On the Boyarsky–Meyers estimates

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Short History

Higher integrability of the gradient or Boyarsky–Meyers estimate has the form

$$\int\limits_{\Omega} |\nabla u|^{2+\delta} dx \leqslant C \int\limits_{\Omega} |f|^{2+\delta}\, dx,$$

where u is a solution to a boundary value problem for the second order linear elliptic equation with "right-hand side" f, in bounded strongly Lipschitz domain Ω and for p-Laplacian

$$\int\limits_{\Omega} |\nabla u|^{p+\delta} dx \leqslant C \int\limits_{\Omega} |f|^{p'(1+\delta/p)}\, dx, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Short History

The following paper
[1] B.V. Bojarskii, Generalized solutions to a system of first-order differential equations of elliptic type with discontinuous coefficients // Math. Sbornik, V. 43(85) (4, 1957). P. 451–503.
is the first publication in the topic. In this article the author showed, that the gradient of the solution to the Dirichlet problem for the divergent uniformly elliptic equations with measurable coefficients in bounded domain, is integrable in the power greater than two.

Short History

Later, in the multidimensional case for equations of the same type, the increased summability of the gradient of the solution of the Dirichlet problem in a domain with a sufficiently regular boundary was established in the work

[2] N. G. Meyers, An L^p–estimate for the gradient of solutions of second order elliptic divergence equations // Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3-e série. T. 17, (3, 1963). P. 189–206.

Subsequently, similar results were obtained for the Neumann problem.

Short History

We also note that higher integrability of the gradient of solutions to the Dirichlet problem in a domain with a Lipshitz boundary for the p-Laplace equation with a variable exponent $p(x)$ satisfying special conditions on the modulus of continuity was obtained in the paper

[3] V.V. Zhikov, On some Variational Problems // Russian Journal of Mathematical physics, V. 5 (1, 1997). P. 105–116.

[4]V. V. Zhikov, Meyers-type estimates for solving the nonlinear Stokes system, Differ. Equ., 33:1 (1997), 108–115.

Short History

Later, in the papers
[5] E. Acerbi, G. Mingione. Gradient estimates for the $p(x)$-Laplacian system. // J. Reine Angew. Math. 2005. V. 584. P. 117–148.
[6] L. Diening, S. Schwarzsacher. Global gradient estimates for the $p(.)$-Laplacian. // Nonlinear Anal. 2014. V. 106. P. 70–85.
this result was strengthened and extended to systems of elliptic equations with variable summability exponent.
Anna Balci et al consider functions of the Muckenhoupt class.

Short History

For the Laplace equation, the mixed Zaremba problem formulated by W. Wirtinger, in a three-dimensional bounded domain with a smooth boundary and inhomogeneous Dirichlet and Neumann conditions was first considered in the work

[7] Zaremba, S.: Sur un problème mixte relatif à l'équation de Laplace (French). Bulletin de l'Académie des sciences de Cracovie, Classe des sciences mathématiques et naturelles, serie A, 313–344 (1910)

The classical solvability of the problem was established by the methods of potential theory under the assumption that the boundary of the open set on which the Neumann data are given also has a certain smoothness.

Short History

The study of the properties of solutions to the Zaremba problem for second-order elliptic equations with variable regular coefficients goes back to the work

[8] G. Fichera. Sul problema misto per le equazioni lineari alle derivate parziali del secondo ordine di tipo ellittico (Italian) // Rev. Roumaine Math. Pures Appl. 1964. V. 9. P. 3–9.

In it, in particular, it was established that at the junction of the Dirichlet and Neumann data, the smoothness of the solutions is lost.

Short History

Homogenization of rapidly oscillating Zaremba problem have been studied in the papers

[10] A. Damlamian, Li Ta-Tsien (Li Daqian). Boundary Homogenization for Elliptic Problems. // J.Math.Pure et Appl. 1987. V. 66. P. 351–361.
[11] G.A. Chechkin. On Boundary — Value Problems for a second — order Elliptic Equation with Oscillating Boundary Conditions. // Nonclassical Partial Differential Equations, Ed. Vladimir N.Vragov. Novosibirsk: IM SOAN SSSR, 1988, P. 95–104. (Reported in Referent. Math., 1989, 12B442, p.62)
[12] M. Lobo, M.E. Pérez. Asymptotic Behavior of an Elastic Body With a Surface Having Small Stuck Regions. // Math Modelling Numerical Anal. V. 22. № 4. 1988. P. 609–624.

Short History

In the papers

[14] Yu.A. Alkhutov, G.A. Chechkin. Increased Integrability of the Gradient of the Solution to the Zaremba Problem for the Poisson Equation. // Russian Academy of Sciencies. Doklady Mathematics 103 (2, 2021): 69–71.

[15] Yu.A. Alkhutov, G.A. Chechkin, The Meyer's Estimate of Solutions to Zaremba Problem for Second-order Elliptic Equations in Divergent Form // C R Mécanique, T. 349 (2, 2021). P. 299–304.

for the linear elliptic equation of the second order, an estimate is obtained for the higher integrability of the gradient of solutions to the Zaremba problem in a domain with a Lipschitz boundary and a rapid change of the Dirichlet and Neumann boundary conditions.

Short History

[16] Yu.A. Alkhutov, G.A. Chechkin, V.G. Maz'ya. On the Boyarsky–Meyers Estimate of a Solution to the Zaremba Problem // Arch Rational Mech Anal, V. 245, No 2 (2022). P. 1197–1211.
[17] Yu.A. Alkhutov, G.A. Chechkin. On Higher Integrability of the Gradient of a Solution to the Zaremba Problem for $p(\cdot)$-Laplace Equation in a Plane Domain // Lobachevskii Journal of Mathematics.- 2023.- v. 44, No 8.- p. 3196–3205.

Linear equations

Linear equation

Setting of the problem

We prove estimates of solutions to the Zaremba problem for elliptic equation in bounded Lipschitz domain $D \in \mathbb{R}^n$, where $n > 1$, of the form

$$\mathcal{L}u := \operatorname{div}(a(x)\nabla u) \tag{1}$$

with uniformly elliptic measurable and symmetric matrix $a(x) = \{a_{ij}(x)\}$, i.e. $a_{ij} = a_{ji}$ and

$$\alpha^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \alpha|\xi|^2 \text{ for almost all } x \in D \text{ and all } \xi \in \mathbb{R}^n. \tag{2}$$

We assume that $F \subset \partial D$ is closed and $G = \partial D \setminus F$.

Setting of the problem

Consider the Zaremba problem

$$\begin{cases} \mathcal{L}u = l \quad \text{in} \quad D, \\ u = 0 \quad \text{on} \quad F, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad G, \end{cases} \tag{3}$$

where $\frac{\partial u}{\partial \nu}$ is the outer conormal derivative of u, and l is a linear functional on $W_2^1(D, F)$, the set of functions from $W_2^1(D)$ with zero trace on F.

Setting of the problem

By the solution of the problem (3) we mean the function $u \in W_2^1(D, F)$ for which the integral identity

$$\int_D a\nabla u \cdot \nabla\varphi \, dx = \int_D f \cdot \nabla\varphi \, dx \qquad (4)$$

holds for all test-functions $\varphi \in W_2^1(D, F)$, the components of the vector-function $f = (f_1, \dots, f_n)$ belong to $L_2(D)$. Here f appears from the representation of the functional l.

Auxiliaries

We are interested in the question of higher integrability of the gradient of solutions to the problem (3). The conditions on the structure of the set of the Dirichlet data support F playes the key role.

For the compact $K \subset \mathbb{R}^n$ we define the capacity $C_q(K)$, $1 < q < n$, by the formula

$$C_q(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla\varphi|^q \, dx : \ \varphi \in C_0^\infty(\mathbb{R}^n), \ \varphi \geqslant 1 \text{ on } K \right\}. \quad (5)$$

Auxiliaries

Suppose $B_r^{x_0}$ is an open ball of the radius r centered in x_0, and $mes_{n-1}(E)$ is $(n-1)$-measure of the set E. Assume also that $q = 2n/(n+2)$ as $n > 2$ and $q = 3/2$ as $n = 2$. We suppose one of the following conditions is fulfilled: for an arbitrary point $x_0 \in F$ as $r \leqslant r_0$ the inequality

$$C_q(F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{n-q} \tag{6}$$

holds true or the inequality

$$mes_{n-1}(F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{n-1} \tag{7}$$

holds, the positive constant c_0 does not depend on x_0 and r. Condition (7) is universal (even for nonlinear equations).

Auxiliaries

The condition (7) is stronger, than (6), but it is clearer. Note that under any of these conditions, the functions $v \in W_2^1(D, F)$ satisfy the Friedrichs inequality

$$\int\limits_D v^2\,dx \leqslant K \int\limits_D |\nabla v|^2\,dx,$$

which, by the Lax-Milgram theorem, implies the unique solvability of the problem (3).

Main result

Theorem

If $f \in L_{2+\delta_0}(D)$, where $\delta_0 > 0$, then there exist positive constants $\delta(n, \delta_0) < \delta_0$ and C, such that for a solution to the problem (3) the estimate

$$\int_D |\nabla u|^{2+\delta} dx \leqslant C \int_D |f|^{2+\delta} \, dx, \tag{8}$$

holds, where C depends only on δ_0, the dimension n, constant c_0 from (6) and (7), and also the constant r_0.

Diffusion with Convection

Linear elliptic operator with drift

Setting of the problem

We prove estimates of solutions to the Zaremba problem for elliptic equation in bounded Lipschitz domain $D \in \mathbb{R}^n$, where $n > 1$, of the form

$$\mathcal{L}u := \operatorname{div}(a(x)\nabla u) + b \cdot \nabla u \tag{9}$$

with uniformly elliptic measurable and symmetric matrix $a(x) = \{a_{ij}(x)\}$, i.e. $a_{ij} = a_{ji}$ and for almost all $x \in D$ and all $\xi \in \mathbb{R}^n$ we have

$$\alpha^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \alpha|\xi|^2, \tag{10}$$

$$b(x) = (b_1(x), \dots, b_n(x)),\ \ b_j(x) \in L_p(D),\ \ p \geqslant n, n > 2, j = 1, \dots, n,$$
$$b(x) = (b_1(x), b_2(x)), \quad b_j(x) \in L_p(D),\ \ p > 2, n = 2, j = 1, 2. \tag{11}$$

Setting of the problem

We assume that $F \subset \partial D$ is closed and $G = \partial D \setminus F$.
Consider the Zaremba problem

$$\begin{cases} \mathcal{L}u = l \quad \text{in} \quad D, \\ u = 0 \quad \text{on} \quad F, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad G, \end{cases} \tag{12}$$

where $\frac{\partial u}{\partial \nu}$ is the outer conormal derivative of u, and l is a linear functional on $W_2^1(D, F)$, the set of functions from $W_2^1(D)$ with zero trace on F.

Setting of the problem

By the solution of the problem (12) we mean the function $u \in W_2^1(D, F)$ for which the integral identity

$$\int\limits_D a\nabla u \cdot \nabla\varphi \, dx - \int\limits_D (b \cdot \nabla u)\varphi \, dx = \int\limits_D f \cdot \nabla\varphi \, dx \qquad (13)$$

holds for all test-functions $\varphi \in W_2^1(D, F)$, the components of the vector-function $f = (f_1, \dots, f_n)$ belong to $L_2(D)$. Here f appears from the representation of the functional l.

Auxiliaries

Suppose $B_r^{x_0}$ is an open ball of the radius r centered in x_0. We suppose one of the following conditions is fulfilled: for an arbitrary point $x_0 \in F$ as $r \leqslant r_0$ the inequality

$$C_q(F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{n-q}, \;\; q = \frac{2n}{n+2}, \;\; n > 2,$$
$$C_q(F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{n-q_1}, \;\; q_1 = \frac{p}{p-1}, \;\; n = 2, \qquad (14)$$

hold true.

Auxiliaries

Note that under any of these conditions, the functions $v \in W_2^1(D, F)$ satisfy the Friedrichs inequality

$$\int_D v^2\,dx \leqslant K \int_D |\nabla v|^2\,dx.$$

Main result

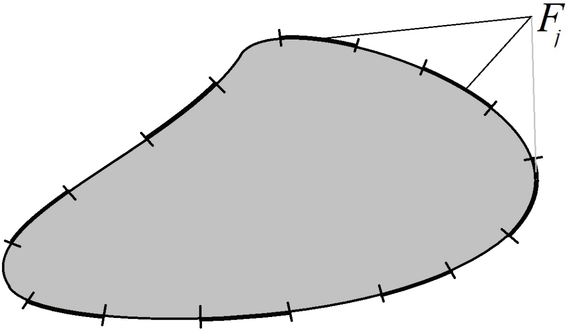
Theorem

If $f \in L_{2+\delta_0}(D)$, where $\delta_0 > 0$, then there exist positive constants $\delta(n, \delta_0) < \delta_0$ and C, such that for a solution to the problem (3) the estimate

$$\int_D |\nabla u|^{2+\delta} dx \leqslant C \int_D |f|^{2+\delta} \, dx, \qquad (15)$$

holds, where C depends only on δ_0, the dimension n, constant c_0 from (14), and also the constant r_0.

How to apply



How to apply

Denote by M_ε the number of the Dirichlet parts F_j, $F = \bigcup_{j=1}^{M_\varepsilon} F_j$.

Consider in D the problem

$$\begin{cases} -\Delta u = f \quad \text{in} \quad D, \\ \frac{\partial u}{\partial n} + au = 0 \quad \text{on} \quad G, \\ u = 0 \quad \text{on} \quad F \end{cases} \tag{16}$$

and the limit problem

$$\begin{cases} -\Delta u_0 = f \quad \text{in} \quad D, \\ \frac{\partial u_0}{\partial n} + au_0 = 0 \quad \text{on} \quad \partial D. \end{cases} \tag{17}$$

How to apply

We estimate the rate of convergence $u \to u_0$ as $\varepsilon \to 0$.

1) The family $\|u\|$ is bounded, hence there exists a weak limit $u \rightharpoonup u_0$.

2) Cut–off $\psi_\varepsilon = \prod\limits_k \psi_\varepsilon^k$, $\psi_\varepsilon^k = \psi\left(\frac{|\ln \varepsilon|}{|\ln r_k|}\right)$, $\psi(s) = \begin{cases} 0, s \leqslant 1, \\ 1, s \geqslant 1+\sigma. \end{cases}$

3) Take $\varphi_\varepsilon = \varphi\psi_\varepsilon$ as a test-function, subtract one integral identity from another. We have

$$\int\limits_D (\psi_\varepsilon \nabla u - \nabla u_0) \cdot \nabla\varphi \, dx + \int\limits_{\partial D} a(u-u_0)\varphi \, ds =$$

$$= \int\limits_D f \cdot \nabla\varphi(\psi_\varepsilon - 1)\, dx + \int\limits_D \nabla u \cdot \nabla\psi_\varepsilon \varphi \, dx + \int\limits_D f \cdot \nabla\psi_\varepsilon\varphi \, dx. \tag{18}$$

How to apply

Keeping in mind the equivalence of the norms in the Sobolev space, we derive

$$\|u - u_0\|^2_{W_2^1(D)} \leqslant C\left(\int\limits_D f \cdot \nabla\varphi(\psi_\varepsilon - 1)\, dx + \int\limits_D \nabla u \cdot \nabla\psi_\varepsilon\, dx \right). \tag{19}$$

The first term in the right hand side of the inequality (19) is estimated by

$$K\, M_\varepsilon^{\frac{1}{2}}\, \varepsilon^{\frac{1}{1+\sigma}}.$$

Here $\varepsilon^{\frac{1}{1+\sigma}}$ is the diameter of the circle, where $\psi_\varepsilon - 1 \neq 0$.

4) Next, we estimate $\int\limits_D (\nabla u, \nabla\psi_\varepsilon)\; dx$.

How to apply

I

$$\int\limits_D (\nabla u, \nabla \psi_\varepsilon)\, dx \leqslant \Big(\int\limits_D |\nabla u|^2\, dx\Big)^{\frac{1}{2}} \Big(\int\limits_D |\nabla \psi_\varepsilon|^2\, dx\Big)^{\frac{1}{2}} \leqslant$$

$$\leqslant K_1 M_\varepsilon^{\frac{1}{2}} |\ln\varepsilon| \Big(\int\limits_{\varepsilon}^{\varepsilon^{\frac{1}{1+\sigma}}} |\ln r|^{-4} d \ln r \Big)^{\frac{1}{2}} \leqslant K_2 M_\varepsilon^{\frac{1}{2}} |\ln\varepsilon|^{-\frac{1}{2}}.$$

$$M_\varepsilon = |\ln\varepsilon|^{1-\theta}, \qquad 0 < \theta < 1.$$

How to apply

II $\qquad p_1 = 2+\delta > 2, \quad p_2 = \frac{2+\delta}{1+\delta} < 2.$

$$\int_D (\nabla u, \nabla \psi_\varepsilon)\, dx \leqslant \Big(\int_D |\nabla u|^{p_1}\, dx\Big)^{\frac{1}{p_1}} \Big(\int_D |\nabla \psi_\varepsilon|^{p_2}\, dx\Big)^{\frac{1}{p_2}} \leqslant$$

$$\leqslant K_1 M_\varepsilon^{\frac{1}{p_2}} \varepsilon^{\frac{2-p_2}{p_2(1+\sigma)}} |\ln \varepsilon| \Big(\int_\varepsilon^{\varepsilon^{\frac{1}{1+\sigma}}} |\ln r|^{-2p_2} d\ln r \Big)^{\frac{1}{p_2}} \leqslant K_2 M_\varepsilon^{\frac{1}{p_2}} \varepsilon^{\frac{2-p_2}{p_2(1+\sigma)}} |\ln \varepsilon|^{\frac{1}{p_2}-1}.$$

$$M_\varepsilon = \varepsilon^{-\frac{\delta}{(1+\delta)(1+\sigma)}} |\ln \varepsilon|^{\frac{1}{1+\delta}-\theta}, \qquad 0 < \theta < \frac{1}{1+\delta}.$$

$p(\cdot)$-Laplacian

$p(\cdot)$-Laplacian

Results from

[19] Yu.A. Alkhutov, G.A. Chechkin. The Boyarsky–Meyers Inequality for the Zaremba Problem for $p(\cdot)$-Laplacian // Journal of Mathematical Sciences, New York, Springer, Vol. 274, No. 4, 2023: 423–441.

Settings

We formulate the Zaremba problem for inhomogeneous $p(\cdot)$-Laplacian in Lipschitz domain $D \subset \mathbb{R}^n$ with variable exponent p, such that

$$1 < \alpha \leqslant p(x) \leqslant \beta < \infty \quad \text{for almost all } x \in D. \tag{20}$$

To set the problem we introduce the functional space

$$W(D) = \{v \in W^1_\alpha(D), |\nabla v|^{p(\cdot)} \in L_1(D)\} \tag{21}$$

with Sobolev-Orlicz norm

$$\|v\|_{W^1_{p(\cdot)}(D)} = \|v\|_{L_\alpha(D)} + \|\nabla v\|_{L_{p(\cdot)}(D)}, \tag{22}$$

Settings

where $\|\cdot\|_{L_{p(\cdot)}(D)}$ is the Luxemburg norm defined by the following formula:

$$\|g\|_{L_{p(\cdot)}(D)} = \inf_{t>0}\Big\{ \int\limits_D |t^{-1}g(x)|^{p(x)}\,dx \leqslant 1 \Big\}. \qquad (23)$$

Settings

Given the norm (22) in the space $W(D)$, we get the reflexive Banach space. Denote it by $W^1_{p(\cdot)}(D)$. Also we denote by $W^1_{p(\cdot)}(D, F)$ the completion of the set of functions from $W^1_{p(\cdot)}(D)$ with support lying outside some neighborhood of the closed set $F \subset \partial D$, by the norm (22).

Settings

Define the space of functions $H^1_{p(\cdot)}(D)$, which is the closure of the set of smooth functions in the norm (22). Similarly, one can introduce the space of functions $H^1_{p(\cdot)}(D, F)$ as a completion in the norm (22) of smooth functions equal to zero in a neighborhood of F.

The density of smooth functions in $W^1_{p(\cdot)}(D)$ is provided by the well-known logarithmic condition

$$|p(x) - p(y)| \leqslant \frac{k_0}{\Big|\ln|x-y|\Big|} \text{ for } x, y \in D, \ |x-y| < \frac{1}{2}, \qquad (24)$$

found by V.V. Zhikov.

Settings

Setting $G = \partial D \setminus F$, consider the Zaremba problem

$$\begin{aligned} \Delta_{p(\cdot)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = l \text{ in } D, \\ u = 0 \text{ on } F, \ \frac{\partial u}{\partial n} = 0 \text{ on } G, \end{aligned} \tag{25}$$

where $\frac{\partial u}{\partial n}$ means the outer normal derivative of the function u, and l is a linear functional in the space dual to $W^1_{p(\cdot)}(D, F)$ or dual to $H^1_{p(\cdot)}(D, F)$, which we describe later. For such a problem, one can define W-solutions and H-solutions.

Settings

The W-solution of the problem (25) is the function $u \in W^1_{p(\cdot)}(D, F)$ for which the integral identity

$$\int\limits_D |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \sum_{i=1}^{n} \int\limits_\Omega f_i \varphi_{x_i} dx, \qquad (26)$$

where $f_i \in L_{p'(\cdot)}(\Omega)$ for $i = 1, \dots, n$ and $p'(x) = \frac{p(x)}{p(x)-1}$, is valid for all test-functions $\varphi \in W^1_{p(\cdot)}(D, F)$.
In analogous way one can define H-solution, for which (26) takes place with test-functions $\varphi \in H^1_{p(\cdot)}(D, F)$.

Settings

Further, it is assumed that the inequality

$$\|v\|_{L_\alpha(D)} \leqslant C\|\nabla v\|_{L_\alpha(D)}, \tag{27}$$

holds, which implies the relation

$$\|v\|_{L_\alpha(D)} \leqslant C\|\nabla v\|_{L_{p(\cdot)}(D)}.$$

Therefore, in the space $W^1_{p(\cdot)}(D,F)$ $(H^1_{p(\cdot)}(D,F))$ we can introduce the norm

$$\|v\|_{W^1_{p(\cdot)}(D,F)} = \|\nabla v\|_{L_{p(\cdot)}(D)}. \tag{28}$$

Conditions

For an arbitrary point $x_0 \in F$ for $r \leqslant r_0$ the inequality

$$C_{q_0}(F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{n-q_0}, \text{ where}$$
$$q_0 = \frac{\alpha' + 1}{2}, \ \alpha' = \min\left(\alpha, \frac{n}{n-1}\right), \tag{29}$$

is valid with constant $\alpha > 1$ from (20).
Note that the condition (29) follows from the following universal condition: for an arbitrary point $x_0 \in F$ for $r \leqslant r_0$ the inequality

$$mes_{n-1}(F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{n-1} \tag{30}$$

holds with a positive constant c_0 independent of x_0 and r.

Inequality

Theorem

Let $|f|^{p'} \in L_{1+\delta_0}(D)$, where $\delta_0 > 0$. Then, there exists a positive constant $\delta < \delta_0$, depending only on δ_0 and α, such that the solution to the problem (25) *satisfies the estimate*

$$\int_D |\nabla u|^{p(x)(1+\delta)}\,dx \leqslant C\Big(\int_D |f|^{p'(x)(1+\delta)}\,dx + 1\Big).$$

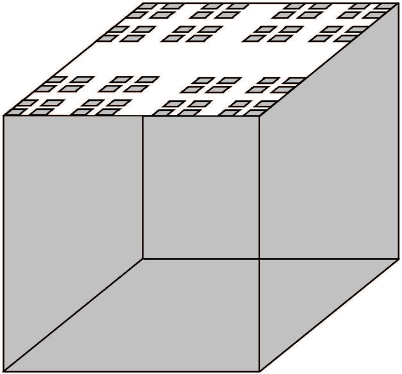
Here the constant C depends only on $p(\cdot)$, δ_0, the value c_0 from the condition on F, the domain D and $\|f^{p'(\cdot)}\|_{L_1(D)}$.

If

$$\alpha \geqslant n + \nu, \ \ \nu > 0,$$

than Theorem is true for $F \neq \emptyset$.

An example of the set F



An example of the set F

Let $\{l_j\}$ is decreasing sequence of positive numbers, $2l_{j+1} < l_j$ ($j = 1, 2, \cdots$) and Δ_1 is a segment of the length $l_1 \leqslant 1$ on the axis Ox_1. Denote by e_1 the union of two closed Δ_2 and Δ_3 of the length l_2, containing both ends of Δ_1

Let $E_1 = e_1 \times e_1$. Repeating the procedure for the segments Δ_2 and Δ_3 (here l_3 plays the role of l_2).

We get four segments of the length l_3. Denote the union of them by e_2.

Then, denoting $E_2 = e_2 \times e_2$, we continue the process.

Finally, we have the two-dimensional Cantor set $F = \bigcap\limits_{j=1}^{\infty} E_j$.

An example of the set F

We consider 3D domain, hence $q = 6/5$. The condition

$$C_{6/5}(F) > 0. \tag{31}$$

is equivalent to

$$\sum_{j=1}^{\infty} 2^{-10j} l_j^{-9} < \infty. \tag{32}$$

We set $l_j = a^{-j+1}$, where $a \in (2, 4^{5/9})$, and hence, $2l_{j+1} < l_j$, then

$$\sum_{j=1}^{\infty} \left(\frac{1}{4}a^{9/5}\right)^{5j} a^{-9} < \infty.$$

An example of the set F

One can show that two-dimensional measure of F equals to zero. Indeed, on the j-th steep we have 4^j closed squares with sides of the length a^{-j+1}.

An example of the set F

For an arbitrary point $x_0 \in F$ and $r \leqslant r_0$ we have

$$C_{6/5}(F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{9/5}, \tag{33}$$

where $B_r^{x_0}$ is a ball of radius r, centered in x_0, the constants $c_0 = \frac{1}{2} a^{-9/5} C_{6/5}(F)$ and $r_0 = \frac{1}{a}$ are positive.
Thus, the Boyarskiy–Meyers estimate is valid in this case.

Examples of the Domains

Fractals

Thanks for your attention!