2/p_1} = \infty.$$

Hence, $\|f_1 * f_2\|_{M^\lambda_p} = \infty$ and, more generally, $\|f_1 * f_2\|_{M^\nu_q} = \infty$ for all $0 < q \le \infty$ and $0 \le \nu \le 1/q$.

For this reason, in the following statement we additionally assume that $f_1 \in L_{p_1}$ and $f_2 \in L_{p_2}$.

Theorem 2. *Let*

$$1 \le p_1, p_2 \le p \le \infty, \qquad \frac{p_1 p_2}{p} \le \theta_1, \theta_2 \le \infty, \qquad 0 \le \alpha_1, \alpha_2 \le 1, \tag{3.8}$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1, \qquad \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = \frac{1}{p}, \qquad \frac{\alpha_1}{\theta_1} + \frac{\alpha_2}{\theta_2} = \frac{1}{\theta}. \tag{3.9}$$

Let, next, $w_1 \in \Omega_{p_1\theta_1}$, $w_2 \in \Omega_{p_2\theta_2}$, *and*

$$w(r) = w_1^{\alpha_1}(r)\, w_2^{\alpha_2}(r), \qquad r > 0. \tag{3.10}$$

Then $w \in \Omega_{p\theta}$; *for all* $f_k \in GM_{p_k\theta_k, w_k(\cdot)} \cap L_{p_k}$, $k = 1, 2$, *the convolution* $f_1 * f_2$ *exists almost everywhere on* $\mathbb{R}^n$; *and*

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1\|^{\alpha_1}_{GM_{p_1\theta_1,w_1(\cdot)}} \|f_1\|^{1-\alpha_1}_{L_{p_1}} \|f_2\|^{\alpha_2}_{GM_{p_2\theta_2,w_2(\cdot)}} \|f_2\|^{1-\alpha_2}_{L_{p_2}}. \tag{3.11}$$

Let us distinguish the following particular cases of inequality (3.11).

1. If $\alpha_1 = 0$ and $p_1 = 1$, then $p_2 = p$, $\alpha_2 = 1$, $\theta_2 = \theta$, and $w_2(\cdot) = w(\cdot)$, and this is inequality (3.4). In this case, it suffices to assume that $f_2 \in GM_{p\theta,w(\cdot)}$. (For $p \ge 1$, the additional assumption $f_2 \in L_p$ is redundant.)

2. If $\alpha_1 = 0$ and $p_1 > 1$, then $p_2 < p$, $\alpha_2 = p_2/p$, and $\theta_2 = (p_2/p)\theta$, where $p_1 \le \theta \le \infty$, $w_2(\cdot) = w^{p/p_2}(\cdot)$, and

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1\|_{L_{p_1}} \|f_2\|^{p_2/p}_{GM_{p_2,(p_2/p)\theta,w^{p/p_2}(\cdot)}} \|f_2\|^{1-p_2/p}_{L_{p_2}} \tag{3.12}$$

for $w \in \Omega_{p\theta}$.

3. If $\theta_1 = \theta_2 = \theta = \infty$, $0 \le \lambda_1 \le n/p_1$, $0 \le \lambda_2 \le n/p_2$, $w_1(r) = r^{-\lambda_1}$, and $w_2(r) = r^{-\lambda_2}$, then $w(r) = r^{-(\alpha_1\lambda_1+\alpha_2\lambda_2)}$ and

$$\|f_1 * f_2\|_{M_p^{\alpha_1\lambda_1+\alpha_2\lambda_2}} \le \|f_1\|^{\alpha_1}_{M^{\lambda_1}_{p_1}} \|f_1\|^{1-\alpha_1}_{L_{p_1}} \|f_2\|^{\alpha_2}_{M^{\lambda_2}_{p_2}} \|f_2\|^{1-\alpha_2}_{L_{p_2}}. \tag{3.13}$$

According to this inequality, for fixed p_1, p_2, λ_1, and λ_2, the maximum value of the parameter λ for which $f_1 * f_2 \in M_p^\lambda$ is equal to $\max\{p_1\lambda_1/p, p_2\lambda_2/p\}$ (the maximum is attained either for $\alpha_1 = 0$ or for $\alpha_2 = 0$).

4. If $\theta_1 = \theta_2 = \infty$, $\alpha_1 = 0$, $\alpha_2 = p_2/p$, and $w_2(r) = r^{-\lambda_2}$, then $\theta = \infty$, $w(r) = r^{-(p_2/p)\lambda_2}$, and

$$\|f_1 * f_2\|_{M_p^{(p_2/p)\lambda_2}} \le \|f_1\|_{L_{p_1}} \|f_2\|^{p_2/p}_{M^{\lambda_2}_{p_2}} \|f_2\|^{1-p_2/p}_{L_{p_2}} \tag{3.14}$$

for $0 \le \lambda_2 \le n/p_2$.

Proof of Theorem 2. 1. The fact that $w \in \Omega_{p\theta}$ was verified in the proof of Lemma 2.

2. Below, without loss of generality, we will assume that f_1 and f_2 are nonnegative functions.

First, we prove inequality (3.12). Note that the condition $w \in \Omega_{p\theta}$ implies the inclusion $w^{p/p_1} \in \Omega_{p_1,(p_1/p)\theta}$. Since $p_2/p + p_2/p_1' = 1$ according to (3.9), applying Hölder's inequality with the exponents p_1 and p_1' yields[3]

$$\begin{aligned}
(f_1 * f_2)(z) = (f_2 * f_1)(z) &= \int_{\mathbb{R}^n} f_2(z-y) f_1(y)\,dy = \int_{\mathbb{R}^n} \bigl(f_1(y) f_2(z-y)^{p_2/p}\bigr) f_2(z-y)^{p_2/p_1'}\,dy \\
&\le \bigl\|f_1(y) f_2(z-y)^{p_2/p}\bigr\|_{L_{p_1,y}(\mathbb{R}^n)} \bigl\|f_2(z-y)^{p_2/p_1'}\bigr\|_{L_{p_1',y}(\mathbb{R}^n)} \\
&= \bigl\|f_1(y) f_2(z-y)^{p_2/p}\bigr\|_{L_{p_1,y}(\mathbb{R}^n)} \|f_2\|^{p_2/p_1'}_{L_{p_2}(\mathbb{R}^n)}
\end{aligned} \tag{3.15}$$

for all $z \in \mathbb{R}^n$.

Since $p_1 \le p$, we can apply the generalized Minkowski inequality for integrals and find that

$$\begin{aligned}
\|f_1 * f_2\|_{L_p(B(x,r))} &\le \bigl\| \|f_1(y) f_2(z-y)^{p_2/p}\|_{L_{p_1,y}(\mathbb{R}^n)} \bigr\|_{L_{p,z}(B(x,r))} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \\
&\le \bigl\| \|f_2(z-y)^{p_2/p}\|_{L_{p,z}(B(x,r))} f_1(y) \bigr\|_{L_{p_1,y}(\mathbb{R}^n)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \\
&= \bigl\| \|f_2^{p_2/p}\|_{L_p(B(x-y,r))} f_1(y) \bigr\|_{L_{p_1,y}(\mathbb{R}^n)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \\
&= \bigl\| \|f_2\|^{p_2/p}_{L_{p_2}(B(x-y,r))} f_1(y) \bigr\|_{L_{p_1,y}(\mathbb{R}^n)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)}
\end{aligned} \tag{3.16}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. (If $r = \infty$, this inequality coincides with inequality (3.1), and we obtain its short proof.)

[3] As usual, p_1' denotes the conjugate exponent of p_1 $(1/p_1 + 1/p_1' = 1)$.

Since $p_1 \le \theta$, another application of the generalized Minkowski inequality for integrals yields

$$\begin{aligned}
\big\|w(r)\|f_1 * f_2\|_{L_p(B(x,r))}\big\|_{L_\theta(0,\infty)} &\le \Big\|w(r)\big\|\|f_2\|^{p_2/p}_{L_{p_2}(B(x-y,r))} f_1(y)\big\|_{L_{p_1,y}(\mathbb{R}^n)}\Big\|_{L_{\theta,r}(0,\infty)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \\
&= \Big\|\big\|\big(w(r)^{p/p_2}\|f_2\|_{L_{p_2}(B(x-y,r))}\big)^{p_2/p} f_1(y)\big\|_{L_{p_1,y}(\mathbb{R}^n)}\Big\|_{L_{\theta,r}(0,\infty)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \\
&\le \Big\|\big\|\big(w(r)^{p/p_2}\|f_2\|_{L_{p_2}(B(x-y,r))}\big)^{p_2/p} f_1(y)\big\|_{L_{\theta,r}(0,\infty)}\Big\|_{L_{p_1,y}(\mathbb{R}^n)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \\
&= \Big\|\big\|w(r)^{p/p_2}\|f_2\|_{L_{p_2}(B(x-y,r))}\big\|^{p_2/p}_{L_{(p_2/p)\theta,r}(0,\infty)} f_1(y)\Big\|_{L_{p_1,y}(\mathbb{R}^n)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \\
&\le \Big(\sup_{u\in\mathbb{R}^n}\big\|w(r)^{p/p_2}\|f_2\|_{L_{p_2}(B(u,r))}\big\|_{L_{(p_2/p)\theta,r}(0,\infty)}\Big)^{p_2/p} \|f_1(y)\|_{L_{p_1,y}(\mathbb{R}^n)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \\
&= \|f_1\|_{L_{p_1}(\mathbb{R}^n)} \|f_2\|^{p_2/p}_{GM_{p_2,(p_2/p)\theta,w^{p/p_2}(\cdot)}(\mathbb{R}^n)} \|f_2\|^{1-p_2/p}_{L_{p_2}(\mathbb{R}^n)} \qquad (3.17)
\end{aligned}$$

for all $x \in \mathbb{R}^n$, which implies inequality (3.12).

3. Let

$$\beta_1 = \frac{\alpha_1 p}{p_1}, \quad \beta_2 = \frac{\alpha_2 p}{p_2}, \quad \eta_1 = \frac{\theta_1 p}{p_1}, \quad \eta_2 = \frac{\theta_2 p}{p_2}, \quad v_1(r) = w_1^{p_1/p}(r), \quad v_2(r) = w_2^{p_2/p}(r), \quad r > 0.$$

By virtue of (3.9) and (3.10),

$$\beta_1 + \beta_2 = 1, \qquad \frac{\beta_1}{\eta_1} + \frac{\beta_2}{\eta_2} = \frac{1}{\theta}, \qquad v_1^{\beta_1}(r) v_2^{\beta_2}(r) = w(r), \quad r > 0.$$

Using inequality (2.6) with $p_1 = p_2 = p$, the parameters θ_1 and θ_2 replaced by η_1 and η_2, the parameters α_1 and α_2 replaced by β_1 and β_2, and the functions $w_1(r)$ and $w_2(r)$ replaced by $v_1(r)$ and $v_2(r)$, we obtain

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1 * f_2\|^{\beta_1}_{GM_{p\eta_1,v_1(\cdot)}} \|f_1 * f_2\|^{\beta_2}_{GM_{p\eta_2,v_2(\cdot)}}.$$

Now, applying inequality (3.12) with θ replaced by η_2 and w replaced by v_2 to the second factor and the same inequality with f_1 and f_2 interchanged, θ replaced by η_1, and w replaced by v_1 to the first factor, we see that

$$\begin{aligned}
\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} &\le \Big(\|f_2\|_{L_{p_2}} \|f_1\|^{p_1/p}_{GM_{p_1,p_1\eta_1/p,v_1^{p/p_1}(\cdot)}} \|f_1\|^{1-p_1/p}_{L_{p_1}}\Big)^{\alpha_1 p/p_1} \\
&\quad\times \Big(\|f_1\|_{L_{p_1}} \|f_2\|^{p_2/p}_{GM_{p_2,p_2\eta_2/p,v_2^{p/p_2}(\cdot)}} \|f_2\|^{1-p_2/p}_{L_{p_2}}\Big)^{\alpha_2 p/p_2} \\
&= \Big(\|f_2\|_{L_{p_2}} \|f_1\|^{p_1/p}_{GM_{p_1\theta_1,w_1(\cdot)}} \|f_1\|^{1-p_1/p}_{L_{p_1}}\Big)^{\alpha_1 p/p_1} \Big(\|f_1\|_{L_{p_1}} \|f_2\|^{p_2/p}_{GM_{p_2\theta_2,w_2(\cdot)}} \|f_2\|^{1-p_2/p}_{L_{p_2}}\Big)^{\alpha_2 p/p_2} \\
&= \|f_1\|^{\alpha_1}_{GM_{p_1\theta_1,w_1(\cdot)}} \|f_1\|^{1-\alpha_1}_{L_{p_1}} \|f_2\|^{\alpha_2}_{GM_{p_2\theta_2,w_2(\cdot)}} \|f_2\|^{1-\alpha_2}_{L_{p_2}},
\end{aligned}$$

since in view of (3.9) we have

$$\Big(1 - \frac{p_1}{p}\Big)\frac{\alpha_1 p}{p_1} + \frac{\alpha_2 p}{p_2} = 1 - \alpha_1, \qquad \frac{\alpha_1 p}{p_1} + \Big(1 - \frac{p_2}{p}\Big)\frac{\alpha_2 p}{p_2} = 1 - \alpha_2. \qquad \square$$

If $f \in GM_{p\theta,w(\cdot)}$ (in particular, if $f \in M_p^\lambda$), then this does not generally imply that $f \in L_p$. For example, if $0 < \lambda < n/p$, then $|x|^{\lambda-n/p} \in M_p^\lambda$, but $|x|^{\lambda-n/p} \notin L_p$.

In this connection, consider the modified global Morrey-type spaces

$$\widehat{GM}_{p\theta,w(\cdot)} = GM_{p\theta,w(\cdot)} \cap L_p$$

with the quasinorm

$$\|f\|_{\widehat{GM}_{p\theta,w(\cdot)}} = \max\bigl\{\|f\|_{GM_{p\theta,w(\cdot)}}, \|f\|_{L_p}\bigr\},$$

including the spaces

$$\widehat{M}_p^\lambda = M_p^\lambda \cap L_p$$

with the quasinorm

$$\|f\|_{\widehat{M}_p^\lambda} = \max\bigl\{\|f\|_{M_p^\lambda}, \|f\|_{L_p}\bigr\}.$$

Corollary 1. *Under the hypotheses of Theorem* 2,

$$\|f_1 * f_2\|_{\widehat{GM}_{p\theta,w(\cdot)}} \le \|f_1\|_{\widehat{GM}_{p_1\theta_1,w_1(\cdot)}} \|f_2\|_{\widehat{GM}_{p_2\theta_2,w_2(\cdot)}}. \tag{3.18}$$

Proof. It suffices to notice that according to inequalities (3.11) and (3.1),

$$\|f_1 * f_2\|_{GM_{p\theta,w(\cdot)}} \le \|f_1\|_{\widehat{GM}_{p_1\theta_1,w_1(\cdot)}} \|f_2\|_{\widehat{GM}_{p_2\theta_2,w_2(\cdot)}}$$

and

$$\|f_1 * f_2\|_{L_p} \le \|f_1\|_{\widehat{GM}_{p_1\theta_1,w_1(\cdot)}} \|f_2\|_{\widehat{GM}_{p_2\theta_2,w_2(\cdot)}}. \qquad \square$$

If $\theta_1 = \theta_2 = \theta = \infty$, $0 \le \lambda_1 \le n/p_1$, $0 \le \lambda_2 \le n/p_2$, $w_1(r) = r^{-\lambda_1}$, and $w_2(r) = r^{-\lambda_2}$, then inequality (3.18) takes the form

$$\|f_1 * f_2\|_{\widehat{M}_p^{\alpha_1\lambda_1+\alpha_2\lambda_2}} \le \|f_1\|_{\widehat{M}_{p_1}^{\lambda_1}} \|f_2\|_{\widehat{M}_{p_2}^{\lambda_2}}. \tag{3.19}$$

Note that the space $\widehat{M}_p^\lambda$ possesses the monotonicity property with respect to the parameter λ: if $0 \le \lambda \le \mu \le n/p$, then

$$\widehat{M}_p^\mu \subset \widehat{M}_p^\lambda \qquad \text{and} \qquad \|f\|_{\widehat{M}_p^\lambda} \le \|f\|_{\widehat{M}_p^\mu}.$$

Indeed,

$$\begin{aligned}
\|f\|_{\widehat{M}_p^\lambda} &= \max\Bigl\{\sup_{r>0}\sup_{x\in\mathbb{R}^n} r^{-\lambda}\|f\|_{L_p(B(x,r))},\ \|f\|_{L_p(\mathbb{R}^n)}\Bigr\}\\
&= \max\Bigl\{\sup_{0<r\le 1}\sup_{x\in\mathbb{R}^n} r^{-\lambda}\|f\|_{L_p(B(x,r))},\ \sup_{r>1}\sup_{x\in\mathbb{R}^n} r^{-\lambda}\|f\|_{L_p(B(x,r))},\ \|f\|_{L_p(\mathbb{R}^n)}\Bigr\}\\
&\le \max\Bigl\{\sup_{0<r\le 1}\sup_{x\in\mathbb{R}^n} r^{-\mu}\|f\|_{L_p(B(x,r))},\ \|f\|_{L_p(\mathbb{R}^n)}\Bigr\} = \|f\|_{\widehat{M}_p^\mu}.
\end{aligned}$$

Therefore, the "best" inequality among those of the form (3.19) is the inequality

$$\|f_1 * f_2\|_{\widehat{M}_p^\lambda} \le \|f_1\|_{\widehat{M}_{p_1}^{\lambda_1}} \|f_2\|_{\widehat{M}_{p_2}^{\lambda_2}} \qquad \text{with} \quad \lambda = \max\Bigl\{\frac{p_1\lambda_1}{p}, \frac{p_2\lambda_2}{p}\Bigr\}.$$

For comparison, we present an analogous inequality for the Nikol'skii spaces. For any p_1, p_2, and p satisfying condition (3.2) and for any $\lambda_1, \lambda_2 > 0$,

$$\|f_1 * f_2\|_{H_p^{\lambda_1+\lambda_2}} \le \|f_1\|_{H_{p_1}^{\lambda_1}} \|f_2\|_{H_{p_2}^{\lambda_2}}. \tag{3.20}$$

It is assumed that in the definition of the Nikol'skii spaces, $\sigma_1 > \lambda_1$ and $\sigma_2 > \lambda_2$ on the right-hand side and $\sigma = \sigma_1 + \sigma_2$ on the left-hand side. Inequality (3.20) is obtained by an application of Young's inequality for convolutions (3.1) to the equality $\Delta_h^{\sigma_1+\sigma_2}(f_1 * f_2) = (\Delta_h^{\sigma_1} f_1) * (\Delta_h^{\sigma_2} f_2)$.

4. ANALOG OF YOUNG'S INEQUALITY FOR TRUNCATED CONVOLUTIONS IN THE CASE OF GENERAL GLOBAL MORREY-TYPE SPACES

Let $\Omega \subset \mathbb{R}^n$ be an open set and $0 < p, \theta \le \infty$. For a function f defined on Ω, we will denote by f° its extension by zero to $\mathbb{R}^n$. For $w \in \Omega_\theta$, by definition, $f \in LM_{p\theta,w(\cdot)}(\Omega)$ if $f^\circ \in LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ and, accordingly, for $w \in \Omega_{p\theta}$, $f \in GM_{p\theta,w(\cdot)}(\Omega)$ if $f^\circ \in GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$. In this case,

$$\|f\|_{LM_{p\theta,w(\cdot)}(\Omega)} \equiv \|f^\circ\|_{LM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \big\|w(r)\|f\|_{L_p(\Omega\cap B(0,r))}\big\|_{L_\theta(0,\infty)}$$

and

$$\|f\|_{GM_{p\theta,w(\cdot)}(\Omega)} \equiv \|f^\circ\|_{GM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \big\|w(r)\|f\|_{L_p(\Omega\cap B(x,r))}\big\|_{L_\theta(0,\infty)}.$$

In the case of local spaces $LM_{p\theta,w(\cdot)}(\Omega)$, it is assumed that $0 \in \Omega$.

Note that if the set Ω is bounded, then

$$LM_{p\theta,w(\cdot)}(\Omega) \subset GM_{p\theta,w(\cdot)}(\Omega) \subset L_p(\Omega)$$

and

$$\|f\|_{L_p(\Omega)} \le \frac{\|f\|_{LM_{p\theta,w(\cdot)}(\Omega)}}{\|w\|_{L_\theta(r_1,\infty)}}, \qquad \|f\|_{L_p(\Omega)} \le \frac{\|f\|_{GM_{p\theta,w(\cdot)}(\Omega)}}{\|w\|_{L_\theta(r_2,\infty)}}, \tag{4.1}$$

where

$$r_1 = \inf\{r > 0\colon\ \Omega \subset B(0,r)\}, \qquad r_2 = \operatorname{diam}\Omega = \inf_{x\in\Omega} \inf\{r > 0\colon\ \Omega \subset B(x,r)\}.$$

Consider a "truncated" convolution

$$(f_1 * f_2)_{\Omega_2}(x) = \int\limits_{\Omega_2} f_1(x-y) f_2(y)\,dy$$

for $x \in \Omega_1$, where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are measurable sets, f_2 is a measurable function on Ω_2, and f_1 is a measurable function on $\Omega_1 - \Omega_2 = \{x - y\colon x \in \Omega_1,\ y \in \Omega_2\}$.

Let f_2° be the zero extension of f_2 to $\mathbb{R}^n \setminus \Omega_2$ and f_1° be the zero extension of f_1 to $\mathbb{R}^n \setminus (\Omega_1 - \Omega_2)$. Then, for $x \in \Omega_1$,

$$(f_1 * f_2)_{\Omega_2}(x) = \int\limits_{\Omega_2} f_1(x-y) f_2(y)\,dy = \int\limits_{\Omega_2} f_1^\circ(x-y) f_2(y)\,dy = \int\limits_{\mathbb{R}^n} f_1^\circ(x-y) f_2^\circ(y)\,dy = (f_1^\circ * f_2^\circ)(x).$$

Since

$$\|f_1^\circ\|_{L_{p_1}(\mathbb{R}^n)} = \|f_1\|_{L_{p_1}(\Omega_1-\Omega_2)}, \qquad \|f_1^\circ\|_{GM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \|f_1\|_{GM_{p\theta,w(\cdot)}(\Omega_1-\Omega_2)},$$

$$\|f_2^\circ\|_{L_{p_1}(\mathbb{R}^n)} = \|f_1\|_{L_{p_1}(\Omega_2)}, \qquad \|f_2^\circ\|_{GM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \|f_2\|_{GM_{p\theta,w(\cdot)}(\Omega_2)},$$

and

$$\|(f_1 * f_2)_{\Omega_2}\|_{GM_{p\theta,w(\cdot)}(\Omega_1)} = \|f_1^\circ * f_2^\circ\|_{GM_{p\theta,w(\cdot)}(\Omega_1)} \le \|f_1^\circ * f_2^\circ\|_{GM_{p\theta,w(\cdot)}(\mathbb{R}^n)},$$

it follows that under the hypotheses of Theorem 2, for all $f_1 \in GM_{p_1\theta_1,w_1(\cdot)}(\Omega_1-\Omega_2)\cap L_{p_1}(\Omega_1-\Omega_2)$ and $f_2 \in GM_{p_2\theta_2,w_2(\cdot)}(\Omega_2)\cap L_{p_2}(\Omega_2)$, the convolution $(f_1*f_2)_{\Omega_2}$ exists almost everywhere on Ω_1 and

$$\begin{aligned}\|(f_1*f_2)_{\Omega_2}\|_{GM_{p\theta,w(\cdot)}(\Omega_1)} &\le \|f_1^\circ * f_2^\circ\|_{GM_{p\theta,w(\cdot)}(\mathbb{R}^n)}\\ &\le \|f_1^\circ\|^{\alpha_1}_{GM_{p_1\theta_1,w_1(\cdot)}(\mathbb{R}^n)}\|f_1^\circ\|^{1-\alpha_1}_{L_{p_1}(\mathbb{R}^n)}\|f_2^\circ\|^{\alpha_2}_{GM_{p_2\theta_2,w_2(\cdot)}(\mathbb{R}^n)}\|f_2^\circ\|^{1-\alpha_2}_{L_{p_2}(\mathbb{R}^n)}\\ &= \|f_1\|^{\alpha_1}_{GM_{p_1\theta_1,w_1(\cdot)}(\Omega_1-\Omega_2)}\|f_1\|^{1-\alpha_1}_{L_{p_1}(\Omega_1-\Omega_2)}\|f_2\|^{\alpha_2}_{GM_{p_2\theta_2,w_2(\cdot)}(\Omega_2)}\|f_2\|^{1-\alpha_2}_{L_{p_2}(\Omega_2)}. \end{aligned}\tag{4.2}$$

If the sets Ω_1 and Ω_2 are bounded, then inequalities (4.2) and (4.1) imply that there exists a $c>0$, depending only on θ_1, θ_2, $\operatorname{diam}\Omega_1$, $\operatorname{diam}\Omega_2$, w_1, and w_2, such that the following inequality is valid for all $f_1 \in GM_{p_1\theta_1,w_1(\cdot)}(\Omega_1-\Omega_2)\cap L_{p_1}(\Omega_1-\Omega_2)$ and $f_2 \in GM_{p_2\theta_2,w_2(\cdot)}(\Omega_2)\cap L_{p_2}(\Omega_2)$:

$$\|(f_1*f_2)_{\Omega_2}\|_{GM_{p\theta,w(\cdot)}(\Omega_1)} \le c\|f_1\|_{GM_{p_1\theta_1,w_1(\cdot)}(\Omega_1-\Omega_2)}\|f_2\|_{GM_{p_2\theta_2,w_2(\cdot)}(\Omega_2)}. \tag{4.3}$$

If $\alpha_1=p_1/p$, $\alpha_2=0$, $\theta_1=(p_1/p)\theta$, and $w_1(\cdot)=w^{p/p_1}(\cdot)$, then inequality (4.2) takes the form

$$\|(f_1*f_2)_{\Omega_2}\|_{GM_{p\theta,w(\cdot)}(\Omega_1)} \le \|f_1\|^{p_1/p}_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}(\Omega_1-\Omega_2)}\|f_1\|^{1-p_1/p}_{L_{p_1}(\Omega_1-\Omega_2)}\|f_2\|_{L_{p_2}(\Omega_2)}. \tag{4.4}$$

Tracing the proof of Theorem 2, we can sharpen this estimate.

Theorem 3. *Let $\Omega_1,\Omega_2\subset\mathbb{R}^n$ be measurable sets,*

$$1\le p_1,p_2\le p\le\infty,\qquad \frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}+1,\qquad p_2\le\theta\le\infty,$$

and $w\in\Omega_{p\theta}$.

Then the convolution $(f_1*f_2)_{\Omega_2}$ exists almost everywhere on Ω_1 and

$$\begin{aligned}&\|(f_1*f_2)_{\Omega_2}\|_{GM_{p\theta,w(\cdot)}(\Omega_1)}\\ &\qquad\le \Bigl(\sup_{y\in\Omega_2}\|f_1\|_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}(\Omega_1-y)}\Bigr)^{p_1/p}\Bigl(\sup_{x\in\Omega_1}\|f_1\|_{L_{p_1}(x-\Omega_2)}\Bigr)^{1-p_1/p}\|f_2\|_{L_{p_2}(\Omega_2)} \end{aligned}\tag{4.5}$$

for all measurable functions f_1 on $\Omega_1-\Omega_2$ and f_2 on Ω_2 for which the right-hand side of this inequality is finite.

Proof. Just as in the proof of Theorem 2, we assume without loss of generality that f_1 and f_2 are nonnegative functions.

Instead of inequality (3.15), we obtain

$$\begin{aligned}(f_1*f_2)_{\Omega_2}(z) &= \int\limits_{\Omega_2} f_1(z-y)f_2(y)\,dy \le \bigl\|f_2(y)f_1(z-y)^{p_1/p}\bigr\|_{L_{p_2,y}(\Omega_2)}\bigl\|f_1(z-y)^{p_1/p_2'}\bigr\|_{L_{p_2',y}(\Omega_2)}\\ &= \bigl\|f_2(y)f_1(z-y)^{p_1/p}\bigr\|_{L_{p_2,y}(\Omega_2)}\|f_1\|^{p_1/p_2'}_{L_{p_1}(z-\Omega_2)}. \end{aligned}\tag{4.6}$$

Since

$$(\Omega_1\cap B(x,r))-y=(\Omega_1-y)\cap(B(x,r)-y)=(\Omega_1-y)\cap B(x-y,r),$$

instead of inequality (3.16) we obtain

$$\|(f_1 * f_2)_{\Omega_2}\|_{L_p(\Omega_1\cap B(x,r))} \le \Big\|\big\|f_2(y)f_1(z-y)^{p_1/p}\big\|_{L_{p_2,y}(\Omega_2)}\|f_1\|^{1-p_1/p}_{L_{p_1}(z-\Omega_2)}\Big\|_{L_{p,z}(\Omega_1\cap B(x,r))}$$

$$\le \big\|\|f_2(y)f_1(z-y)^{p_1/p}\|_{L_{p_2,y}(\Omega_2)}\big\|_{L_{p,z}(\Omega_1\cap B(x,r))}\Big(\sup_{z\in\Omega_1}\|f_1\|_{L_{p_1}(z-\Omega_2)}\Big)^{1-p_1/p}$$

$$\le \big\|\|f_1(z-y)^{p_1/p}\|_{L_{p,z}(\Omega_1\cap B(x,r))}f_2(y)\big\|_{L_{p_2,y}(\Omega_2)}\sup_{z\in\Omega_1}\|f_1\|^{1-p_1/p}_{L_{p_1}(z-\Omega_2)}$$

$$= \big\|\|f_1\|^{p_1/p}_{L_{p,z}((\Omega_1-y)\cap B(x-y,r))}f_2(y)\big\|_{L_{p_2,y}(\Omega_2)}\sup_{z\in\Omega_1}\|f_1\|^{1-p_1/p}_{L_{p_1}(z-\Omega_2)}. \qquad (4.7)$$

Finally, instead of inequality (3.17), we find that for all $x \in \mathbb{R}^n$

$$\big\|w(r)\|(f_1 * f_2)_{\Omega_2}\|_{L_p(\Omega_1\cap B(x,r))}\big\|_{L_\theta(0,\infty)}$$

$$\le \Big\|\big\|\big(w(r)^{p/p_1}\|f_1\|_{L_{p,z}((\Omega_1-y)\cap B(x-y,r))}\big)^{p_1/p}f_2(y)\big\|_{L_{p_2,y}(\mathbb{R}^n)}\Big\|_{L_{\theta,r}(0,\infty)}\sup_{z\in\Omega_1}\|f_1\|^{1-p_1/p}_{L_{p_1}(z-\Omega_2)}$$

$$\le \Big\|\big\|w(r)^{p/p_1}\|f_1\|_{L_{p,z}((\Omega_1-y)\cap B(x-y,r))}\big\|^{p_1/p}_{L_{(p_1/p)\theta,r}(0,\infty)}f_2(y)\Big\|_{L_{p_2,y}(\Omega_2)}\sup_{z\in\Omega_1}\|f_1\|^{1-p_1/p}_{L_{p_1}(z-\Omega_2)}$$

$$\le \sup_{y\in\Omega_2}\Big(\sup_{u\in\mathbb{R}^n}\big\|w(r)^{p/p_1}\|f_1\|_{L_p((\Omega_1-y)\cap B(u,r))}\big\|_{L_{(p_1/p)\theta,r}(0,\infty)}\Big)^{p_1/p}\|f_2(y)\|_{L_{p_2,y}(\Omega_2)}\sup_{z\in\Omega_1}\|f_1\|^{1-p_1/p}_{L_{p_1}(z-\Omega_2)}$$

$$= \Big(\sup_{y\in\Omega_2}\|f_1\|_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}(\Omega_1-y)}\Big)^{p_1/p}\Big(\sup_{z\in\Omega_1}\|f_1\|_{L_{p_1}(z-\Omega_2)}\Big)^{1-p_1/p}\|f_2\|_{L_{p_2}(\Omega_2)}, \qquad (4.8)$$

which implies inequality (4.5). ☐

Remark 5. Since

$$(f_1 * f_2)_{\Omega_2}(x) = \int\limits_{x-\Omega_2} f_2(x-y)f_1(y)\,dy, \qquad (4.9)$$

this does not allow us to obtain a variant of inequality (4.5) in which Ω_1, f_1 and Ω_2, f_2 are interchanged.

Remark 6. Theorem 3 remains valid if we define the global Morrey-type spaces $GM_{p\theta,w(\cdot)}$ as the spaces of all measurable functions f on Ω such that

$$\|f\|^{(1)}_{GM_{p\theta,w(\cdot)}(\Omega)} = \sup_{x\in\Omega}\big\|w(r)\|f\|_{L_p(\Omega\cap B(x,r))}\big\|_{L_\theta(0,\infty)} < \infty.$$

In the proof, only the arguments used in the derivation of inequality (4.8) should be slightly changed. In this case, for all $x \in \Omega_1$,

$$\big\|w(r)\|(f_1 * f_2)_{\Omega_2}\|_{L_p(\Omega_1\cap B(x,r))}\big\|_{L_\theta(0,\infty)}$$

$$\le \sup_{y\in\Omega_2}\Big(\sup_{u\in\Omega_1-y}\big\|w(r)^{p/p_1}\|f_1\|_{L_{p,z}((\Omega_1-y)\cap B(u,r))}\big\|_{L_{(p_1/p)\theta,r}(0,\infty)}\Big)^{p_1/p}\sup_{z\in\Omega_1}\|f_1\|^{1-p_1/p}_{L_{p_1}(z-\Omega_2)}\|f_2(y)\|_{L_{p_2,y}(\Omega_2)}$$

$$= \Big(\sup_{y\in\Omega_2}\|f_1\|^{(1)}_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}(\Omega_1-y)}\Big)^{p_1/p}\Big(\sup_{z\in\Omega_1}\|f_1\|_{L_{p_1}(z-\Omega_2)}\Big)^{1-p_1/p}\|f_2\|_{L_{p_2}(\Omega_2)}, \qquad (4.10)$$

which implies inequality (4.5) with

$$\Big(\sup_{y\in\Omega_2}\|f_1\|^{(1)}_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}(\Omega_1-y)}\Big)^{p_1/p} \qquad \text{instead of} \qquad \Big(\sup_{y\in\Omega_2}\|f_1\|_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}(\Omega_1-y)}\Big)^{p_1/p}.$$

Remark 7. All the results of this section remain valid if we replace the balls $B(0,r)$ and $B(x,r)$ in Definition 1 of the spaces $LM_{p\theta,w(\cdot)}$ and $GM_{p\theta,w(\cdot)}$ with the cubes $Q(0,r)$ and $Q(x,r)=\{y\in\mathbb{R}^n\colon |x_j-y_j|<r\}$, respectively.

5. ANALOG OF YOUNG'S INEQUALITY FOR MORREY-TYPE SPACES OF PERIODIC FUNCTIONS

Definition 3. Let $0<p,\theta\le\infty$ and $T>0$, and let w be a nonnegative Lebesgue measurable function on the interval $(0,T/2)$. The *periodic global Morrey-type space* $GM^*_{p\theta,w(\cdot)}$ is the space of all Lebesgue measurable T-periodic functions f on $\mathbb{R}^n$ with finite quasinorm

$$\|f\|^*_{GM_{p\theta,w(\cdot)}}=\sup_{x\in Q_T}\big\|w(r)\|f\|_{L_p(Q(x,r))}\big\|_{L_\theta(0,T/2)},$$

where $Q_T\equiv Q(0,T/2)$.

It is assumed that $w\in\Omega^*_{p\theta}$. This means that the zero extension w° of w to $(T/2,\infty)$ belongs to $\Omega_{p\theta}$.

Note that

$$\|f\|^*_{GM_{p\theta,w(\cdot)}}\ge\big\|w(r)\|f\|_{L_p(Q(\varepsilon T/4,r))}\big\|_{L_\theta(T/4,T/2)}\ge\|f\|_{L_p(Q(\varepsilon T/4,T/4))}\|w\|_{L_\theta(T/4,T/2)},$$

where $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)$ and $\varepsilon_j=1$ or -1 for all $j=1,\dots,n$, which implies that

$$\|f\|^*_{L_p}\equiv\|f\|_{L_p(Q_T)}=\Bigg(\sum_{\varepsilon\colon\,\varepsilon_j\in\{-1,1\}}\|f\|^p_{L_p(Q(\varepsilon T/4,T/4))}\Bigg)^{1/p}\le 2^{n/p}\|w\|^{-1}_{L_\theta(T/4,T/2)}\|f\|^*_{GM_{p\theta,w(\cdot)}}. \tag{5.1}$$

Lemma 3. *For all* $0<p,\theta\le\infty$, $T>0$, $w\in\Omega^*_{p\theta}$, *and* $f\in GM^*_{p\theta,w(\cdot)}$,

$$\|f\|^*_{GM_{p\theta,w(\cdot)}}=\|f\|^{**}_{GM_{p\theta,w(\cdot)}}\equiv\sup_{x\in\mathbb{R}^n}\big\|w(r)\|f\|_{L_p(Q(x,r))}\big\|_{L_\theta(0,T/2)}. \tag{5.2}$$

Proof. The inequality $\|f\|^*_{GM_{p\theta,w(\cdot)}}\le\|f\|^{**}_{GM_{p\theta,w(\cdot)}}$ is obvious. Let us prove the inequality $\|f\|^{**}_{GM_{p\theta,w(\cdot)}}\le\|f\|^*_{GM_{p\theta,w(\cdot)}}$. To this end, for any $\xi\in\mathbb{R}^n$, set

$$I(\xi)=\big\|w(r)\|f\|_{L_p(Q(\xi,r))}\big\|_{L_\theta(0,T/2)}.$$

If $\xi\in Q_T$, then we obviously have $I(\xi)\le\|f\|^*_{GM_{p\theta,w(\cdot)}}$. Now, let $\xi\in\partial Q_T$. Consider a sequence of points $\xi_k\in Q_T$, $k\in\mathbb{N}$, such that $\xi_k\to\xi$ as $k\to\infty$. Then, for any function $f\in GM^*_{p\theta,w(\cdot)}$ and any $0<r\le T/2$, we have

$$\lim_{k\to\infty}\|f\|_{L_p(Q(\xi_k,r))}=\|f\|_{L_p(Q(\xi,r))}$$

and, since the function f is periodic,

$$\|f\|_{L_p(Q(\xi_k,r))}\le\|f\|_{L_p(Q(\xi_k,T/2))}=\|f\|_{L_p(Q_T)}$$

for all $k \in \mathbb{N}$. Since $w \in L_\theta(\varepsilon, T/2)$ for every $0 < \varepsilon < T/2$, we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{k\to\infty} \big\| w(r) \|f\|_{L_p(Q(\xi_k,r))} \big\|_{L_\theta(\varepsilon,T/2)} = \big\| w(r) \|f\|_{L_p(Q(\xi,r))} \big\|_{L_\theta(\varepsilon,T/2)}.$$

Hence, for all $\xi \in \partial Q_T$,

$$I(\xi) = \sup_{0<\varepsilon<T/2} \big\| w(r) \|f\|_{L_p(Q(\xi,r))} \big\|_{L_\theta(\varepsilon,T/2)} \le \sup_{0<\varepsilon<T/2} \sup_{k\in\mathbb{N}} \big\| w(r) \|f\|_{L_p(Q(\xi,r))} \big\|_{L_\theta(\varepsilon,T/2)}$$

$$\le \sup_{k\in\mathbb{N}} \big\| w(r) \|f\|_{L_p(Q(\xi,r))} \big\|_{L_\theta(0,T/2)} \le \sup_{x\in Q_T} \big\| w(r) \|f\|_{L_p(Q(\xi,r))} \big\|_{L_\theta(0,T/2)} = \|f\|^*_{GM_{p\theta,w(\cdot)}}.$$

This implies that

$$\sup_{x\in\overline{Q_T}} I(\xi) \le \|f\|^*_{GM_{p\theta,w(\cdot)}}. \tag{5.3}$$

Let, finally, $\xi \notin \overline{Q_T}$. Let $k = (k_1, \dots, k_n)$ with integers k_j such that $-T/2 \le \xi_j + k_j T \le T/2$; hence, $\xi + kT \in \overline{Q_T}$. Since f is periodic,

$$\|f(y)\|_{L_p(Q(\xi,r))} = \|f(y+kT)\|_{L_p(Q(\xi,r))} = \|f(z)\|_{L_p(Q(\xi+kT,r))};$$

therefore, for any $\xi \notin \overline{Q_T}$,

$$I(\xi) = I(\xi + kT) \le \sup_{x\in\overline{Q_T}} I(x).$$

This and inequality (5.3) imply that

$$\|f\|^{**}_{GM_{p\theta,w(\cdot)}} = \sup_{\xi\in\mathbb{R}^n} I(\xi) \le \|f\|^*_{GM_{p\theta,w(\cdot)}}. \quad \square$$

Let us present analogs of the results of the previous section for the convolution of T-periodic functions f_1 and f_2:

$$(f_1 * f_2)(x) = \int_{Q_T} f_1(x-y) f(y)\, dy, \qquad x \in \mathbb{R}^n.$$

Theorem 4. *Suppose that the hypotheses of Theorem 2 concerning the numerical parameters p_1, p_2, p, θ_1, θ_2, θ, α_1, and α_2 are satisfied. Let, next, $w_1 \in \Omega^*_{p_1\theta_1}$, $w_2 \in \Omega^*_{p_2\theta_2}$, and*

$$w(r) = w_1^{\alpha_1}(r) w_2^{\alpha_2}(r), \qquad 0 < r \le \frac{T}{2}. \tag{5.4}$$

Then $w \in \Omega^*_{p\theta}$; for all $f_k \in GM^*_{p_k\theta_k,w_k(\cdot)} \cap L^*_{p_k}$, $k = 1,2$, the convolution $f_1 * f_2$ exists almost everywhere on $\mathbb{R}^n$; and

$$\|f_1 * f_2\|^*_{GM_{p\theta,w(\cdot)}} \le \big(\|f_1\|^*_{GM_{p_1\theta_1,w_1(\cdot)}}\big)^{\alpha_1} \big(\|f_1\|^*_{L_{p_1}}\big)^{1-\alpha_1} \big(\|f_2\|^*_{GM_{p_2\theta_2,w_2(\cdot)}}\big)^{\alpha_2} \big(\|f_2\|^*_{L_{p_2}}\big)^{1-\alpha_2}. \tag{5.5}$$

Proof. We will follow the scheme of proof of Theorem 2. First, we prove an analog of inequality (3.12) for T-periodic functions f_1 and f_2, namely,

$$\|f_1 * f_2\|^*_{GM_{p\theta,w(\cdot)}} \le \Big(\|f_1\|^*_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}}\Big)^{p_1/p} \big(\|f_1\|^*_{L_{p_1}}\big)^{1-p_1/p} \|f_2\|^*_{L_{p_2}}. \tag{5.6}$$

Without loss of generality, we will assume that the functions f_1 and f_2 are nonnegative.

To obtain an analog of inequality (3.15), we should replace $\mathbb{R}^n$ by Q_T and take into account that for periodic functions

$$\big\|f_1(z-y)^{p_1/p_2'}\big\|_{L_{p_2',y}(Q_T)} = \|f_1\|^{p_1/p_2'}_{L_{p_1}(z-Q_T)} = \|f_1\|^{p_1/p_2'}_{L_{p_1}(Q_T)} = \big(\|f_1\|^*_{L_{p_1}}\big)^{p_1/p_2'}.$$

As a result, we obtain

$$(f_1*f_2)(z) \le \big\|f_2(y)f_1(z-y)^{p_1/p}\big\|_{L_{p_2,y}(Q_T)}\big(\|f_1\|^*_{L_{p_1}}\big)^{p_1/p_2'}.$$

To get an analog of inequality (3.16), we should replace $B(x,r)$ by $Q(x,r)$ and $\mathbb{R}^n$ by Q_T; this leads to the inequality

$$\|f_1*f_2\|_{L_p(Q(x,r))} \le \big\|\|f_1\|^{p_1/p}_{L_p(Q(x-y,r))}f_2(y)\big\|_{L_{p_2,y}(Q_T)}\big(\|f_1\|^*_{L_{p_1}}\big)^{1-p_1/p}.$$

Finally, to obtain an analog of inequality (3.17), we should replace $B(x,r)$ by $Q(x,r)$, $\mathbb{R}^n$ by Q_T, and $L_\theta(0,\infty)$ by $L_\theta(0,T/2)$. Then, for all $x\in Q_T$, we have

$$\begin{aligned}
&\big\|w(r)\|f_1*f_2\|_{L_p(Q(x,r))}\big\|_{L_\theta(0,T/2)}\\
&\qquad\le \Big\|\big\|w(r)^{p/p_1}\|f_1\|_{L_p(Q(x-y,r))}\big\|^{p_1/p}_{L_{(p_1/p)\theta,r}(0,T/2)}f_2(y)\Big\|_{L_{p_2,y}(\mathbb{R}^n)}\big(\|f_1\|^*_{L_{p_1}}\big)^{1-p_1/p}\\
&\qquad\le \Big(\sup_{u\in Q_{2T}}\big\|w(r)^{p/p_1}\|f_1\|_{L_p(Q(u,r))}\big\|_{L_{(p_1/p)\theta,r}(0,\infty)}\Big)^{p_1/p}\|f_2(y)\|^*_{L_{p_2}}\big(\|f_1\|^*_{L_{p_1}}\big)^{1-p_1/p}\\
&\qquad\le \Big(\|f_1\|^*_{GM_{p_1,(p_1/p)\theta,w^{p/p_1}(\cdot)}}\Big)^{p_1/p}\big(\|f_1\|^*_{L_{p_1}}\big)^{1-p_1/p}\|f_2\|^*_{L_{p_2}}
\end{aligned}$$

according to Lemma 3, which implies inequality (5.6).

For measurable T-periodic functions f_1 and f_2, equality (4.9) reduces to

$$(f_1*f_2)_{\Omega_2}(x) = \int\limits_{Q_T} f_2(x-y)f_1(y)\,dy;$$

therefore, we can interchange f_1 and f_2 in inequality (5.6). This allows us to obtain inequality (5.5) by using step 3 in the proof of Theorem 2. The only change is that we should apply inequality (5.6) instead of (3.12). $\square$

Corollary 2. *There exists a $c>0$, depending on p_1, p_2, θ_1, θ_2, α_1, α_2, w_1, and w_2, such that under the hypotheses of Theorem 4*

$$\|f_1*f_2\|^*_{GM_{p\theta,w(\cdot)}} \le c\|f_1\|^*_{GM_{p_1\theta_1,w_1(\cdot)}}\|f_2\|^*_{GM_{p_2\theta_2,w_2(\cdot)}} \tag{5.7}$$

for all $f_1\in GM^*_{p_1\theta_1,w_1(\cdot)}$ and $f_2\in GM^*_{p_2\theta_2,w_2(\cdot)}$.

Proof. It suffices to apply inequality (5.1) to inequality (5.5); this yields inequality (5.7) with

$$c = 2^n\|w_1\|^{\alpha_1-1}_{L_{\theta_1}(T/4,T/2)}\|w_2\|^{\alpha_2-1}_{L_{\theta_2}(T/4,T/2)}. \qquad \square$$

Consider separately the periodic Morrey space $(M_p^\lambda)^*$, where $0\le\lambda\le n/p$ and $0<p\le\infty$, which consists of all T-periodic functions $f\in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|^*_{M_p^\lambda} = \sup_{x\in Q_T}\ \sup_{0<r\le T/2} r^{-\lambda}\|f\|_{L_p(Q(x,r))}$$

is finite.

Note that

$$\|f\|^*_{M_p^\lambda} \ge \left(\frac{T}{2}\right)^{-\lambda} \|f\|_{L_p(Q(0,T/2))};$$

hence,

$$\|f\|^*_{L_p} \equiv \|f\|_{L_p(Q_T)} \le \left(\frac{T}{2}\right)^{\lambda} \|f\|^*_{M_p^\lambda}. \tag{5.8}$$

In addition, $(M_p^{\lambda_2})^* \subset (M_p^{\lambda_1})^*$ for $0 \le \lambda_1 < \lambda_2 \le n/p$, and

$$\|f\|^*_{M_p^{\lambda_1}} \le \left(\frac{T}{2}\right)^{\lambda_2-\lambda_1} \|f\|^*_{M_p^{\lambda_2}}.$$

For the spaces $(M_p^\lambda)^*$, we obtain the following results.

Corollary 3. *Let*

$$1 \le p_1, p_2 \le p \le \infty, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1, \qquad 0 \le \lambda_1 \le \frac{n}{p_1}, \qquad 0 \le \lambda_2 \le \frac{n}{p_2},$$

$$0 \le \alpha_1, \alpha_2 \le 1, \qquad \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = \frac{1}{p}.$$

Then, for all $f_1 \in (M_{p_1}^{\lambda_1})^*$ and $f_2 \in (M_{p_2}^{\lambda_2})^*$, the convolution $f_1 * f_2$ exists almost everywhere on $\mathbb{R}^n$ and

$$\|f_1 * f_2\|^*_{M_p^{\alpha_1\lambda_1+\alpha_2\lambda_2}} \le c\|f_1\|^*_{M_{p_1}^{\lambda_1}} \|f_2\|^*_{M_{p_2}^{\lambda_2}}, \tag{5.9}$$

where

$$c = \left(\frac{T}{2}\right)^{\lambda_1(1-\alpha_1)+\lambda_2(1-\alpha_2)} \le \max\left\{1, \left(\frac{T}{2}\right)^n\right\}.$$

Proof. Inequality (5.9) follows from inequalities (5.5) and (5.8). If $T \le 2$, then obviously $c \le 1$, while if $T > 2$, then, according to the assumptions on the parameters,

$$c \le \left(\frac{T}{2}\right)^{\frac{n}{p_1}(1-\alpha_1)+\frac{n}{p_2}(1-\alpha_2)} = \left(\frac{T}{2}\right)^{n\left(\frac{1}{p_1}+\frac{1}{p_2}-\left(\frac{\alpha_1}{p_1}+\frac{\alpha_2}{p_2}\right)\right)} = \left(\frac{T}{2}\right)^n. \qquad \square$$

Corollary 4. *Let*

$$1 \le p_1, p_2 \le p \le \infty, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1, \qquad 0 \le \lambda_1 \le \frac{n}{p_1}, \qquad 0 \le \lambda_2 \le \frac{n}{p_2}.$$

Then

$$\left(M_{p_1}^{\lambda_1}\right)^* \cap \left(M_{p_2}^{\lambda_2}\right)^* \subset \left(M_p^{\lambda}\right)^*, \tag{5.10}$$

where

$$\lambda = \max\left\{\frac{\lambda_1 p_1}{p}, \frac{\lambda_2 p_2}{p}\right\}. \tag{5.11}$$

Proof. It suffices to apply inequality (5.9) and take into account that the maximum of the expression $\alpha_1\lambda_1 + \alpha_2\lambda_2$ under the conditions $0 \le \alpha_1, \alpha_2 \le 1$ and $\alpha_1/p_1 + \alpha_2/p_2 = 1/p$ is equal to $\lambda_1 p_1/p$ if $\lambda_1 p_1 \ge \lambda_2 p_2$ (it is attained for $\alpha_1 = p_1/p$ and $\alpha_2 = 0$) and to $\lambda_2 p_2/p$ if $\lambda_2 p_2 \ge \lambda_1 p_1$ (it is attained for $\alpha_1 = 0$ and $\alpha_2 = p_2/p$). $\square$

ACKNOWLEDGMENTS

We are grateful to M.L. Goldman and E.D. Nursultanov for useful remarks.

The work of V.I. Burenkov (Sections 1–4) is supported by the Russian Science Foundation under grant 14-11-00443 and performed in Steklov Mathematical Institute of Russian Academy of Sciences. Section 5 is written by T.V. Tararykova.

REFERENCES

1. O. V. Besov, V. P. Il'in, and S. M. Nikol'skii, *Integral Representations of Functions and Embedding Theorems*, 2nd ed. (Nauka, Moscow, 1996) [in Russian]; Engl. transl. of the 1st ed.: *Integral Representations of Functions and Embedding Theorems* (J. Wiley & Sons, New York, 1978, 1979).
2. V. I. Burenkov, "Sharp estimates for integrals over small intervals for functions possessing some smoothness," in *Progress in Analysis: Proc. 3rd Int. ISAAC Congr., Berlin, 2001* (World Sci., River Edge, NJ, 2003), Vol. 1, pp. 45–56.
3. V. I. Burenkov, "Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. I, II," Eurasian Math. J. **3** (3), 8–27 (2012); **4** (1), 21–45 (2013).
4. V. I. Burenkov and W. D. Evans, "The weight Hardy inequality for differences and the complete continuity of the embedding of Sobolev spaces for domains with arbitrary strong degeneracy," Dokl. Akad. Nauk **355** (5), 583–585 (1997) [Dokl. Math. **56** (1), 565–567 (1997)].
5. V. I. Burenkov and W. D. Evans, "Weighted Hardy-type inequalities for differences and the extension problem for spaces with generalized smoothness," J. London Math. Soc., Ser. 2, **57** (1), 209–230 (1998).
6. V. I. Burenkov, W. D. Evans, and M. L. Goldman, "On weighted Hardy and Poincaré-type inequalities for differences," J. Inequal. Appl. **1** (1), 1–10 (1997).
7. V. I. Burenkov, A. Gogatishvili, V. S. Guliyev, and R. Ch. Mustafayev, "Boundedness of the fractional maximal operator in local Morrey-type spaces," Complex Var. Elliptic Eqns. **55** (8–10), 739–758 (2010).
8. V. I. Burenkov, A. Gogatishvili, V. S. Guliyev, and R. Ch. Mustafayev, "Boundedness of the Riesz potential in local Morrey-type spaces," Potential Anal. **35** (1), 67–87 (2011).
9. V. I. Burenkov and M. L. Goldman, "Exact analogues of the Hardy inequality for differences in the case of related weights," Dokl. Akad. Nauk **366** (2), 155–157 (1999) [Dokl. Math. **59** (3), 372–374 (1999)].
10. V. I. Burenkov and M. L. Goldman, "Hardy-type inequalities for moduli of continuity," Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk **227**, 92–108 (1999) [Proc. Steklov Inst. Math. **227**, 87–103 (1999)].
11. V. I. Burenkov and M. L. Goldman, "Necessary and sufficient conditions for the boundedness of the maximal operator from Lebesgue spaces to Morrey-type spaces," Math. Inequal. Appl. **17** (2), 401–418 (2014).
12. V. I. Burenkov and G. V. Guliev, "Necessary and sufficient conditions for the boundedness of the maximal operator in local Morrey-type spaces," Dokl. Akad. Nauk **391** (5), 591–594 (2003) [Dokl. Math. **68** (1), 107–110 (2003)].
13. V. I. Burenkov and H. V. Guliyev, "Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces," Stud. Math. **163** (2), 157–176 (2004).
14. V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, "Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces," J. Comput. Appl. Math. **208** (1), 280–301 (2007).
15. V. I. Burenkov and V. S. Guliyev, "Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces," Potential Anal. **30** (3), 211–249 (2009).
16. V. I. Burenkov, V. S. Guliev, and G. V. Guliev, "Necessary and sufficient conditions for the boundedness of the fractional maximal operator in local Morrey-type spaces," Dokl. Akad. Nauk **409** (4), 443–447 (2006) [Dokl. Math. **74** (1), 540–544 (2006)].
17. V. I. Burenkov, V. S. Guliev, and G. V. Guliev, "Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces," Dokl. Akad. Nauk **412** (5), 585–589 (2007) [Dokl. Math. **75** (1), 103–107 (2007)].
18. V. I. Burenkov, V. S. Guliyev, A. Serbetci, and T. V. Tararykova, "Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces," Eurasian Math. J. **1** (1), 32–53 (2010).
19. V. I. Burenkov, V. S. Guliev, T. V. Tararykova, and A. Serbetci, "Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces," Dokl. Akad. Nauk **422** (1), 11–14 (2008) [Dokl. Math. **78** (2), 651–654 (2008)].
20. V. I. Burenkov, P. Jain, and T. V. Tararykova, "On boundedness of the Hardy operator in Morrey-type spaces," Eurasian Math. J. **2** (1), 52–80 (2011).

21. V. I. Burenkov, E. D. Nursultanov, and D. K. Chigambayeva, "Description of the interpolation spaces for a pair of local Morrey-type spaces and their generalizations," Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk **284**, 105–137 (2014) [Proc. Steklov Inst. Math. **284**, 97–128 (2014)].
22. V. I. Burenkov and R. Oinarov, "Necessary and sufficient conditions for boundedness of the Hardy-type operator from a weighted Lebesgue space to a Morrey-type space," Math. Inequal. Appl. **16** (1), 1–19 (2013).
23. V. I. Burenkov and Y. Sawano, "Necessary and sufficient conditions for the boundedness of classical operators of real analysis in general Morrey-type spaces," in *Proc. Symp. on Real Analysis, Ibaraki Univ., 2012* (Ibaraki Univ., Mito, 2013), pp. 81–90.
24. V. S. Guliyev, "Generalized weighted Morrey spaces and higher order commutators of sublinear operators," Eurasian Math. J. **3** (3), 33–61 (2012).
25. K. Kurata, S. Nishigaki, and S. Sugano, "Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators," Proc. Am. Math. Soc. **128** (4), 1125–1134 (2000).
26. B. Kuttner, "Some theorems on fractional derivatives," Proc. London Math. Soc., Ser. 3, **3**, 480–497 (1953).
27. Yu. V. Kuznetsov, "On the pasting of functions from the spaces $W^r_{p,\theta}$," Tr. Mat. Inst. im. V.A. Steklova, Akad. Nauk SSSR **140**, 191–200 (1976) [Proc. Steklov Inst. Math. **140**, 209–220 (1976)].
28. P. G. Lemarié-Rieusset, "The role of Morrey spaces in the study of Navier–Stokes and Euler equations," Eurasian Math. J. **3** (3), 62–93 (2012).
29. T. Mizuhara, "Boundedness of some classical operators on generalized Morrey spaces," in *Harmonic Analysis: Proc. Conf. Sendai (Japan), 1990, ICM-90 Satell. Conf. Proc.*, Ed. by S. Igari (Springer, Tokyo, 1991), pp. 183–189.
30. E. Nakai, "Recent topics of fractional integrals," Sugaku Expo. **20** (2), 215–235 (2007).
31. S. M. Nikol'skii, "On a property of the classes $H_p^{(r)}$," Ann. Univ. Sci. Budapest. Eötvös, Sect. Math. **3–4**, 205–216 (1960/1961).
32. S. M. Nikol'skii, "On imbedding, continuation and approximation theorems for differentiable functions of several variables," Usp. Mat. Nauk **16** (5), 63–114 (1961) [Russ. Math. Surv. **16** (5), 55–104 (1961)].
33. S. M. Nikol'skii, *Approximation of Functions of Several Variables and Embedding Theorems*, 2nd ed. (Nauka, Moscow, 1977) [in Russian]; Engl. transl. of the 1st ed.: *Approximation of Functions of Several Variables and Imbedding Theorems* (Springer, Berlin, 1975).
34. H. Rafeiro, N. Samko, and S. Samko, "Morrey–Campanato spaces: An overview," in *Operator Theory, Pseudo-differential Equations, and Mathematical Physics* (Birkhäuser, Basel, 2013), Oper. Theory Adv. Appl. **228**, pp. 293–323.
35. M. A. Ragusa, "Operators in Morrey type spaces and applications," Eurasian Math. J. **3** (3), 94–109 (2012).
36. W. Sickel, "Smoothness spaces related to Morrey spaces—a survey. I, II," Eurasian Math. J. **3** (3), 110–149 (2012); **4** (1), 82–124 (2013).
37. T. V. Tararykova, "Comments on definitions of general local and global Morrey-type spaces," Eurasian Math. J. **4** (1), 125–134 (2013).
38. H. Triebel, *Theory of Function Spaces* (Birkhäuser, Basel, 1983).

Translated by I. Nikitin

2. Неравенство Юнга для сверток функций

Теорема 1. Пусть

$$1 \le p, r \le q \le \infty, \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1.$$

Если $f \in L_r(\mathbb{R}^n)$, $g \in L_p(\mathbb{R}^n)$, то интеграл $\int_{\mathbb{R}^n} f(x-y)g(y)\,dy$ существует и конечен для почти всех $x \in \mathbb{R}^n$ и

$$\|f*g\|_{L_q(\mathbb{R}^n)} = \left\|\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right\|_{L_q(\mathbb{R}^n)} \le \|f\|_{L_r(\mathbb{R}^n)} \|g\|_{L_p(\mathbb{R}^n)}.$$

Доказательство. Так как

$$\frac{r}{q} + \frac{r}{p'} = r\left(\frac{1}{q} + 1 - \frac{1}{p}\right) = 1,$$

то согласно неравенству Гёльдера

$$|(f*g)(x)| = \left|\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right| \le \int_{\mathbb{R}^n} |f(x-y)|\cdot|g(y)|\,dy$$

$$= \int_{\mathbb{R}^n} \left(|g(y)|\cdot|f(x-y)|^{\frac{r}{q}}\right) |f(x-y)|^{\frac{r}{p'}}\,dy$$

$$\le \left\| |g(y)|\cdot|f(x-y)|^{\frac{r}{q}} \right\|_{L_{p,y}(\mathbb{R}^n)} \left\| |f(x-y)|^{\frac{r}{p'}} \right\|_{L_{p',y}(\mathbb{R}^n)}$$

$$= \left\| |g(y)|\cdot|f(x-y)|^{\frac{r}{q}} \right\|_{L_{p,y}(\mathbb{R}^n)} \|f\|^{\frac{r}{p'}}_{L_r(\mathbb{R}^n)}.$$

Так как $p \le q$, то согласно обобщенному неравенству Минковского

$$\|f*g\|_{L_q(\mathbb{R}^n)} \le \left\| \left\| |g(y)|\cdot|f(x-y)|^{\frac{r}{q}} \right\|_{L_{p,y}(\mathbb{R}^n)} \right\|_{L_{q,x}(\mathbb{R}^n)} \|f\|^{\frac{r}{p'}}_{L_r(\mathbb{R}^n)}$$

$$\le \left\| \left\| |g(y)|\,|f(x-y)|^{\frac{r}{q}} \right\|_{L_{q,x}(\mathbb{R}^n)} \right\|_{L_{p,y}(\mathbb{R}^n)} \|f\|^{\frac{r}{p'}}_{L_r(\mathbb{R}^n)}$$

$$= \|f\|^{\frac{r}{q}}_{L_r(\mathbb{R}^n)} \|g\|_{L_p(\mathbb{R}^n)} \|f\|^{\frac{r}{p'}}_{L_r(\mathbb{R}^n)} = \|f\|_{L_r(\mathbb{R}^n)} \|g\|_{L_p(\mathbb{R}^n)}.$$

Стандартное доказательство $(f,g\geq 0)$ основывается на представлении

$$(f*g)(x)=\int_{\mathbb{R}^n} f(x-y)g(y)\,dy$$

$$=\int_{\mathbb{R}^n}\left(f(x-y)^r g(y)^p\right)^{\frac{1}{q}}\left(f(x-y)^r\right)^{\frac{1}{r}-\frac{1}{q}}\left(g(y)^p\right)^{\frac{1}{p}-\frac{1}{q}}dy$$

$$\underline{\qquad}\cdot\underline{\qquad}\cdot\underline{\qquad}$$

"Грубое" следствие из неравенства Юнга

$$\left\|\int_B f(x-y)g(y)\,dy\right\|_{L_q(A)}\leq\|f\|_{L_r(B-A)}\|g\|_{L_p(B)}$$

Теорема 2. Пусть

$$1 \le p, r \le q \le \infty, \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1,$$

$A, B \subset \mathbb{R}^n$ – измеримые по Лебегу множества.

Если $g \in L_p(B)$ и

$$\sup_{y \in B} \|f\|_{L_r(A-y)} < \infty, \quad \sup_{x \in A} \|f\|_{L_r(x-B)} < \infty,$$

то интеграл $\int_B f(x-y)g(y)\,dy$ существует и конечен для почти всех $x \in A$ и

$$\|(f*g)_B\|_{L_q(A)} = \left\|\int_B f(x-y)g(y)\,dy\right\|_{L_q(A)}$$

$$\le \left(\sup_{y \in B} \|f\|_{L_r(A-y)}\right)^{\frac{r}{q}} \left(\sup_{x \in A} \|f\|_{L_r(x-B)}\right)^{1-\frac{r}{q}} \|g\|_{L_p(B)}. \quad (*)$$

Доказательство. Так как $\frac{r}{q} + \frac{r}{p'} = 1$, то согласно неравенству Гёльдера для любых $x \in A$

$$|(f*g)_B(x)| = \left|\int_B f(x-y)g(y)\,dy\right| \le \int_B |f(x-y)|\cdot|g(y)|\,dy$$

$$= \int_B \left(|g(y)|\cdot|f(x-y)|^{\frac{r}{q}}\right)|f(x-y)|^{\frac{r}{p'}}\,dy$$

$$\le \left\||g(y)|\cdot|f(x-y)|^{\frac{r}{q}}\right\|_{L_{p,y}(B)} \left\||f(x-y)|^{\frac{r}{p'}}\right\|_{L_{p',y}(B)}.$$

Здесь

$$\left\||f(x-y)|^{\frac{r}{p'}}\right\|_{L_{p',y}(B)} = (x-y=z) = \left\||f(z)|^{\frac{r}{p'}}\right\|_{L_{p',z}(x-B)}$$

$$= \|f\|_{L_r(x-B)}^{\frac{r}{p'}} \le \left(\sup_{x \in A} \|f\|_{L_r(x-B)}\right)^{\frac{r}{p'}} = \left(\sup_{x \in A} \|f\|_{L_r(x-B)}\right)^{1-\frac{r}{q}}$$

Итак, для любых $x \in A$

$$|(f*g)_B(x)| \le \left\||g(y)|\cdot|f(x-y)|^{\frac{r}{q}}\right\|_{L_{p,y}(B)} \left(\sup_{x \in A} \|f\|_{L_r(x-B)}\right)^{1-\frac{r}{q}}$$

Так как $p \le q$, то согласно обобщенному неравенству Минковского

$$\|(f*g)_B\|_{L_q(A)} \le \Big\| \big\| |g(y)| \cdot |f(x-y)|^{\frac{r}{q}} \big\|_{L_{p,y}(B)} \Big\|_{L_{q,x}(A)} \Big(\sup_{x\in A} \|f\|_{L_r(x-B)}\Big)^{1-\frac{r}{q}}$$

$$\le \Big\| \big\| |g(y)| \cdot |f(x-y)|^{\frac{r}{q}} \big\|_{L_{q,x}(A)} \Big\|_{L_{p,y}(B)} \Big(\sup_{x\in A} \|f\|_{L_r(x-B)}\Big)^{1-\frac{r}{q}}$$

Здесь

$$\big\||g(y)| \cdot |f(x-y)|^{\frac{r}{q}}\big\|_{L_{q,x}(A)} = |g(y)| \, \big\||f(x-y)|^{\frac{r}{q}}\big\|_{L_{q,x}(A)} = (x-y=z)$$

$$= |g(y)| \, \big\||f(z)|^{\frac{r}{q}}\big\|_{L_{q,z}(A-y)} = |g(y)| \, \|f\|^{\frac{r}{q}}_{L_r(A-y)}$$

$$\le |g(y)| \Big(\sup_{y\in B} \|f\|_{L_r(A-y)}\Big)^{\frac{r}{q}},$$

следовательно

$$\Big\| \big\| |g(y)| \cdot |f(x-y)|^{\frac{r}{q}} \big\|_{L_{q,x}(A)} \Big\|_{L_{p,y}(B)}$$

$$\le \|g\|_{L_p(B)} \Big(\sup_{y\in B} \|f\|_{L_r(A-y)}\Big)^{\frac{r}{q}}$$

и

$$\|(f*g)_B\|_{L_q(A)} \le \Big(\sup_{y\in B} \|f\|_{L_r(A-y)}\Big)^{\frac{r}{q}} \Big(\sup_{x\in A} \|f\|_{L_r(x-B)}\Big)^{1-\frac{r}{q}} \|g\|_{L_p(B)}.$$

В.И. Буренков, Т.В. Тарарыкова. Аналог неравенства Юнга для сверток функций для общих пространств типа Морри. Труды МИ РАН 293 (2016), 113–132.