The *Energy Spreading* PONS Transform and its Applications

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https://prometheus-us.com/PONS-papers/

Mathematical Features

- Hadamard matrix
- quadrature mirror filter
- low crest factor array; every row of every PONS matrix has the uniform crest factor $\sqrt{2}$
- yields optimal uncertainty principle bounds
- excellent correlation properties

Mathematical Features II

- Each row of every PONS matrix is an energy spreading second order Reed-Muller codeword
- unlike the Walsh-Hadamard matrices, all PONS matrices are symmetric and all even-size PONS matrices have equal row (hence column) sums

Why Spread Energy?

- Covert transmission - signals appear as white noise
- Naturally low probability of intercept, low probability of detection, anti-jam (LPI/LPD/AJ) waveforms
- Robust transmission - gradual signal degradation when corrupted
- Signal can face extreme interference and usable data can still be reconstructed

Computational Behavior

- fast transform: $2log_2n$ operations per sample for window size n
- integer computations; no floating point operations
- in-place algorithm
- highly parallelizable
- FPGA built with minimal # of gates
- Algorithms operate to compress and decompress signals in real time

Theorem (H.S. Shapiro — corollary to a compactness result of Kolmogorov):

The following is impossible: $\mathcal{F} \subset L^2(\mathbf{R})$ an infinite orthonormal set, $p > \frac{1}{2}$, and $\forall f \in \mathcal{F}$

(*) $\quad |f(x)| < \dfrac{C_1}{(1+|x|)^p}, \quad |\hat{f}(\xi)| < \dfrac{C_2}{(1+|\xi|)^p}$

Theorem

$\exists$ a CONS $\{\phi_n\}$ for $C[-\pi,\pi]$ such that

1) $\forall n, x,\ \phi_n(x)$ takes on only ± 1

2) $\left| \int_{-\pi}^{\pi} \phi_n(x) e^{-ix\xi} dx \right| \leq \dfrac{18}{\sqrt{\pi}\sqrt{1+|\xi|}}$

Corollary (a Global Uncertainty Principle)

$\exists$ a CONS $\mathcal{S}$ for $L^2(\mathbf{R})$ such that (*) with p = $\frac{1}{2}$ is satisfied for all $f \in \mathcal{S}$

Shapiro Polynomials

$$P_0(z) = Q_0(z) = 1$$

$$P_{n+1}(z) = P_n(z) + z^{2^n} Q_n(z)$$

$$Q_{n+1}(z) = P_n(z) - z^{2^n} Q_n(z)$$

P_n and Q_n are polynomials of degree 2^n -1 with coeffients ±1.

P_0		1		
Q_0		1		
P_1		1	1	
Q_1		1	-1	
P_2	1	1	1	-1
Q_2	1	1	-1	1

⋮

Properties of Shapiro Polynomials

For $|z|=1$, $$|P_{n+1}(z)|^2+|Q_{n+1}(z)|^2$$
$$=2(|P_n(z)|^2+|Q_n(z)|)^2$$
$$=2^{n+2}$$

$$|P_n(z)|\le\sqrt{2}\sqrt{2^n} \quad \text{or} \quad \|P_n\|_\infty\le\sqrt{2}\|P_n\|_{L^2}$$

(same for Q_n)

This choice of ±1's gives an excellent bound ($\sqrt{2}$) for the "peak factor" (peak-to-average ratio), thereby spreading the "energy" of these polynomials almost equally around the unit circle.

Shapiro Sequences

+

+

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\+ + + − − − + −

⋮

Three ways of thinking of these sequences:

(a) sequences of ±1's of length 2^n

(b) coefficients of polynomials

(c) values of piecewise constant functions on $[-\pi,\pi]$

Note: P_n and Q_n are orthogonal in the sense of (c)

Shapiro Sequences are Incomplete

Want:

A collection of functions of type (c) which form a basis for length 2^n digital signals.

Have:

2 such functions for each n

Need:

2^n such functions for each n

PONS Sequences

+	+
+	-

+	+	+	-
+	+	-	+
+	-	+	+
-	+	+	+

+	+	+	-	+	+	-	+
+	+	+	-	-	-	+	-
+	+	-	+	+	+	+	-
-	-	+	-	+	+	+	-
+	-	+	+	-	+	+	+
+	-	+	+	+	-	-	-
-	+	+	+	+	-	+	+
+	-	-	-	+	-	+	+

⋮

Mathematical Properties

$$P(z) = \sum_{k=0}^{2^n-1} \varepsilon_k z^k, \quad z = e^{it}, \quad \{\varepsilon_k\} \text{ a PONS sequence}$$

$$|P(z)| \leq \sqrt{2}\sqrt{2^n}$$

$$P(z)P(1/z) + P(-z)P(-1/z) \equiv 2^{n+1}$$

$$\left| \int_{-\pi}^{\pi} P(e^{it}) e^{-i\omega t} dt \right| \leq \frac{18}{\sqrt{\pi}\sqrt{1+|\omega|}}, \quad \omega \in \mathbf{R}$$

Walsh Functions

The Walsh functions are

- piecewise constant
- take on only the values ± 1
- form a basis

These properties make the Walsh functions useful in

- signal processing
- digital filtering
- communications
- coding

This usefulness is hampered because

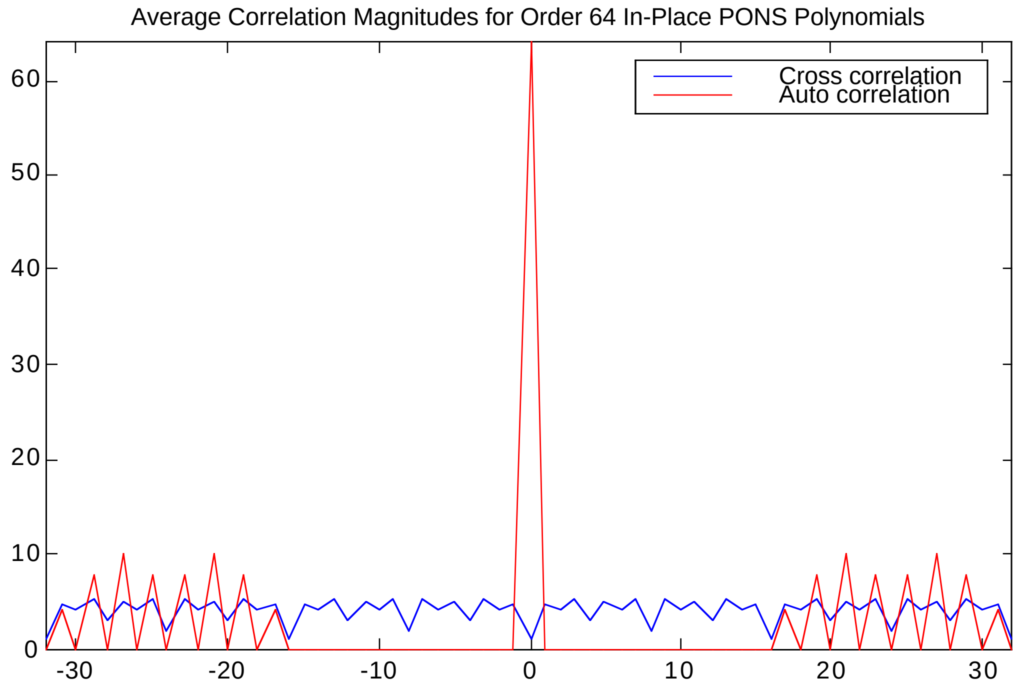
- Walsh polynomials have the worst possible crest factor
- they are not quadrature mirror filters
- their Fourier transforms decay as slowly as possible

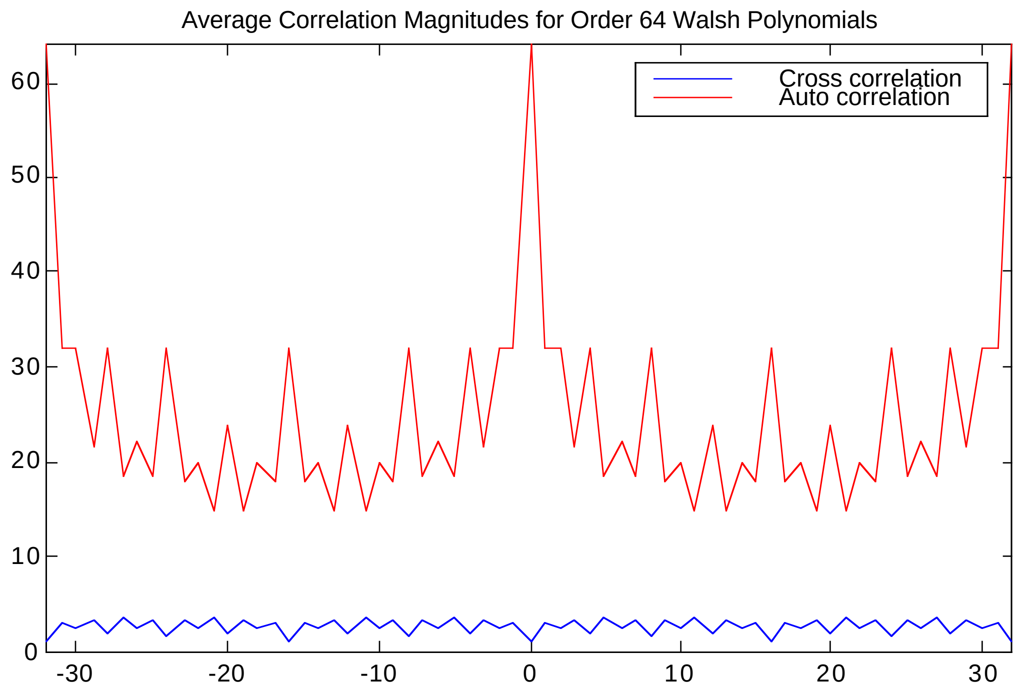
PONS = Walsh+

PONS satisfies all useful Walsh properties *plus*

- PONS polynomials have uniformly low crest factor
- they *are* quadrature mirror filters
- PONS Fourier transforms decay as *quickly* as possible
- they are optimal w.r.t. the *Global Uncertainty Principle*

Average Correlation Magnitudes





Energy Spreading in Mathematical Terms

A an $N \times N$ Hadamard matrix, x a discrete signal of length N, $\|\bullet\|_\infty$ the sup norm, $\|\bullet\|_{ME}$ the Beurling minimal extrapolation norm, S a set of such x's.

Trivial bound: $\|Ax\|_\infty \leq \sqrt{N}\|x\|_\infty$ for all x

Claim: If, for some constant M not too much larger than 1, $\|Ax\|_\infty \leq M\|x\|_\infty$ for all $x \in S$, then A gives good energy spreading on S.

Energy Spreading — Theorems

Theorem: If A is a PONS matrix, then

$$\|Ax\|_\infty \;\le\; \sqrt{2}\|x\|_{ME} \quad \text{for all } x.$$

Example: $x = [x' \mid x'']$, $k + l = N$, A PONS

$$x' = [a\cos(\omega_1 + b), \ldots, a\cos(k\omega_1 + b)]$$

$$x'' = [c\cos(\omega_2 + d), \ldots, c\cos(l\omega_2 + d)]$$

Then $\|Ax\|_\infty \;\le\; 3\sqrt{6/5}\sqrt{a^2 + c^2}$