Semiclassical Approximation with Complex Phases for Constructing Effective Plancherel-Rotach type asymptotics of 1-D and 2-D orthogonal polynomials.

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partially supported by the Government program
AAAA-A20-120011690131-7 and
the Russian Science Foundation (project 21-11-00341)

Seminar on Analysis,
Differential Equations and Mathematical Physics

Rostov-on-Don
12 May 2022, online

FORMULATION OF THE PROBLEM:

HERMITIAN TYPE ORTHOGONAL POLYNOMIALS

(P. Deift, A. I. Aptekarev, P. M. Bleher, A. Branquinho, A.R.Its, A. B. J. Kuijlaars, T. Kriecherbauer, K. T.-R. McLaughlin, W. Van Assche, S. Venakides, X. Zhou ....)

Recurrent equations

$$\mathrm{H}_{\mathrm{n}_1+1,\mathrm{n}_2}(\mathrm{z},\alpha) = \mathrm{z}\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2}(\mathrm{z},\alpha) + \alpha\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2}(\mathrm{z},\alpha) - \frac{1}{2}\left(\mathrm{n}_1\mathrm{H}_{\mathrm{n}_1-1,\mathrm{n}_2}(\mathrm{z},\alpha) + \mathrm{n}_2\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2-1}(\mathrm{z},\alpha)\right),$$

$$\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2+1}(\mathrm{z},\alpha) = \mathrm{z}\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2}(\mathrm{z},\alpha) - \alpha\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2}(\mathrm{z},\alpha) - \frac{1}{2}\left(\mathrm{n}_1\mathrm{H}_{\mathrm{n}_1-1,\mathrm{n}_2}(\mathrm{z},\alpha) + \mathrm{n}_2\mathrm{H}_{\mathrm{n}_1,\mathrm{n}_2-1}(\mathrm{z},\alpha)\right)$$

Initial data

$$\mathrm{H}_{0,0}(\mathrm{z},\alpha) = 1, \quad \mathrm{H}_{\mathrm{n},-1}(\mathrm{z},\alpha) = \mathrm{H}_{-1,\mathrm{n}}(\mathrm{z},\alpha) = 0, \mathrm{n} \in \mathbb{Z}^+, \mathrm{n} > 1$$

The aim:
to construct the asymptotics of dioganal polynomial $H_{n,n}(z,\alpha)$ as $n \to \infty$

Recurrent equations for $H_{n,n}(z,\alpha), H_{n,n-1}(z,\alpha)$

$$\begin{pmatrix} \mathrm{H}_{\mathrm{n,n}} \\ \mathrm{H}_{\mathrm{n+1,n}} \\ \mathrm{H}_{\mathrm{n+1,n+1}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha \mathrm{n} & -\mathrm{n} & \mathrm{z}+\alpha \\ \alpha \mathrm{n}(\mathrm{z}-\alpha) & -\mathrm{nz} & \mathrm{z}^2-\alpha^2-\frac{1}{2}(2\mathrm{n}+1) \end{pmatrix} \begin{pmatrix} \mathrm{H}_{\mathrm{n-1,n-1}} \\ \mathrm{H}_{\mathrm{n,n-1}} \\ \mathrm{H}_{\mathrm{n,n}} \end{pmatrix}$$

Initial data

$$\mathrm{H}_{0,0}(\mathrm{z},\alpha) = 1, \quad \mathrm{H}_{1,0}(\mathrm{z},\alpha) = \mathrm{z}+\alpha, \quad \mathrm{H}_{1,1}(\mathrm{z},\alpha) = \mathrm{z}^2-\alpha^2-1/2$$

Introduce small artificial parameter $h = \dfrac{1}{n}$,

thus we are looking for asymptotics as $h \to +0$

Two approaches:

(1) based on the construction of decompositions of bases of homogeneous difference equations (A.I.Aptekarev and D.M.Tulyakov part)

(2) "real semiclassics for asymptotics with complex-valued phases"

A lot of results: E.Hilb, M. Plancherel, W. Rotach, F. Olver, P.K.Suetin, S.P.Suetin, X. Zhou, Z. Wang, R. Wong, X-Sh. Wang, P. Deift, A.R.Its, A. B. J. Kuijlaars, T. Kriecherbauer, K. T.-R. McLaughlin, A.I.Aptekarev, D.M.Tulyakov, P. M. Bleher, A. Branquinho, W. Van Assche, S. Venakides, V.Yu.Novokshonov, D.R.Yafaev, I.T.Yakubov.....

Retreat: excursion to the polynomials defined recurrence equations of the second order

Uniform Plancherel-Rotach type asymptotics of Hermitian polynomials

$$\mathrm{H}_{n+1}(z) = 2z\mathrm{H}_n(z) - 2n\mathrm{H}_{n-1}(z),\ \mathrm{H}_0(z) = 1,\ \mathrm{H}_1(z) = 2z$$

Following V.P.Maslov introduce the smooth function

$$\psi(x,z) : \psi(kh,z) = H_k(z),$$

the shift operators $\mathrm{e}^{\pm i\hat{p}}\psi(\mathrm{x}) = \psi(\mathrm{x} \pm \mathrm{h}), \quad \hat{\mathrm{p}} = -i\mathrm{h}\frac{\partial}{\partial \mathrm{x}}$

and rewrite the difference equation in a pseudodifferential form

$$\left(\frac{1}{2}\mathrm{e}^{i\hat{p}} + \frac{\mathrm{x}}{\mathrm{h}}\mathrm{e}^{-i\hat{p}} - \frac{\mathrm{y}}{\sqrt{\mathrm{h}}}\right)\psi(\mathrm{x},\mathrm{y}) = 0, \quad \mathrm{y} = \sqrt{\mathrm{h}}\mathrm{z}$$

with complex-valued symbol $\mathcal{H}(\mathrm{x},\mathrm{p};\mathrm{h}) = \frac{1}{2}\mathrm{e}^{i\mathrm{p}} + \frac{\mathrm{x}}{\mathrm{h}}\mathrm{e}^{-i\mathrm{p}} - \frac{\mathrm{y}}{\sqrt{\mathrm{h}}}$

$$f = h^{-x/2}\psi : \quad \hat{\mathbf{H}}f = 0, \quad \mathbf{H} = (\frac{1}{2} + x)\cos p - y + i(\frac{1}{2} - x)\sin p$$

General case:

$$a_n u_{n+1}(z) + b_n(z) u_n(z) + c_n u_{n-1}(z) = 0, \quad n = 0, 1, \ldots, N, \ldots, \quad z \in \mathbb{R},$$

$$b_n = b_n^0 + z b_n^1, \quad u_0(z) = v^0, u_1(z) = v^1 + v^2 z$$

Pseudodifferential equation for $f(x; z), \quad u_n = f(nh; z)$:

$$\alpha(x, h; z) \longrightarrow a_n,\ \beta(x, h; z) \longrightarrow b_n(z),\ \gamma(x, h; z) \longrightarrow c_n$$

$$\alpha(x, h; z) e^{i\hat{p}} f(x; z) + \beta(x, h; z) f(x; z) + \gamma(x, h; z) e^{-i\hat{p}} f(x; z) =$$

$$((\alpha + \gamma)\cos\hat{p} + \beta + i(\alpha - \gamma)\sin\hat{p}) f(x; z) = 0,$$

x is a variable and z is a parameter

WKB-anzatz: $f = \sum A_j(x; z, h) e^{\frac{i}{h}(S^j(x;z) + i\Phi^j(x;z))}$

Complex-valued Hamiltonians $\Longrightarrow$ complex-valued phases

$$\alpha(x,h;z)e^{i\hat{p}}f(x;z)+\beta(x,h;z)f(x;z)+\gamma(x,h;z)e^{-i\hat{p}}f(x;z)=$$

$$((\alpha+\gamma)\cos\hat{p}+\beta+i(\alpha-\gamma)\sin\hat{p})f(x;z)=0,$$

WKB-anzatz: $f=\sum A_j(x;z,h)e^{\frac{i}{h}(S^j(x;z)+i\Phi^j(x;z))}$

The characteristic equation (second-order algebraic curve)

$$\mathcal{R}(x,z,\lambda,h)=\alpha\lambda^2+\beta\lambda+\gamma=\alpha(\lambda-\lambda_1(x,z))(\lambda-\lambda_2(x,z))\quad\Longrightarrow$$

The Hamilton-Jacoby equation $\qquad \mathcal{R}(x,z,e^{i\frac{\partial S}{\partial x}},0)=0$ or

$$((\alpha+\gamma)\cos\frac{\partial S}{\partial x}+\beta+i(\alpha-\gamma)\sin\frac{\partial S}{\partial x})f(x;z)=0,$$

Approaches with complex-valued phases:

WKB-method with transition to the complex plane:

second-order differential equations (Stokes,..., Olver,..., Fedoryuk,..., Shkalikov, Shafarevich, Alliluyeva, S. Stepin, S. Suetin...), above-barrier scattering (Pokrovsky...), "momentum" tunneling (Dobrokhotov, Shafarevich)
discrete Schrödinger equation (Buslaev, Fedotov, Klopp, Shetka ...)

Limitations:
1-D case, equations of the type of continuous and discrete Schrödinger equation

complex rays (Maslov $\Longrightarrow$ Khudyakov, Kravtsov-Orlov),

almost-analytic continuation for the case $\mathrm{Im} S \geq 0$
(Maslov, Kucherenko, Sjöstrand),
complex germ theory (Maslov)== approximate finding of complex phases and the corresponding geometry+linearization of complex Lagrangian manifolds and Hamiltonian systems (application to difference schemes – Maslov, Danilov)
special case (quadratic complex germ) – oscillatory approximation; Gaussian beams; V. M. Babich's asymptotic eigenfunctions concentrated in the vicinity of a closed geodesic (the relativistic Sokolov-Ternov electron in accelerators); trajectory-coherent states, etc.

purely imaginary phases $\mathrm{Re} S = 0$): tunneling problems and probabilistic problems, thermodynamics (*tropical mathematics*)

Our approach for asymptotics of polynomials

First, we explain the idea by the example of a second-order differential equation:

$$-h^2\alpha(x)\frac{d^2u}{dx^2} + h\beta(x)\frac{du}{dx} + \gamma(x)u = 0 \Leftrightarrow \alpha\hat{p}^2u + i\beta\hat{p}u + \gamma u = 0, \quad \hat{p} = -ih\frac{d}{dx}.$$

The symbol $\mathcal{H} = \alpha(x)p^2 + i\beta(x)p + \gamma(x)$ of this equation is complex.
A standard well-known replacement

$$u = w\psi, \quad w = e^{\frac{\Phi_0(x)}{h}}, \quad \Phi_0 = \int_{x_0}^{x} \frac{\beta}{2\alpha}dx$$

reduces original equation to the Schrödinger type equation with a real potential

$$\mathcal{H}_1(\hat{p}, x)\psi \equiv -h^2\frac{d^2\psi}{dx^2} + (\mathcal{V}_0(x) + h\mathcal{V}_1(x))\psi = 0,$$

here $\mathcal{H}_1 = p^2 + \mathcal{V}_0(x) + h\mathcal{V}_1(x)$, $\quad \mathcal{V}_0(x) = \frac{\beta^2}{4\alpha^2} + \frac{\gamma}{\alpha}$, $\quad \mathcal{V}_1(x) = -\frac{d}{dx}\left(\frac{\beta}{2\alpha}\right)$.

The equation for w: $\mathcal{H}_0(\hat{\xi}, x)w = 0$, $\hat{\xi} = h\frac{d}{dx}$ with real symbol $\mathcal{H}_0 = \xi - \frac{\beta}{2\alpha}$.

Discrete case $$\mathrm{f(x,h)} = \mathrm{e}^{\frac{S_0(\mathrm{x,h})}{\mathrm{h}}}\mathrm{g(x,h)} = \mathrm{e}^{\frac{\mathrm{S_0(x)}}{\mathrm{h}}}\mathrm{A_0(x,h)g(x,h)}.$$

We use various Feynman-Maslov operator formulas in semi-classical approximation problems, such as commutation formulas for a pseudodifferential operator with a rapidly changing exponent $A(x)e^{\frac{i}{h}S(x)}$, the transition from generating operators to the product of their symbols, etc. It requires justification if $S(x)$ is a complex-valued function with $\mathrm{Im}S$ of indeterminate sign. In the case of general pseudo-differential operators, the justification of these formulas requires a complicated and delicate analysis, based in particular on the saddle point method. Moreover, these formulas are, generally speaking, may prove to be incorrect. In the theory of the complex Maslov germ, it is assumed that $\mathrm{Im}S \geq 0$, the corresponding formulas are approximate and work in a small neighborhood of the set $\mathrm{Im} = 0$. The situation here is *radically* different, the considerations used in the complex germ are not enough at all. But the class of operators here is also very narrow- we work with shift operators, and the corresponding formulas are easily proved using Taylor series expansion. Moreover, these considerations are transferred to the multidimensional case. Also such formulas

$$\mathrm{e}^{\pm\mathrm{i}\hat{\mathrm{p}}}(\mathrm{f_1(x)f_2(x)}) = \mathrm{f_1(x\pm h)f_2(x\pm h)}, \quad \mathrm{e}^{\pm\mathrm{i}\hat{\mathrm{p}}}(\mathrm{F(f(x))} = (\mathrm{F(f(x\pm h))}$$

are true (although they are not correct for general pseudodifferential operators)

Results

$$f(x,h) = e^{\frac{S_0(x,h)}{h}} g(x,h) = e^{\frac{S_0(x)}{h}} (A_0(x) + O(h)) g(x,h)$$

$$S_0(x) = \frac{1}{2}\int \log \frac{\gamma(y,0)}{\alpha(y,0)} dy, \quad A_0(x) = \exp\left[\frac{1}{2}\int \frac{\partial}{\partial h} \log \frac{\gamma(y,h)}{\alpha(y,h)}\Big|_{h=0} dy\right]$$

Discrete Schrödinger type equation

$$\hat{\mathcal{H}}_1 g(x,h) = \left(\cos\hat{p} + V(x,h)\right) g(x,h) = 0$$

$$V(x,h) = \frac{\beta(x,h)}{2\gamma(x,h)} e^{\frac{1}{1+e^{i\hat{p}}} \log \frac{\gamma(x,h)}{\alpha(x,h)}} = \frac{\beta(x,h)}{2\gamma(x,h)} \sqrt{\frac{\gamma(x-\frac{h}{2},h)}{\alpha(x-\frac{h}{2},h)}} (1 + O(h^2))$$

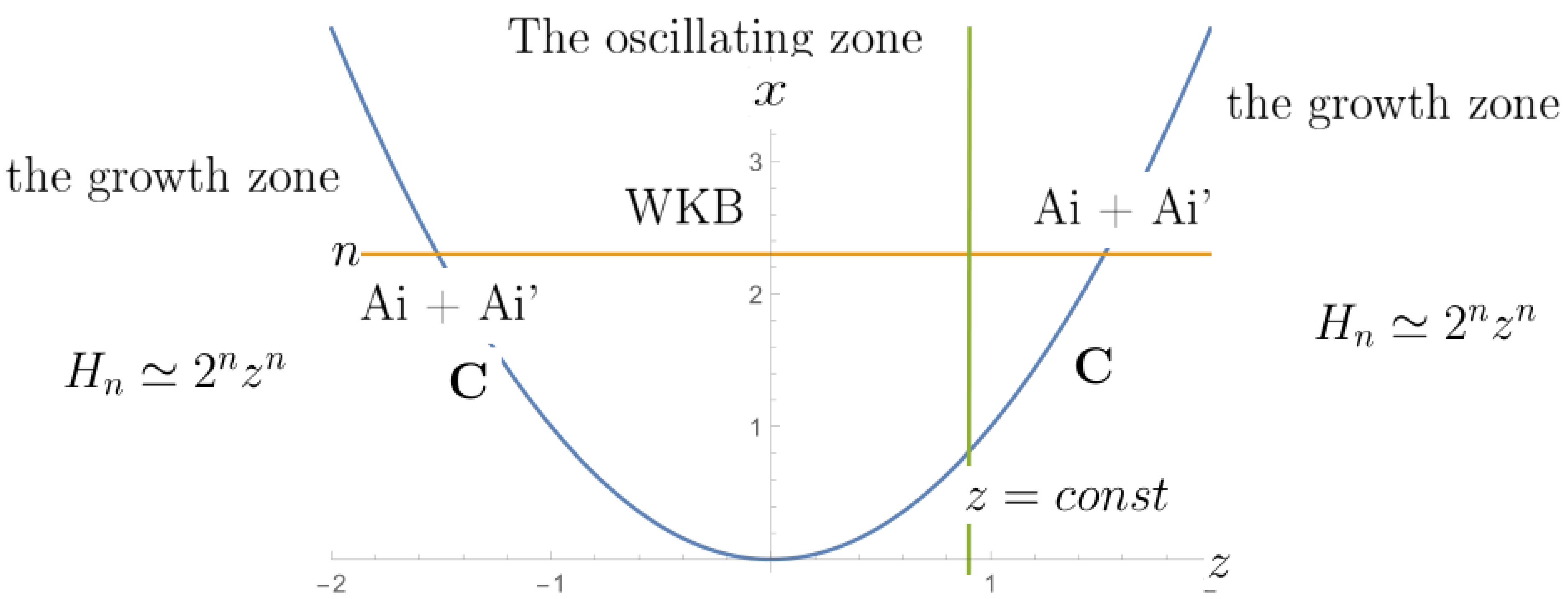
The structure of solutions for polynomials: then $V = V(x, z, h)$

The oscillating zone on the half plane $\{x > 0, z \in \mathbb{R}\colon |V(x, z, 0)| \leq 1,$

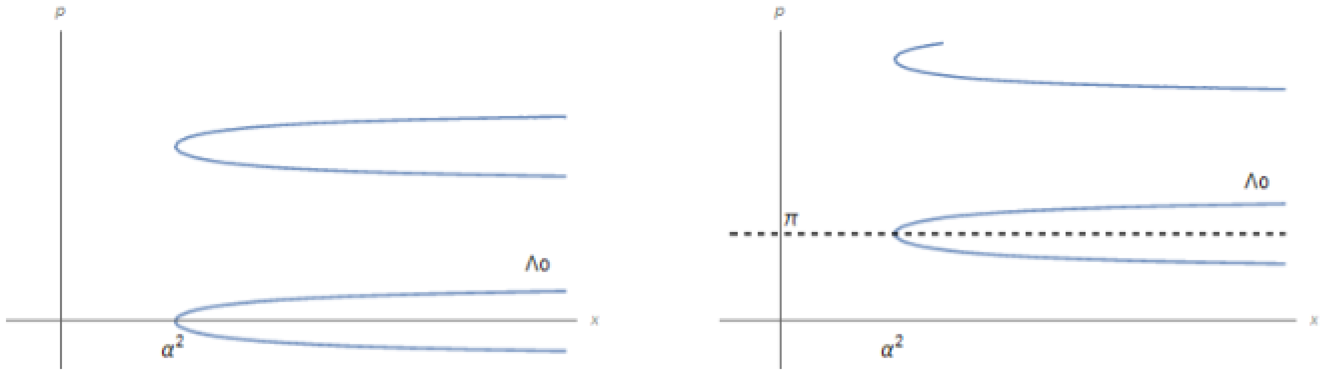
the boundary of the oscillating zone $\mathbf{C} = \{|V(x, z, 0)| = 1 \quad x = \mathcal{X}(z)\},$

$\mathbf{C}$ be a set of turning points (a caustic) if $\left.\frac{\partial V(x,z,0)}{\partial x}\right|_{\mathbf{C}} \neq 0$

The the example of Hermitian polynomials



Lagrangian manifolds=invariant sets: $\Lambda_k = \{\cos p + V(x,z,0) = 0,\ z = const\}$



$$V = -\frac{x}{\sqrt{y}},\ x = nh,\ y = \sqrt{h}z$$

The answer $g = q(z)K_{\Lambda_0}\mathcal{A}^0,\ z > 0,\quad g = q(z)K_{\Lambda_1}\mathcal{A}^1,\ z < 0,$
here K_{Λ_j} is the Maslov canonical operator,
analytical function $q(z)$ is the constant of integration

Boundary conditions: $H_n \simeq 2^n z^n$

+ **ANALYTICITY**

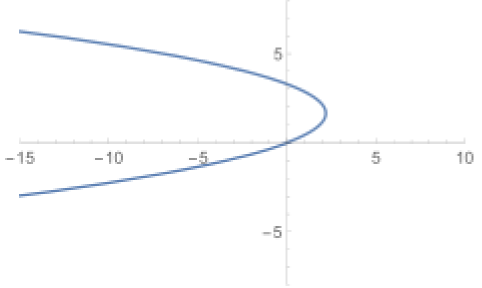
(compare with S.Manakov trick)

Using the Maslov canonical operator and new uniform formulas for its realisation in the wide neighborhood of simple caustics in the form of Airy functions (A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, Theoret. and Math. Phys., 2019) we obtain the generalised Plancherel-Rotach asymptotics (S. Yu. Dobrokhotov, A. V. Tsvetkova, Math. Notes, 2018)

x is real and not complex!

General approach:

The focal point and uniform asymptotics in the form of Airy function:
"naive" constructive approach



Let $x = X(\alpha^*)$ be nondegenerated focal point:

$$\frac{\partial X}{\partial \alpha}(\alpha^*) = 0, \quad \frac{\partial^2 X}{\partial \alpha^2}(\alpha^*) \neq 0.$$

$$\Lambda = \{p = P^* + P'^*(\alpha - \alpha^*) + O((\alpha-\alpha^*)^2), x = X^* + \frac{1}{2}X''^*(\alpha-\alpha^*)^2 + O((\alpha-\alpha^*)^3)\}$$

looks like a "horizontal" parabola.

Then for small $x - X^*$ one can show (at least on the physical level of rigor) that the asymptotic of the integral is expressed in terms of the **Airy function and its derivative**. This implies the following anzats:

$$\psi \approx e^{i\frac{Q(x)}{h}}\big(A_1(x,h)\mathrm{Ai}(\Phi(x,h)) + A_2(x,h)\mathrm{Ai}'(\Phi(x,h))\big),$$

here phases $Q(x)$, $\Phi(x,h)$ and amplitudes $A_j(x,h)$ are unknown functions,

Why Airy ′ ?

$$\alpha^* = 0 \qquad A(\alpha) = \frac{A(\alpha)+A(-\alpha)}{2} + \alpha\frac{A(\alpha)-A(-\alpha)}{2\alpha} = \underbrace{g_1(X)}_{\text{Airy}} + \underbrace{\alpha g_2(X)}_{\text{Airy}'}$$

We can write for $\Phi(x) \ll -1$

$$\psi \approx \frac{e^{i\frac{Q(x)}{h}}}{\sqrt{\pi}}\left(\frac{A_1(x,h)}{\sqrt[4]{z}}\sin(z+\frac{\pi}{4})+A_2(x,h)\sqrt[4]{z}\cos(z+\frac{\pi}{4})\right), \quad z=\frac{2}{3}(-\Phi)^{3/2}.$$

From the other side the Maslov canonical operator gives:

$$\psi \approx a_+(x)e^{-i\frac{\pi}{2}}e^{\frac{iS_+(x)}{h}} + a_-(x)e^{\frac{iS_-(x)}{h}} =$$

$$e^{-\frac{i\pi}{4}}e^{\frac{i}{h}(S_++S_-)}(a_++a_-)\cos(\frac{S_--S_+}{h}+\frac{\pi}{4})+i(a_--a_+)\sin(\frac{S_--S_+}{h}+\frac{\pi}{4}),$$

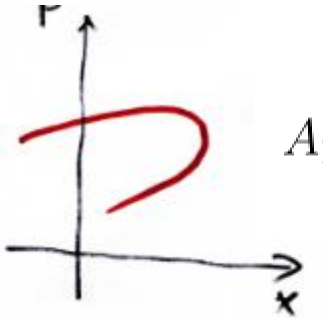
where

$$S_\pm = \int_{\alpha_0}^{\alpha_\pm(x)} P\,dX, \quad a_\pm = \frac{A(\alpha_\pm(x))}{\sqrt{|J(\alpha_\pm(x))}}$$

and $\alpha_\pm(x)$ are two solutions to the equation $X(\alpha)=x$.

This gives

$$Q=(S_-+S_+), \quad \Phi = -\frac{3(S_--S_+)^{2/3}}{2h^{2/3}},$$

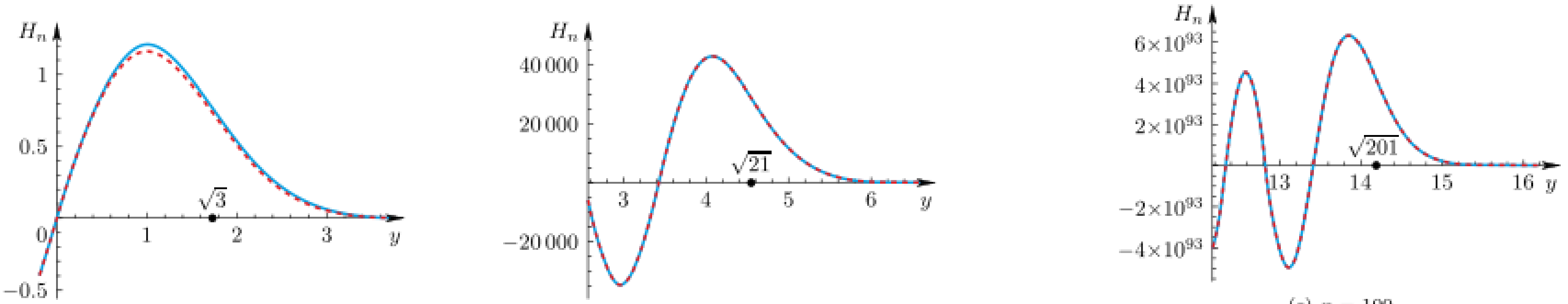


$$A_1 = \frac{ie^{-\frac{i\pi}{4}}}{\sqrt[6]{h}\sqrt{\pi}}(a_+-a_-)\sqrt[3]{S_+-S_-}, \quad A_2 = \frac{\sqrt[6]{h}e^{-\frac{i\pi}{4}}}{\sqrt{\pi}}\frac{(a_++a_-)}{\sqrt[3]{S_+-S_-}}$$

Uniform Plancherel-Rotach type formula for the Hermitian polinomials

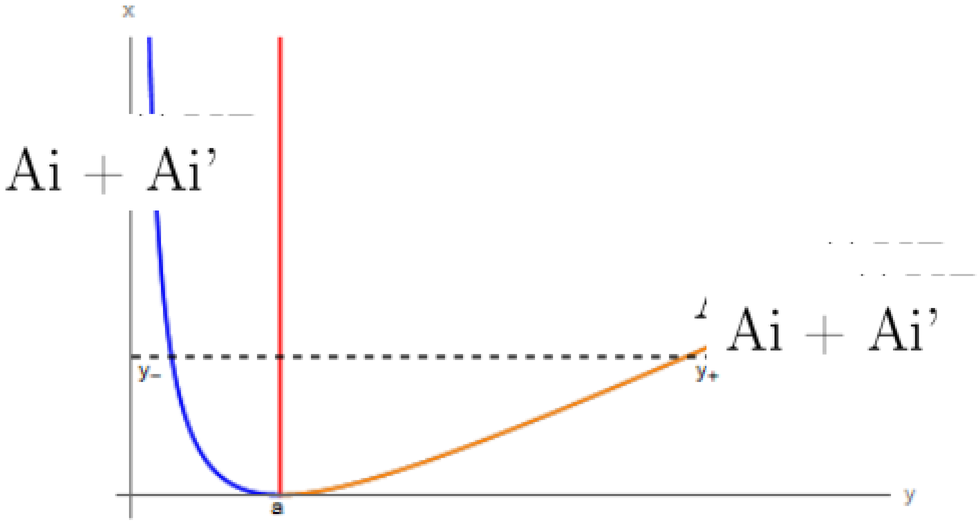
$$\mathrm{H_n(z)} \simeq \mathrm{sign(z^n)e^{\frac{z^2}{2}}}\sqrt{2\pi}\left(\frac{\mathrm{2n}}{\mathrm{e}}\right)^{\frac{\mathrm{n}}{2}}\left|1-\frac{\mathrm{z^2-1}}{\mathrm{2n}}\right|^{-\frac{1}{4}}\left|\mathrm{F_n(z)}\right|^{\frac{1}{4}}\mathrm{Ai}\left(\mathrm{F_n}\left(\mathrm{z}\right)\right)$$

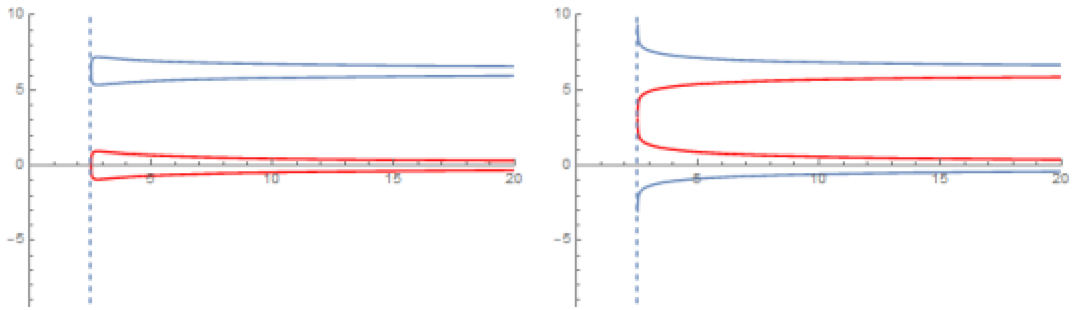
$$\mathrm{F_n(z)} = \begin{cases} -\left(\frac{3}{2}\left(\frac{\pi}{4}+\frac{\pi}{2}\mathrm{n}-\frac{|\mathrm{z}|}{2}\sqrt{\mathrm{2n+1-z^2}}-\frac{\mathrm{2n+1}}{2}\arcsin\frac{|\mathrm{z}|}{\sqrt{\mathrm{2n+1}}}\right)\right)^{\frac{2}{3}} \\ \mathrm{z^2} \leq \mathrm{2n+1}, \\ \left(\frac{3}{2}\left(\frac{|\mathrm{z}|}{2}\sqrt{\mathrm{z^2-2n-1}}-\frac{\mathrm{2n+1}}{2}\ln\left(\frac{|\mathrm{z}|+\sqrt{\mathrm{z^2-2n-1}}}{\sqrt{\mathrm{2n+1}}}\right)\right)\right)^{\frac{2}{3}}, \\ \mathrm{z^2} > \mathrm{2n+1}. \end{cases}$$

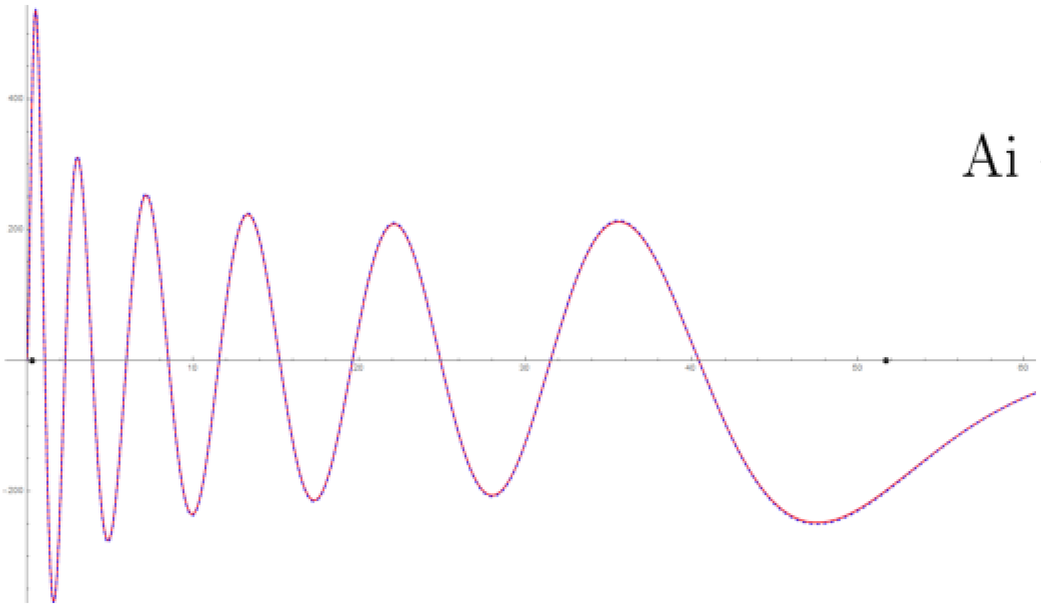


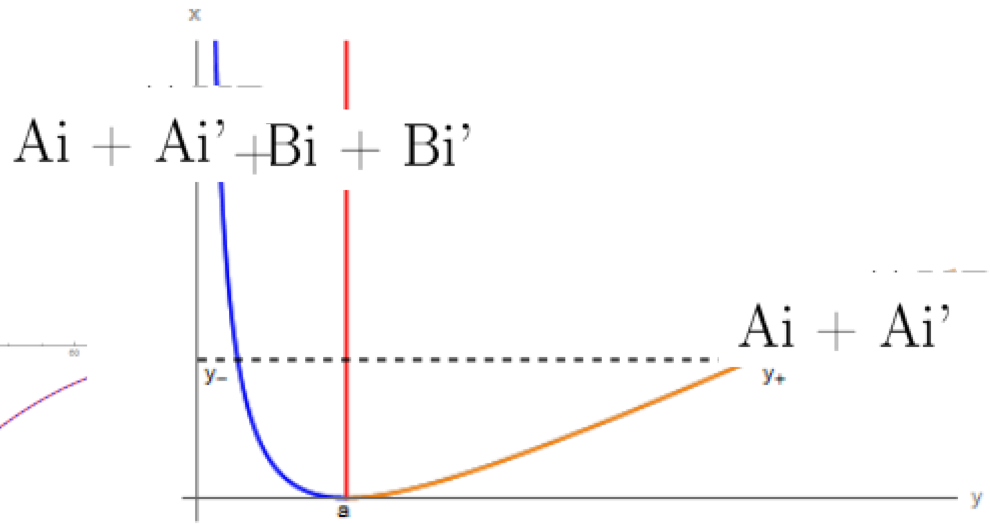
(a) $n = 1$ (b) $n = 10$ (c) $n = 100$

Generalized Laguerre polynomials

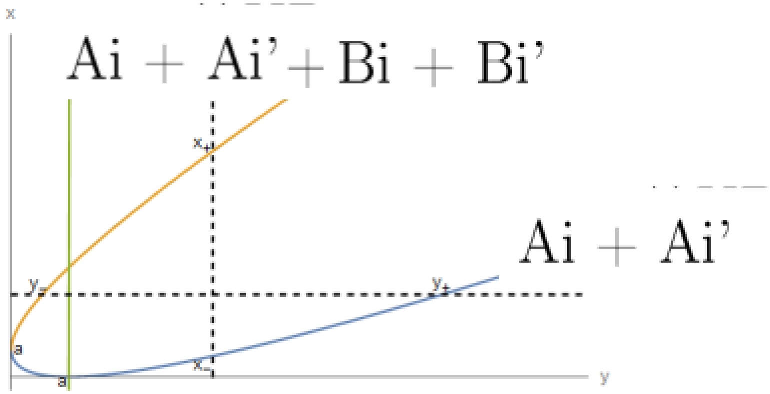


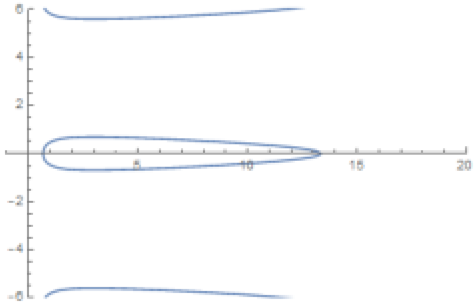




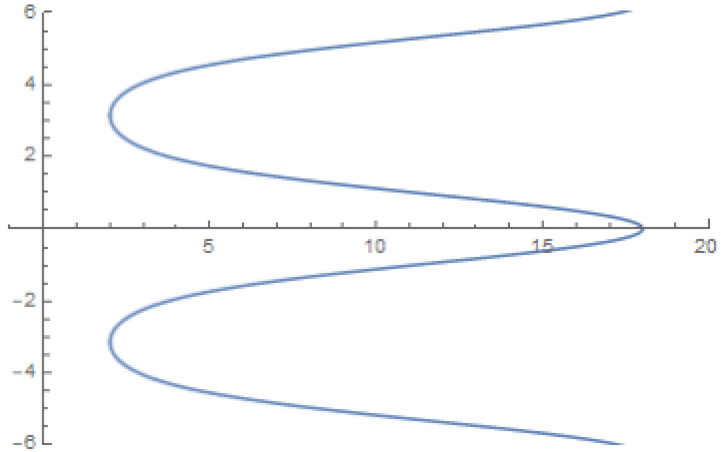


The Charlier Polynomials

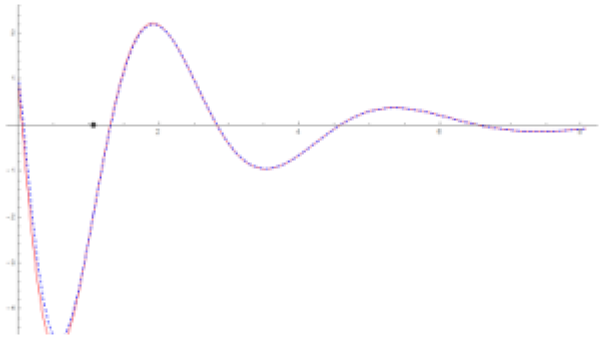


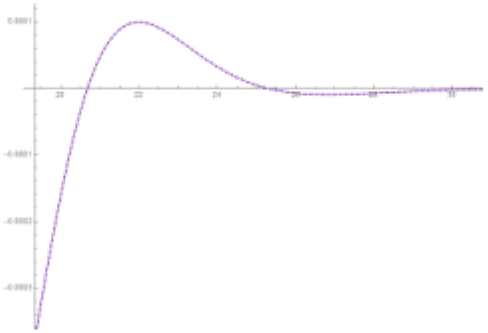


при $0 < y < a$



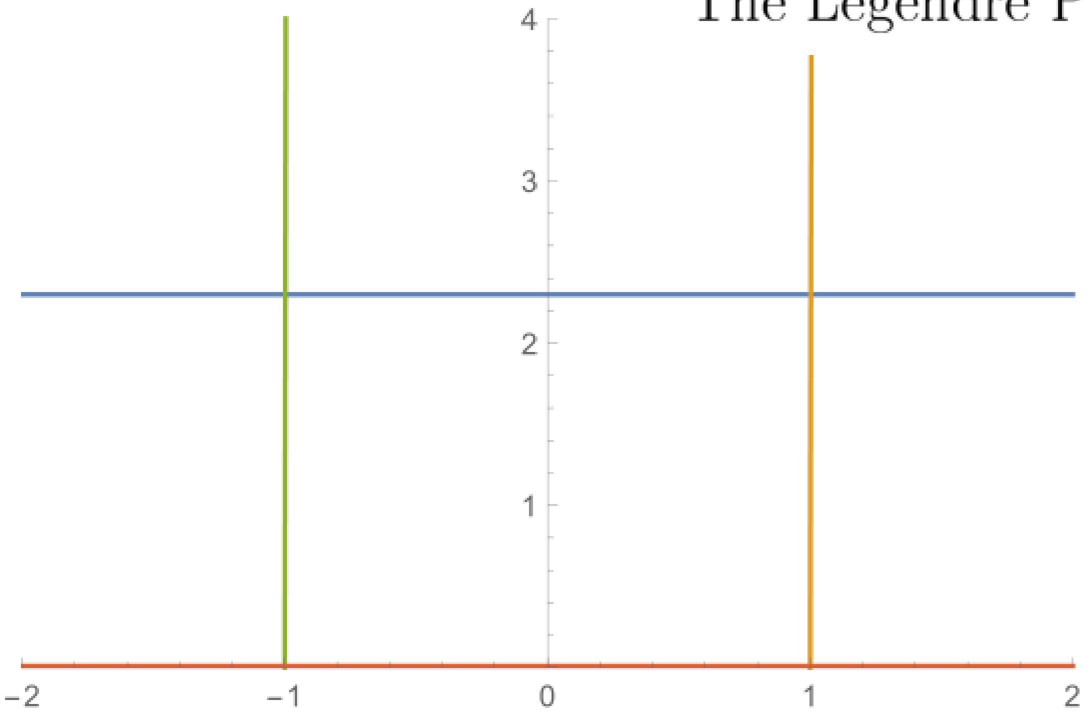
No the discrete Schrödinger operator,
no the Maslov canonical operator (or modified Maslov canonical operator)





The Chebyshev Polynomials

The Legendre Polynomials



Lagrangian manifolds=invariant sets:

$$\Lambda_k = \{\cos p - z) = 0, \; z = const\}$$

Trigonometric functions

The Bessel functions J_0, J_1

Recurrent equations for $H_{n,n}(z,\alpha), H_{n,n-1}(z,\alpha)$

$$\begin{pmatrix} \mathrm{H}_{n,n} \\ \mathrm{H}_{n+1,n} \\ \mathrm{H}_{n+1,n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha n & -n & z+\alpha \\ \alpha n(z-\alpha) & -nz & z^2-\alpha^2-\frac{1}{2}(2n+1) \end{pmatrix} \begin{pmatrix} \mathrm{H}_{n-1,n-1} \\ \mathrm{H}_{n,n-1} \\ \mathrm{H}_{n,n} \end{pmatrix}$$

Initial data

$$\mathrm{H}_{0,0}(z,\alpha)=1, \quad \mathrm{H}_{1,0}(z,\alpha)=z+\alpha, \quad \mathrm{H}_{1,1}(z,\alpha)=z^2-\alpha^2-1/2$$

Introduce small artificial parameter $h=\dfrac{1}{n}$,

thus we are looking for asymptotics as $h \to +0$

Diagonal polynomials and pseudodifferential equations

$$\psi(\mathrm{nh}; \mathrm{z}, \alpha) = \mathrm{h}^{\mathrm{n}} \mathrm{H}_{\mathrm{n},\mathrm{n}}(\mathrm{z}, \alpha), \qquad \theta(\mathrm{nh}; \mathrm{z}, \alpha) = \mathrm{h}^{\mathrm{n}} \mathrm{z} \mathrm{H}_{\mathrm{n},\mathrm{n}-1}(\mathrm{z}, \alpha)$$

The system

$$e^{i\hat{p}}\psi(x) = a(y-a)xe^{-i\hat{p}}\psi(x) - x\theta(x) + \left(y^2 - a^2 - x - \frac{h}{2}\right)\psi(x)$$

$$e^{i\hat{p}}\theta(x) = axye^{-i\hat{p}}\psi(x) - x\theta(x) + (y+a)y\psi(x),$$

$$y = z\sqrt{h},\ a = \alpha\sqrt{h},\ \hat{p} = -ih\frac{\partial}{\partial x}$$

The scalar (formal) equation for $\psi(x) := \psi(x, y, a)$

$(y, a$ are parameters)

$$\hat{H}(x, \hat{p}; h)\psi(x) =$$

$$\left[e^{i\hat{p}} + a^2xe^{-i\hat{p}} + xy^2(e^{i\hat{p}} + x)^{-1} - \left(y^2 - a^2 - x - \frac{h}{2}\right)\right]\psi(x) = 0$$

The boundary condition (large z, y)

$$\mathrm{H_{n,n}(z,\alpha) = z^{2n} - n\left(\alpha^2 + n - \frac{1}{2}\right) z^{2n-2} + O(z^{2n-4})}$$

$$\psi(\mathrm{x;y,a}) \approx \mathrm{h}^{\frac{\mathrm{x}}{\mathrm{h}}} \left(\frac{\mathrm{a}}{\sqrt{\mathrm{h}}}\right)^{2\frac{\mathrm{x}}{\mathrm{h}}} \left(\mathrm{q}^{\frac{2\mathrm{x}}{\mathrm{h}}} - \frac{\mathrm{x}}{\mathrm{h}}\left(1 + \frac{\mathrm{x}}{\mathrm{a}^2} - \frac{\mathrm{h}}{2\mathrm{a}^2}\right)\mathrm{q}^{\frac{2\mathrm{x}}{\mathrm{h}}-2}\right), \quad \mathrm{q} = \frac{\mathrm{y}}{\mathrm{a}} = \frac{\mathrm{z}}{\alpha}$$

+ ANALYTICITY

Ordered form

$$\hat{\mathcal{H}}\psi = \mathcal{H}(\overset{2}{\mathrm{x}}, \overset{1}{\hat{\mathrm{p}}}, \mathrm{y}, \mathrm{a}, \mathrm{h})\psi = 0$$

$$\mathcal{H}(\mathrm{x}, \mathrm{p}; \mathrm{h}) = \mathcal{H}_0(\mathrm{x}, \mathrm{p}; \mathrm{h}) + \mathrm{h}\mathcal{H}_1(\mathrm{x}, \mathrm{p}; \mathrm{h}) + \mathrm{O}(\mathrm{h}^2) = \mathrm{e}^{2\mathrm{ip}} + \mathrm{e}^{\mathrm{ip}}(\mathrm{a}^2 - \mathrm{y}^2 + 2\mathrm{x}) + (2\mathrm{a}^2\mathrm{x} + \mathrm{x}^2) + \mathrm{a}^2\mathrm{x}^2\mathrm{e}^{-\mathrm{ip}} + \mathrm{h}\left[\frac{\mathrm{e}^{\mathrm{ip}} + \mathrm{x}}{2} + \mathrm{e}^{\mathrm{ip}} + \mathrm{a}^2 + \frac{\mathrm{y}^2\mathrm{e}^{\mathrm{ip}}}{\mathrm{e}^{\mathrm{ip}} + \mathrm{x}}\right] + \mathrm{O}(\mathrm{h}^2).$$

Splitting a pseudo-differential equation for ψ

WKB-solutions $\quad \eta = \sum_{k} e^{\frac{iS_k(x,y,a)}{h}} A_k(x,y,a), \qquad S_k = \mathcal{S}_k + is_k$

The Hamilton-Jacobi equations with complex-valued Hamiltonian

$$\mathcal{H}_0\left(x, \frac{\partial S_k}{\partial x}\right) = 0, \quad \mathcal{H}_0(x,p) = e^{-ip}\mathcal{R}(x, e^{ip})$$

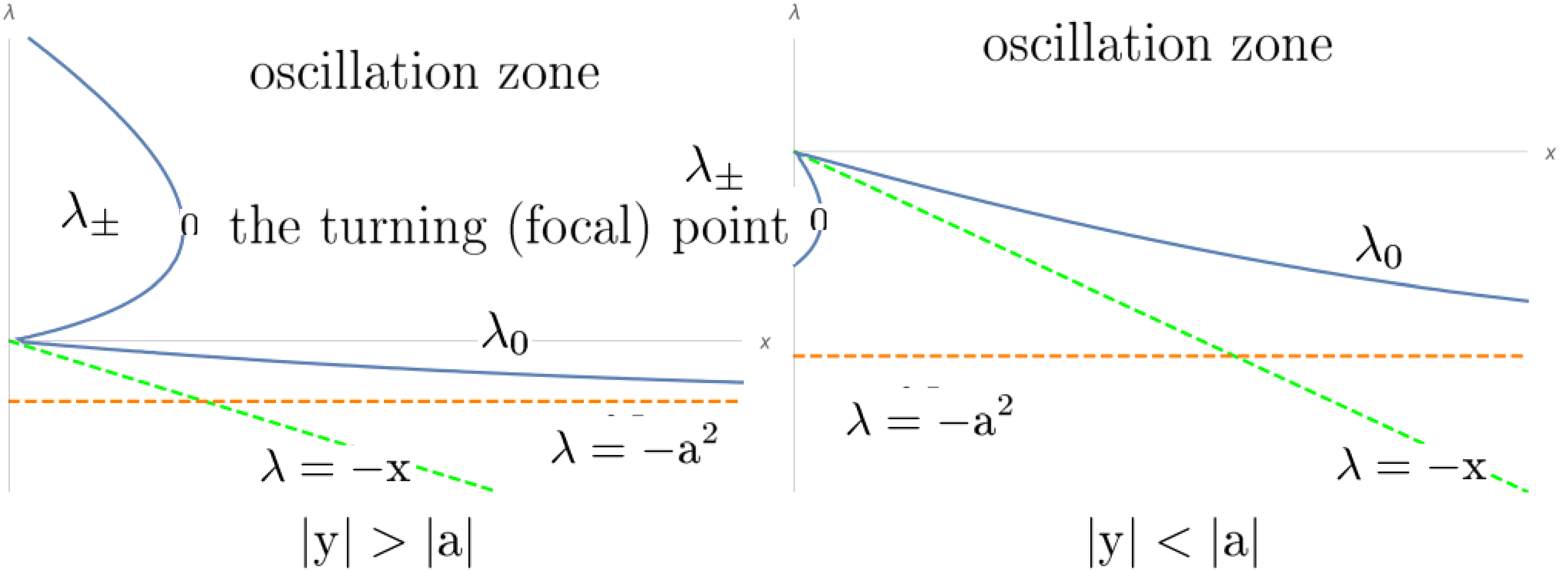
The characteristic equation (***third***-order algebraic curve)

$$\mathcal{R}(x,\lambda,y,a) = \lambda^3 + \lambda^2(a^2 - y^2 + 2x) + \lambda(2a^2x + x^2) + a^2x^2 \equiv$$
$$(\lambda + x)^2(\lambda + a^2) - \lambda^2 y^2.$$

The phases and the structure of the roots

$$\lambda_0, \lambda_\pm \quad : \qquad \mathcal{R}(x, \lambda) = \big(\lambda - \lambda_0(x)\big)\big(\lambda^2 - A(x)\lambda + B(x)\big) = 0,$$

$$S_{0,\pm} = -i \int \log \lambda_{0,\pm}(x, y, a) dx$$



The stucture of the “fundamental” solution

The domain where $\lambda_0, \lambda_\pm$ are real is the “decreasing” zone of the solution (with a suitable weight multiplier)

The domain where $\lambda_\pm$ are complex is the is the zone of solution oscillations

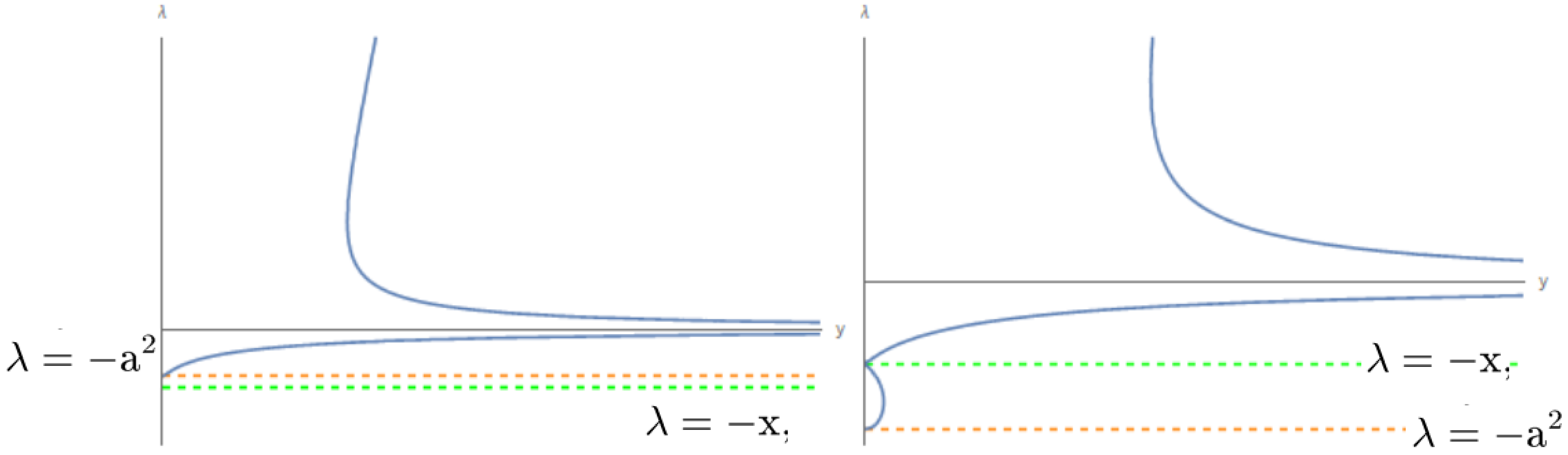
Appropriate representation outside of turning (focal) points:

$$\psi = A_0(x, y, a) e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x,y,a)}{h}} + A_1(x, y, a) e^{\frac{\Phi_1(x,y,a)}{h}} \left(A_2(x, y, a) \sin \left(\frac{\Phi_2(x, y, a)}{h} + g(x, y, a) \right) \right)$$

Focal (turning) points (transition $\lambda_\pm$ from complex to real)

$$4\tilde{x}^3 - (12 + q^2)\tilde{x}^2 + (20q^2 + 12)\tilde{x} - 4(1 - q^2)^2 = 0, \quad \tilde{x} = x/a^2, q = y/a.$$

Roots on the plane (y, λ)



$x > a^2$ $\quad$ $x < a^2$

Factorization of the characteristic equation and the parametrisation $\lambda_\pm$ via λ_0

$$\mathcal{R}(x, \lambda) = (\lambda - \lambda_0(x))(\lambda^2 - A(x)\lambda + B(x))$$

here

$$A(x; y, a) \equiv A(x) = \lambda_+ + \lambda_- = y^2 - a^2 - 2x - \lambda_0,$$

$$B(x; y, a) \equiv B(x) = \lambda_+\lambda_- = -\frac{a^2x^2}{\lambda_0} \equiv$$

$$x(2a^2 + x) + (a^2 + 2x - y^2)\lambda_0 + \lambda_0^2 > 0$$

$$\lambda_+ = \frac{A + \sqrt{A^2 - 4B}}{2}, \quad \lambda_- = \frac{A - \sqrt{A^2 - 4B}}{2}.$$

$\lambda_\pm$ are real if $A^2 - 4B \geq 0, \quad \lambda_- \leq \lambda_+,$ and are complex if $A^2 - 4B < 0$

Real-valued Hamiltonians

We put: $S_0 = -i\Phi_0 + \pi x, \qquad S_\pm = -i\Phi_1 \pm \Phi_2,$

$$e^{i\frac{\partial S_0}{\partial x}} - e^{i\pi}(-\lambda_0) = 0 \quad \Leftrightarrow \quad e^{\frac{\partial \Phi_0}{\partial x}} + \lambda_0 = 0$$

$$e^{\frac{\partial \Phi_1}{\partial x}} e^{\pm i\frac{\partial \Phi_2}{\partial x}} - A(x) + B(x)e^{-\frac{\partial \Phi_1}{\partial x}} e^{\mp i\frac{\partial \Phi_2}{\partial x}} = 0 \Rightarrow$$

$$e^{\frac{\partial \Phi_1}{\partial x}} - \sqrt{B} = 0,$$

$$2\sqrt{B(x)}\cos\left(\frac{\partial \Phi_2}{\partial x}\right) - A(x) = 0 \quad \Longleftrightarrow \quad \cos\left(\frac{\partial \Phi_2}{\partial x}\right) - \frac{A(x)}{2\sqrt{B(x)}} = 0$$

Focal (turning) points $\quad A^2 = 4B$

Three real-valude Hamiltonians

$$H_0(x, \xi) = e^{\xi} + \lambda_0$$

Asymptotics with purely imaginary phases

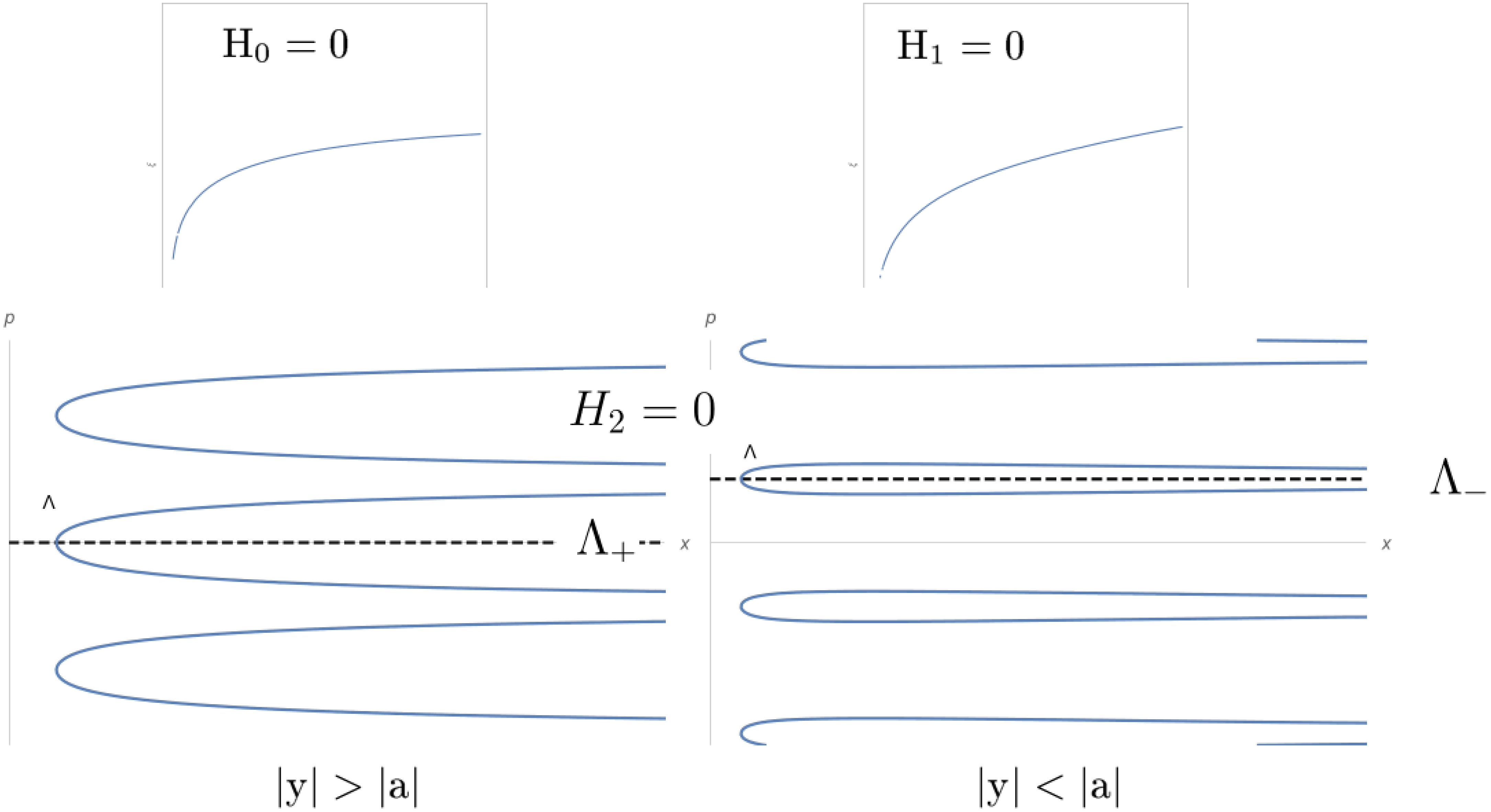
$$H_1 = e^{\xi} - \sqrt{B}$$

oscillating asymptotics

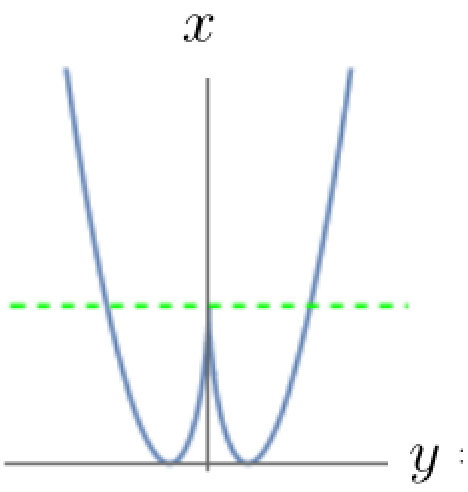
$$H_2 = \cos p - \frac{A(x)}{2\sqrt{B(x)}}$$

quntization $\xi \to \hat{\xi} = h\frac{\partial}{\partial x}, \quad p \to \hat{p} = -ih\frac{\partial}{\partial x} \quad + \textbf{corrections}$

Lagrangian manifolds (level- cut lines of the Hamiltonians)



The set of turning points on the half-plane $x > 0, y \in \mathbb{R}$



$$\frac{A(x,y)}{2\sqrt{B(x,y)}} = \pm 1$$

Splitting of the solution and 3 reduced pseudodifferential equations

ansatz:

$$\psi = \mathrm{C}_0(\mathrm{y},\mathrm{a})\Psi_0 + \mathrm{C}_1(\mathrm{y},\mathrm{a})\Psi(\mathrm{x},\mathrm{y},\mathrm{a}),$$

$$\Psi_0 = \mathrm{e}^{\frac{\mathrm{i}\pi \mathrm{x}}{\mathrm{h}}}\psi_0(\mathrm{x},\mathrm{y},\mathrm{a}), \quad \Psi = \psi_1(\mathrm{x},\mathrm{y},\mathrm{a})\psi_2(\mathrm{x},\mathrm{y},\mathrm{a}),$$

The aim: final equations

$$\big(\mathrm{e}^{\hat{\xi}} + \lambda_0(\mathrm{x}) + \mathrm{h}\mathrm{V}_0(\mathrm{x}) + \mathrm{O}(\mathrm{h}^2)\big)\psi_0 = 0,$$

$$\big(\mathrm{e}^{\hat{\xi}} - \sqrt{\mathrm{B}(\mathrm{x})} + \mathrm{h}\mathrm{V}_1(\mathrm{x}) + \mathrm{O}(\mathrm{h}^2)\big)\psi_1 = 0,$$

$$\Big(\cos\hat{\mathrm{p}} - \frac{\mathrm{A}(\mathrm{x})}{2\sqrt{\mathrm{B}(\mathrm{x})}} + \mathrm{h}\mathrm{V}_2(\mathrm{x}) + \mathrm{O}(\mathrm{h}^2)\Big)\psi_2 = 0.$$

$$\hat{\xi} = \mathrm{h}\frac{\partial}{\partial \mathrm{x}}$$

$$\hat{\mathrm{p}} = -\mathrm{i}\mathrm{h}\frac{\partial}{\partial \mathrm{x}}$$

requirement: V_0, V_1, V_2 are smooth real-valued

Splitting of the solution and 3 reduced pseudodifferential equations

$$\hat{\mathcal{H}}\psi = \mathcal{H}(\overset{2}{x}, \overset{1}{\hat{p}}, y, a, h)\psi = 0.$$

$$\mathcal{H}(x, p; h) = \mathcal{H}_0(x, p; h) + h\mathcal{H}_1(x, p; h) + O(h^2) = e^{2ip} + e^{ip}(a^2 - y^2 + 2x) +$$
$$(2a^2 x + x^2) + a^2 x^2 e^{-ip} + h\left[\frac{e^{ip} + x}{2} + e^{ip} + a^2 + \frac{y^2 e^{ip}}{e^{ip} + x}\right] + O(h^2).$$

We act on the original equation with the operators (very useful technical trick)

$$(e^{i\hat{p}} - \lambda_0)^{-1} \qquad \text{and} \qquad (e^{i\hat{p}} - A(x) + B(x)e^{-i\hat{p}})^{-1}$$

+ Elementary tricks from the Feynman-Maslov calculus

this gives

$$\widehat{\mathcal{H}^0}\Psi_0 := [e^{i\hat{p}} - \lambda_0(x) + h\mathcal{L}_0(\overset{2}{x}, e^{i\overset{1}{\hat{p}}}) + O(h^2)]\Psi_0 = 0$$

$$\mathcal{L}_0(x,\lambda) = \Big[\frac{3\lambda + x + 2a^2}{2(\lambda - A + B\lambda^{-1})} + \frac{y^2\lambda}{(\lambda + x)(\lambda - A + B\lambda^{-1})} - \frac{2(1 - B\lambda^{-2})(\lambda + x)(\lambda + a^2)}{(\lambda - A + B\lambda^{-1})^2}\Big].$$

and

$$\widehat{\mathcal{H}^1}\Psi := [e^{i\hat{p}} - A(x) + B(x)e^{-i\hat{p}} + h\mathcal{L}_1(\overset{2}{x}, e^{i\overset{1}{\hat{p}}}) + O(h^2)]\Psi = 0,$$

$$\mathcal{L}_1 = \frac{3\lambda + x + 2a^2}{2(\lambda - \lambda_0)} + \frac{y^2\lambda}{(\lambda + x)(\lambda - \lambda_0)} - \frac{2(\lambda + x)(\lambda + a^2)}{(\lambda - \lambda_0)^2}$$

The construction of Ψ_0

$$\Psi_0 = e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}} A_0(x)$$

$$e^{\pm i\hat{p}} f(x) = f(x \pm h) = f(x) \pm h \frac{\partial f}{\partial x}(x) + \frac{h^2}{2} \frac{\partial^2 f}{\partial x^2}(x) + O(h^2)$$

$$e^{\pm i\hat{p}}(f_1(x) f_2(x)) = f_1(x \pm h) f_2(x \pm h).$$

and

$$\left[e^{i\hat{p}} - \lambda_0(x) + h\mathcal{L}_0(\overset{2}{x}, e^{i\overset{1}{\hat{p}}})\right] \left(e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}} A_0(x)\right) =$$

$$-e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x+h)}{h}} A_0(x+h) + \left[-\lambda_0(x) + h\mathcal{L}_0(\overset{2}{x}, e^{i\overset{1}{\hat{p}}})\right] \left(e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}} A_0(x)\right) = -e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}}$$

$$\cdot \left[e^{\frac{\partial \Phi_0}{\partial x}} \left(1 + \frac{h}{2} \frac{\partial^2 \Phi_0}{\partial x^2}(x)\right) \left(A_0 + h \frac{\partial A_0}{\partial x}\right) + \lambda_0 A_0 - h\mathcal{L}_0(x, \lambda_0(x)) A_0 + O(h^2)\right] = 0.$$

this gives

$$\Psi_0 = \frac{e^{\frac{i\pi x}{h}} e^{\frac{\Phi_0(x)}{h}}}{\sqrt{-\lambda_0(x)}} \exp\left(-\int_{x_0}^{x} \frac{V_0(\xi)}{\lambda_0(\xi)} d\xi\right), \quad \Phi_0(x) = \int_{x_0}^{x} \log(-\lambda_0(\xi)) d\xi,$$

here x_0 is the integration constant

The construction of Ψ (an important ideological fact)

$$\Psi(x) = \psi_1(x)\psi_2(x),$$

where

$$\psi_1 = e^{\frac{\Phi_1(x)}{h}} A_1(x), \quad e^{\frac{\partial \Phi_1}{\partial x}} = \sqrt{B(x)} \quad \Longleftrightarrow \quad \Phi_1 = \frac{1}{2}\int_{x_1}^{x} \log(B(x))dx,$$

$A_1(x)$ is still unknown , x_1 is a constant

We seek the function ψ_2 in the form the Maslov canonical operator acting to the unknown function A_2 on $\Lambda_\pm$

$$\psi_2 = K_{\Lambda_\pm} A_2.$$

Substitute function $\psi_1\psi_2$ into equation $\widehat{\mathcal{H}^1}\Psi = 0$

$$(e^{i\hat{p}} - A + Be^{-i\hat{p}})(\psi_1\psi_2) = e^{\frac{\Phi_1(x)}{h}}\left[A_1\left(e^{\frac{\partial\Phi_1}{\partial x}}e^{i\hat{p}} - A + Be^{-\frac{\partial\Phi_1}{\partial x}}e^{-i\hat{p}}\right) + \right.$$

$$\left. h\Big(e^{\frac{\partial\Phi_1}{\partial x}}\left(\frac{\partial A_1}{\partial x} + \frac{1}{2}\frac{\partial^2\Phi_1}{\partial x^2}A_1\right)e^{i\hat{p}} + e^{-\frac{\partial\Phi_1}{\partial x}}B\left(-\frac{\partial A_1}{\partial x} + \frac{1}{2}\frac{\partial^2\Phi_1}{\partial x^2}A_1\right)e^{-i\hat{p}}\right]\psi_2 =$$

$$e^{\frac{\Phi_1(x)}{h}}\left[A_1(2\sqrt{B}\cos\hat{p} - A) + h\Big(A_1\frac{\partial\sqrt{B}}{\partial x}\cos\hat{p} + 2i\frac{\partial A_1}{\partial x}\sqrt{B}\sin\hat{p}\Big) + O(h^2)\right]\psi_2.$$

Take into account the equality

$$e^{\frac{\Phi_1(x)}{h}}\Big(A_1\mathcal{L}_1(x, e^{\frac{\partial\Phi_1}{\partial x}}e^{i\hat{p}}) + O(h)\Big)\psi_2 =$$

$$e^{\frac{\Phi_1(x)}{h}}A_1\left[\mathcal{L}_1^{Re}(x,\cos\hat{p}) + i\sin\hat{p}\,\mathcal{L}_1^{Im}(x,\cos\hat{p}) + O(h)\right]\psi_2 = 0$$

this gives

$$e^{\frac{\Phi_1(x)}{h}}\left[A_1\left(2\sqrt{B}\cos\hat{p}-A\right)+h\left(A_1\frac{\partial\sqrt{B}}{\partial x}\cos\hat{p}\ +2i\frac{\partial A_1}{\partial x}\sqrt{B}\sin\hat{p}+\right.\right.$$

$$\left.\left.A_1\mathcal{L}_1^{\mathrm{Re}}\left(x,\frac{A}{2\sqrt{B}}\right)+iA_1\mathcal{L}_1^{\mathrm{Im}}\left(x,\frac{A}{2\sqrt{B}}\right)\sin\hat{p}\right)+O(h^2)\right]\psi_2=0,$$

$$2\frac{\partial A_1}{\partial x}\sqrt{B}+A_1\mathcal{L}_1^{\mathrm{Im}}\left(x,\frac{A}{2\sqrt{B}}\right)=0$$

Imaginary part $=0 \quad \Longrightarrow$

$$A_1(x;y,a)=\exp\left[-\int_{x_1}^{x}\frac{1}{2\sqrt{B(\xi;y,a)}}\mathcal{L}_1^{\mathrm{Im}}\left(\xi,\frac{A}{2\sqrt{B}}(\xi;y,a)\right)d\xi\right]$$

$$\text{Real part} = 0 \quad \Longrightarrow$$

$$\left(\cos\hat{p} - \frac{A(x)}{2\sqrt{B(x)}} + hV_2(x) + O(h^2)\right)\psi_2 = 0,$$

$$V_2(x; y, a) = \frac{A}{8B\sqrt{B}}\frac{\partial B}{\partial x}(x; y, a) + \frac{1}{2\sqrt{B(x; y, a)}}\mathcal{L}_1^{\mathrm{Re}}\left(x, \frac{A}{2\sqrt{B}}(x; y, a)\right).$$

and

$$\psi_2 = \sqrt{\pi}\left(\frac{v_1}{\sqrt[6]{h}}\,\mathrm{Ai}\left(\mathrm{sign}(x_\pm^* - x)\left(\frac{3\Phi_2^\pm}{2h}\right)^{2/3}\right) + \sqrt[6]{h}\,v_2\mathrm{Ai}'\left(\mathrm{sign}(x_\pm^* - x)\left(\frac{3\Phi_2^\pm}{2h}\right)^{2/3}\right)\right).$$

(A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, Uniform asymptotic solution in the form of an Airy function for semiclassical bound states in one-dimensional and radially symmetric problems, Theoret. and Math. Phys., 201:3 (2019), 1742-1770)

Phases and amplitudes

$$J(x;y,a) = \sqrt{\left|1 - \frac{A^2(x;y,a)}{4B(x;y,a)}\right|}, \quad g_\pm(x;y,a) = \pm \int_{x_\pm^*}^{x} \frac{V_2(x;y,a)}{\sqrt{\left|1 - \frac{A^2(x;y,a)}{4B(x;y,a)}\right|}} dx.$$

for $\quad x > x_\pm^*$

$$\Phi_2^\pm(x;y,a) = \int_{x_\pm^*}^{x} \arccos\left(\pm \frac{A(x;y,a)}{2\sqrt{B(x;y,a)}}\right) dx,$$

$$v_1(x;y,a) = \frac{\cos g^\pm}{\sqrt{J}} \left(\frac{3\Phi_2^\pm}{2}\right)^{1/6}, \quad v_2(x;y,a) = -\frac{\sin g^\pm}{\sqrt{J}} \left(\frac{3\Phi_2^\pm}{2}\right)^{-1/6}$$

for $\quad 0 < x < x_\pm^*$

$$\Phi_2^\pm(x;y,a) = \int_{x_\pm^*}^{x} \log\left(\pm \frac{A(x)}{2\sqrt{B(x)}} + \frac{1}{2\sqrt{B(x)}}\sqrt{A^2(x) - 4B(x)}\right) dx,$$

$$v_1(x;y,a) = \frac{\cosh g^\pm}{\sqrt{J}} \left(\frac{3\widetilde{\Phi}_2^\pm}{2}\right)^{1/6}, \quad v_2(x;y,a) = \frac{\sinh g^\pm}{\sqrt{J}} \left(\frac{3\widetilde{\Phi}_2^\pm}{2}\right)^{-1/6}$$

Finding constants of integration

$$\psi = \mathrm{C}_0(\mathrm{y},\mathrm{a})\Psi_0 + \mathrm{C}_1(\mathrm{y},\mathrm{a})\Psi(\mathrm{x},\mathrm{y},\mathrm{a}),$$

$$\Psi_0 = \mathrm{e}^{\frac{\mathrm{i}\pi\mathrm{x}}{\mathrm{h}}}\psi_0(\mathrm{x},\mathrm{y},\mathrm{a}), \quad \Psi = \psi_1(\mathrm{x},\mathrm{y},\mathrm{a})\psi_2(\mathrm{x},\mathrm{y},\mathrm{a}),$$

\+ **ANALYTICITY**

Parametrization (how one can use obtained asymptotic formulas)

Parametrization via root λ_0

$$\mathrm{x} = -\lambda_0\left(1 + \frac{|\mathrm{y}|}{\sqrt{\mathrm{a}^2+\lambda_0}}\right)$$

Parametrization via artificial parameter μ

$$\mathrm{a}^2 + \lambda_0 = \mathrm{a}^2\mu^2.$$

$$\lambda_0 = \mathrm{a}^2(\mu^2 - 1), \quad \mathrm{x} = \mathrm{a}^2(1-\mu^2)\left(1 + \frac{|\mathrm{q}|}{\mu}\right), \quad \mathrm{q} = \frac{\mathrm{y}}{\mathrm{a}} \equiv \frac{\mathrm{z}}{\alpha}$$

Comparison for large $q = \frac{z}{\alpha}$ coefficients for the highest degree $\mathrm{h}^{\frac{x}{h}} \left(\frac{a}{\sqrt{h}}\right)^{\frac{2x}{h}} \mathrm{q}^{\frac{2x}{h}}$

of the polynomial $H_{n,n}$ and the function $\psi = \mathrm{C}_0(\mathrm{y}, \mathrm{a})\Psi_0 + \mathrm{C}_1(\mathrm{y}, \mathrm{a})\Psi(\mathrm{x}, \mathrm{y}, \mathrm{a})$,

We have $$\mu(\mathrm{x}, \mathrm{q}) \sim 1 - \frac{\mathrm{x}}{2\mathrm{a}^2\mathrm{q}} + \frac{4\mathrm{a}^2\mathrm{x} + \mathrm{x}^2}{8\mathrm{a}^4|\mathrm{q}|^2} - \frac{\mathrm{a}^2\mathrm{x} + \mathrm{x}^2}{2\mathrm{a}^4|\mathrm{q}|^3}, \quad \mathrm{q} \to \infty,$$

$$\mu^*(\mathrm{q}) \sim \frac{4}{|\mathrm{q}|} - \frac{32}{|\mathrm{q}|^3}.$$

and $\mathrm{q} \to \infty$ + **ANALYTICITY** similar to the S.Manakov trick

$$e^{\frac{\Phi_0(\mu)}{h}} \sim e^{\frac{x}{h}\log\left(\frac{x}{q}\right)} e^{-x-\frac{x}{q}} = h^{\frac{x}{h}}(q)^{-\frac{x}{h}} e^{\frac{x}{h}\left(\log\frac{x}{h} - 1 - \frac{1}{q}\right)} \quad \Longrightarrow \quad C_0 \approx 0$$

$$\Psi_1 \sim h^{\frac{x}{h}} \left(\frac{a|q|}{\sqrt{h}}\right)^{\frac{2x}{h}} \exp\left[-\frac{a^2}{2h}(|q| - 1)^2 + \frac{1}{4}\log\frac{1}{2(1 + |q|)}\right]$$

and $$\mathrm{C}_1(\mathrm{a}, \mathrm{q}) = \exp\left[\frac{a^2}{2h}(|q| - 1)^2 - \frac{1}{4}\log\frac{1}{2(1 + |q|)}\right]$$

Parametric form

При $n \to \infty$

$$H_{n,n}(z,\alpha) \approx (\mathrm{sign}(z^2-\alpha^2))^n \sqrt{2\pi} e^{\frac{z^2}{2}} \left(\frac{\mu(|\alpha|\mu+|z|)^2}{2\alpha^2\mu^3+|\alpha z|(1+\mu^2)}\right)^{1/4} \cdot$$

$$\exp\left[\frac{\alpha^2}{2}\left(\frac{(1-\mu^2)(|\alpha|\mu+|z|)}{|\alpha|\mu}\log\left(B\left(\mu;\frac{z}{\alpha}\right)\right)+\mu^2-\frac{2|z|}{|\alpha|\mu}\right)\right] \cdot$$

$$\left(v_1^{\pm}\left(\mu;\frac{z}{\alpha}\right) \mathrm{Ai}\left(\mathrm{sign}(x_{\pm}^{*}-x)\left(\frac{3\Phi_2^{\pm}\left(\mu;\frac{z}{\alpha}\right)}{2}\right)^{2/3}\right)+\right.$$

$$\left. v_2^{\pm}\left(\mu;\frac{z}{\alpha}\right) \mathrm{Ai}'\left(\mathrm{sign}(x_{\pm}^{*}-x)\left(\frac{3\Phi_2^{\pm}\left(\mu;\frac{z}{\alpha}\right)}{2}\right)^{2/3}\right)\right), \qquad (68)$$

$$z(\mu) = \pm\left(\frac{n\mu}{|\alpha|(1-\mu^2)}-|\alpha|\mu\right) \qquad \mu \in (0,1]$$

$$\mathrm{A}(\mu;\mathrm{q}) = \alpha^2 \frac{(\mu + |\mathrm{q}|)}{\mu} \left(\mu^2 - 2 + |\mathrm{q}|\mu\right),$$

$$\mathrm{B}(\mu;\mathrm{q}) = \alpha^4 \frac{(\mu + |\mathrm{q}|)^2}{\mu^2} (1 - \mu^2),$$

$$\Phi_2^{\pm}(\mu;\mathrm{q}) = \begin{cases} \int_{\mu^*}^{\mu} \arccos\left|\frac{\mathrm{A}}{2\sqrt{\mathrm{B}}}(\mu;\mathrm{q})\right| \left(-\frac{\alpha^2(2\mu^3+\mu^2|\mathrm{q}|+|\mathrm{q}|)}{\mu^2}\right) \mathrm{d}\mu, & \mathrm{x} \geq \mathrm{x}^*, \\ \int_{\mu^*}^{\mu} \log\left(\left|\frac{\mathrm{A}}{2\sqrt{\mathrm{B}}}\right| + \frac{\sqrt{\mathrm{A}^2-4\mathrm{B}}}{2\sqrt{\mathrm{B}}}\right) \left(-\frac{\alpha^2(2\mu^3+\mu^2|\mathrm{q}|+|\mathrm{q}|)}{\mu^2}\right) \mathrm{d}\mu, & \mathrm{x} < \mathrm{x}^*, \end{cases}$$

$$\mathrm{v}_1^{\pm}(\mu;\mathrm{q}) = \frac{1}{\sqrt{\mathrm{J}(\mu;\mathrm{q})}} \left(\frac{3\Phi_2^{\pm}(\mu;\mathrm{q})}{2}\right)^{1/6} \begin{cases} \cos\left(\int_{\mu^*}^{\mu(\mathrm{x})} \pm\mathcal{V}_2(\mu;\mathrm{q})\mathrm{d}\mu\right), \ \mathrm{x} \geq \mathrm{x}^*, \\ \cosh\left(\int_{\mu^*}^{\mu(\mathrm{x})} \pm\mathcal{V}_2(\mu;\mathrm{q})\mathrm{d}\mu\right), \ \mathrm{x} < \mathrm{x}^*, \end{cases}$$

$$\mathrm{v}_2^{\pm}(\mu;\mathrm{q}) = \frac{1}{\sqrt{\mathrm{J}(\mu;\mathrm{q})}} \left(\frac{3\Phi_2^{\pm}(\mu;\mathrm{q})}{2}\right)^{-1/6} \begin{cases} -\sin\left(\int_{\mu^*}^{\mu(\mathrm{x})} \pm\mathcal{V}_2(\mu;\mathrm{q})\mathrm{d}\mu\right), & \mathrm{x} \geq \mathrm{x}^*, \\ \sinh\left(\int_{\mu^*}^{\mu(\mathrm{x})} \pm\mathcal{V}_2(\mu;\mathrm{q})\mathrm{d}\mu\right), & \mathrm{x} < \mathrm{x}^*, \end{cases}$$

$$\mu^*(|\alpha|\mu^* + |\mathrm{z}|)^2 - 4|\alpha \mathrm{z}| = 0$$

The question about integrable system with complex Hamiltonians analytical with respect momenta $p = \begin{pmatrix} p_1^1 + ip_1^2 \\ p_2^1 + ip_2^2 \end{pmatrix}$:

$$\mathcal{H}_1(p,x) = \mathcal{H}_1(p^1 + ip^2, x) = H_1^1(p^1, p^2, x) + iH_1^2(p^1, p^2, x),$$
$$\mathcal{H}_2(p,x) = \mathcal{H}_2(p^1 + ip^2, x) = H_2^1(p^1, p^2, x) + iH_2^2(p^1, p^2, x)$$

We assume:

The complex Poisson brackets:

$$\{\mathcal{H}_1(p,x), \mathcal{H}_2(p,x)\} = 0$$

Also

$$\mathcal{H}_1(p,x)|_M = \mathcal{H}_2(p,x)|_M = 0$$

Denote $p_1^2 = P_1(p_1^1, p_2^1, x), p_2^2 = P_2(p_1^1, p_2^1, x)$ the solutions to the system

$$H_1^2(p^1, p^2, x) = 0, \quad H_2^2(p^1, p^2, x) = 0$$

and introduce real-valued Hamiltonians

$$\mathbf{H}_1(p^1, x) = H_1^1(p^1, P(p^1), x), \quad \mathbf{H}_2(p^1, x) = H_2^1(p^1, P(p^1), x), \quad P(p^1) = \begin{pmatrix} P_1(p_1^1, p_2^1, x) \\ P_2(p_1^1, p_2^1, x) \end{pmatrix}$$

Then

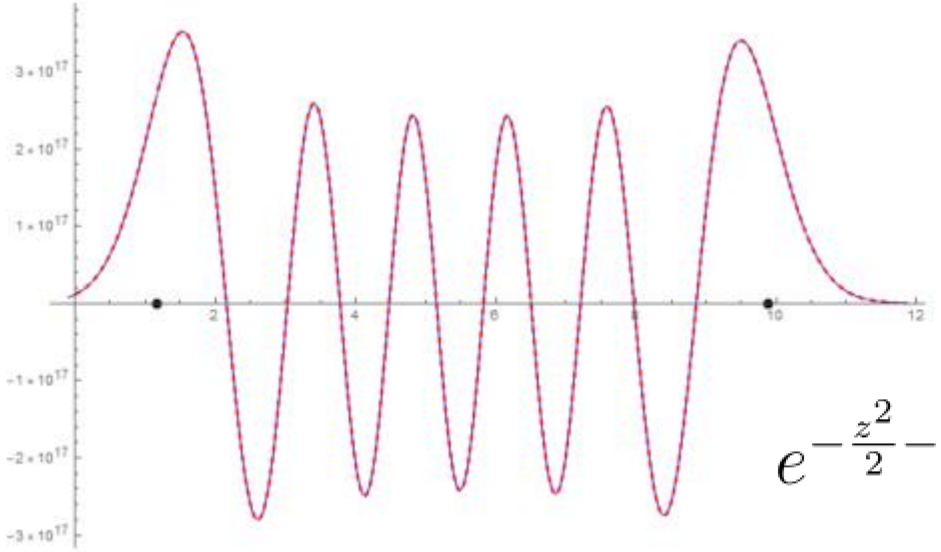
$$\mathbf{H}_1(p, x)|_\Lambda = \mathbf{H}_2(p, x)|_\Lambda = 0, \quad \text{and} \quad \{\mathbf{H}_1(p, x), \mathbf{H}_2(p, x)\} = 0$$

Hence we have “real” integrability of reduced systems.

Is it known anything about such a problem?

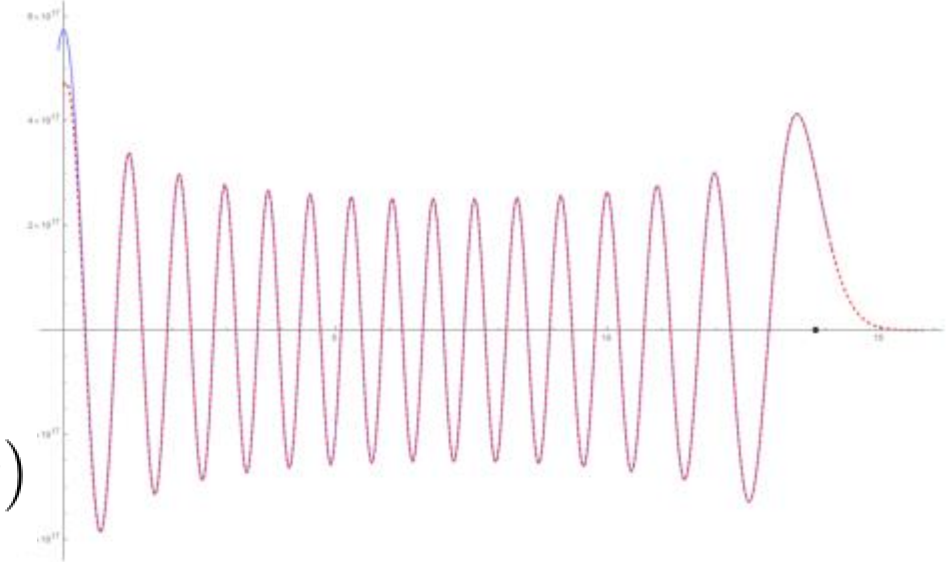
$$\mathcal{V}_2(\mu;\mathrm{q}) = \frac{\alpha^6(\mu+|\mathrm{q}|)^3}{8\mu^3(2\mu^3+|\mathrm{q}|+\mu^2|\mathrm{q}|)\sqrt{\mathrm{B}^3(\mu)}\mathrm{J}(\mu;\mathrm{q})}\cdot$$
$$\cdot\left(4\mu^6+9\mu|\mathrm{q}|-12\mu^3|\mathrm{q}|+3\mu^5|\mathrm{q}|-|\mathrm{q}|^2-4\mu^2|\mathrm{q}|^2+\mu^4|\mathrm{q}|^2\right),$$

$$\mathrm{J}(\mu;\mathrm{q}) = \sqrt{\left|\frac{\mu(\mu(|\mathrm{q}|+\mu)^2-4|\mathrm{q}|)}{4(1-\mu^2)}\right|}.$$



a = 5, n = 10

$$e^{-\frac{z^2}{2}-\Phi_1}H_{n,n}(z,\alpha)$$



a = 5, n = 30

The question about integrable system with complex Hamiltonians analytical with respect momenta $p = \begin{pmatrix} p_1^1 + ip_1^2 \\ p_2^1 + ip_2^2 \end{pmatrix}$:

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$$\mathbf{H}_1(p^1, x) = H_1^1(p^1, P(p^1), x), \quad \mathbf{H}_2(p^1, x) = H_2^1(p^1, P(p^1), x), \quad P(p^1) = \begin{pmatrix} P_1(p_1^1, p_2^1, x) \\ P_2(p_1^1, p_2^1, x) \end{pmatrix}$$

Then

$$\mathbf{H}_1(p, x)|_\Lambda = \mathbf{H}_2(p, x)|_\Lambda = 0, \quad \text{and} \quad \{\mathbf{H}_1(p, x), \mathbf{H}_2(p, x)\} = 0$$

Hence we have “real” integrability of reduced systems.

Is it known anything about such a problem?

Relationship with 3-d order ODE for $u = H_{n_1,n_2}$

(A.Aptekarev, A. Branquinho, W. Van Assche, 2003)

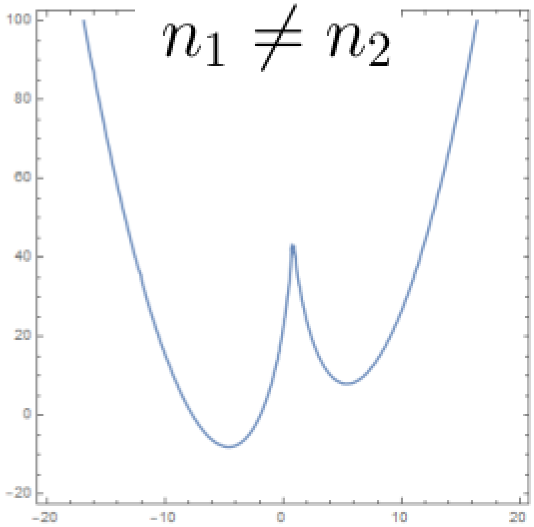
$$\frac{d^3u}{dz^3} - 4z\frac{d^2u}{dz^2} + (4z^2 - 4\alpha^2 + 2(n_1+n_2-1))\frac{du}{dz} - 4(z(n_1+n_2) - \alpha(n_1-n_2))u = 0$$

the oscillating factor

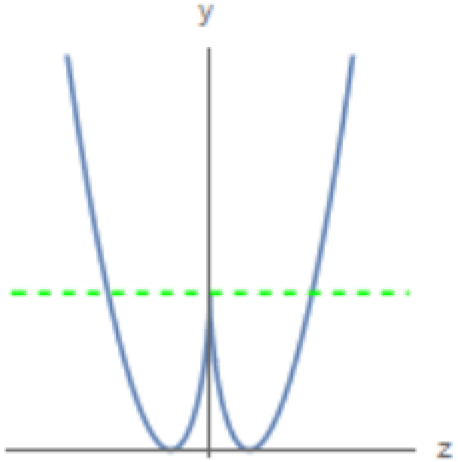
Ansatz $u = C_0\Psi(z,\alpha,n_1,n_2) + C_1\psi_1(z,\alpha,n_1,n_2)\psi_2(z,\alpha,n_1,n_2)$

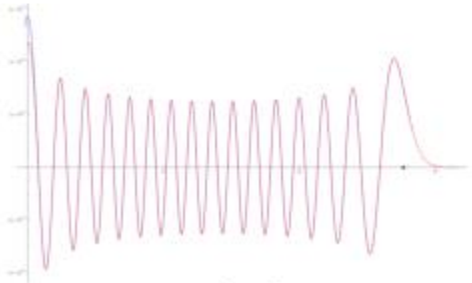
Double-well problem for the oscillating factor

$n_1 \neq n_2$

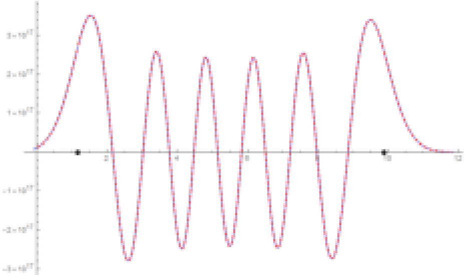


$n_1 = n_2$





a = 5, n = 30



a = 5, n = 10

The differential equation of the third order

The multiple orthogonal Hermite polynomials $H = H_{n_1,n_2}(z,\alpha)$ are solutions of the differential equation

$$\frac{\partial^3 H}{\partial z^3} - 4z\frac{\partial^2 H}{\partial z^2} + (4z^2 - 4\alpha^2 + 2(n_1+n_2-1))\frac{\partial H}{\partial z} - 4(z(n_1+n_2) - \alpha(n_1-n_2))H = 0. \tag{4}$$

Initial/boundary conditions?

We will use

- The polynomials values at points $z = \pm\alpha$
- Behavior of polynomials as $|z| \to \infty$

We introduce $m = n_1 + n_2$, $k = n_1 - n_2$, a small parameter $h \sim 1/\sqrt{m^2 + k^2}$ and new variables

$$y = z\sqrt{h}, \quad a = \alpha\sqrt{h}, \quad E = mh, \quad I = kh. \tag{1}$$

.

The differential equation is rewritten as follows

$$\hat{\mathcal{L}}\psi(y,a) = \Big(-i\hat{p}^3 + 4y\hat{p}^2 + (4(y^2 - a^2) + 2E - 2h)i\hat{p} - 4(yE - aI)\Big)\psi(y,a) = 0, \tag{2}$$

где $\psi(y,a) = H\left(\frac{y}{\sqrt{h}}, \frac{a}{\sqrt{h}}\right)$, $\hat{p} = -ih\frac{\partial}{\partial y} \quad \Longleftrightarrow \quad h\frac{\partial}{\partial y} = i\hat{p}$.

The symbol is complex-valued

$$\mathcal{L}(y,p) = -ip^3 + 4yp^2 + \big(4(y^2 - a^2) + 2E - 2h\big)ip - 4\big(yE - aI\big). \tag{3}$$

The principal part of the symbol

$$\mathcal{L}(y,p) = -ip^3 + 4yp^2 + \big(4(y^2 - a^2) + 2E\big)ip - 4\big(yE - aI\big) - 2hip \quad (59)$$

defines the characteristic polynomial in $\lambda = ip$

$$\lambda^3 - 4y\lambda^2 + \big(4(y^2 - a^2) + 2E\big)\big)\lambda - 4\big(yE - aI\big) = 0. \quad (4)$$

In the domain, where the characteristic polynomial has three real roots, the asymptotics can be presented as sum of three exponents

$$\psi \approx e^{\frac{i\mathcal{S}_1(y)}{h}}\mathcal{A}_1(y) + e^{\frac{i\mathcal{S}_2(y)}{h}}\mathcal{A}_2(y) + e^{\frac{i\mathcal{S}_3(y)}{h}}\mathcal{A}_3(y). \quad (5)$$

The Hamilton-Jacobi equation has the form $\mathcal{L}_0\left(y, \frac{\partial \mathcal{S}_j}{\partial y}\right) = 0$, i.e. in the considered domain phases are purely imaginary

$$\mathcal{S}_j(y) = -i\int_{y_0}^{y} \lambda_j(\xi)d\xi, \quad \lambda_j \text{ is a root of } (4). \quad (6)$$

We can try to split off the root!

- The characteristic polynomial of the third degree $\Rightarrow$

- For every fixed y the polynomial has a real root $\Rightarrow$

- Split off the real root λ_0 of the polynomial and divide the differential equation into two $\hat{L}^0 \psi_0 = 0$ и $\hat{L}^1 \psi_1 = 0$.

The principal part of the symbol L^0 corresponds to the linear part $\lambda - \lambda_0$ of the characteristic polynomial.
The principal part L^1 corresponds to quadratic part $\lambda^2 + g_1 \lambda + g_2 \lambda$ of the characteristic polynomial, where

$$g_1 = -4y + \lambda_0, \quad g_2 = \frac{4(yE - aI)}{\lambda_0} = \lambda_0^2 - 4y\lambda_0 + (4(y^2 - a^2) + 2E).$$

The idea of the approach

Thus, the asymptotics of the solution ψ can be represented in the form of the sum $\psi = C_0\psi_0 + C_1\psi_1$.

The function ψ_0 is an asymptotics of the solution

$$\hat{L}^0\psi_0 = 0, \quad L^0(y,p) = ip - \lambda_0 + O(h), \tag{7}$$

which can be represented in the form

$$e^{\frac{iS_0(y)}{h}}A_0(y), \quad S_0 = -i\int_{y_0}^{y}\lambda_0(\xi)d\xi.$$

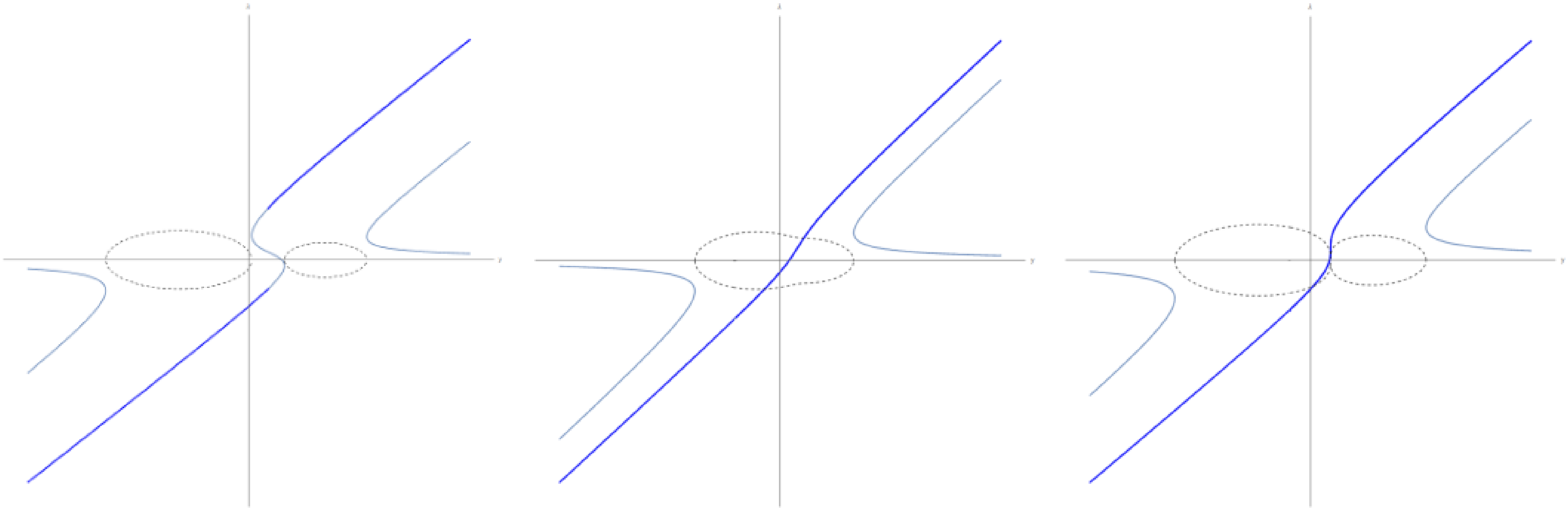
The function ψ_1 is an asymptotics of the solution

$$\hat{L}^1\psi_1 = 0, \quad L^1(y,p) = -p^2 + ig_1 p + g_2 + O(h). \tag{8}$$

By a certain substitution, this equation is reduced to the Schrödinger equation $\Rightarrow$ the asymptotics is represented in the form of the Airy function.

How to choose the root λ_0?

The idea: in the region where the real root is unique, we choose it uniquely, and then extend by continuity



$54a^2I^2 > (E - 2a^2)(E + 16a^2)^2$ $\quad$ $54a^2I^2 < (E - 2a^2)(E + 16a^2)^2$ $\quad$ $54a^2I^2 = (E - 2a^2)(E + 16a^2)^2$

Figure: The roots of the characteristic polynomial (4) on the plane (y, λ). The root λ_0 is in bold. The dotted line shows the imaginary parts of complex roots

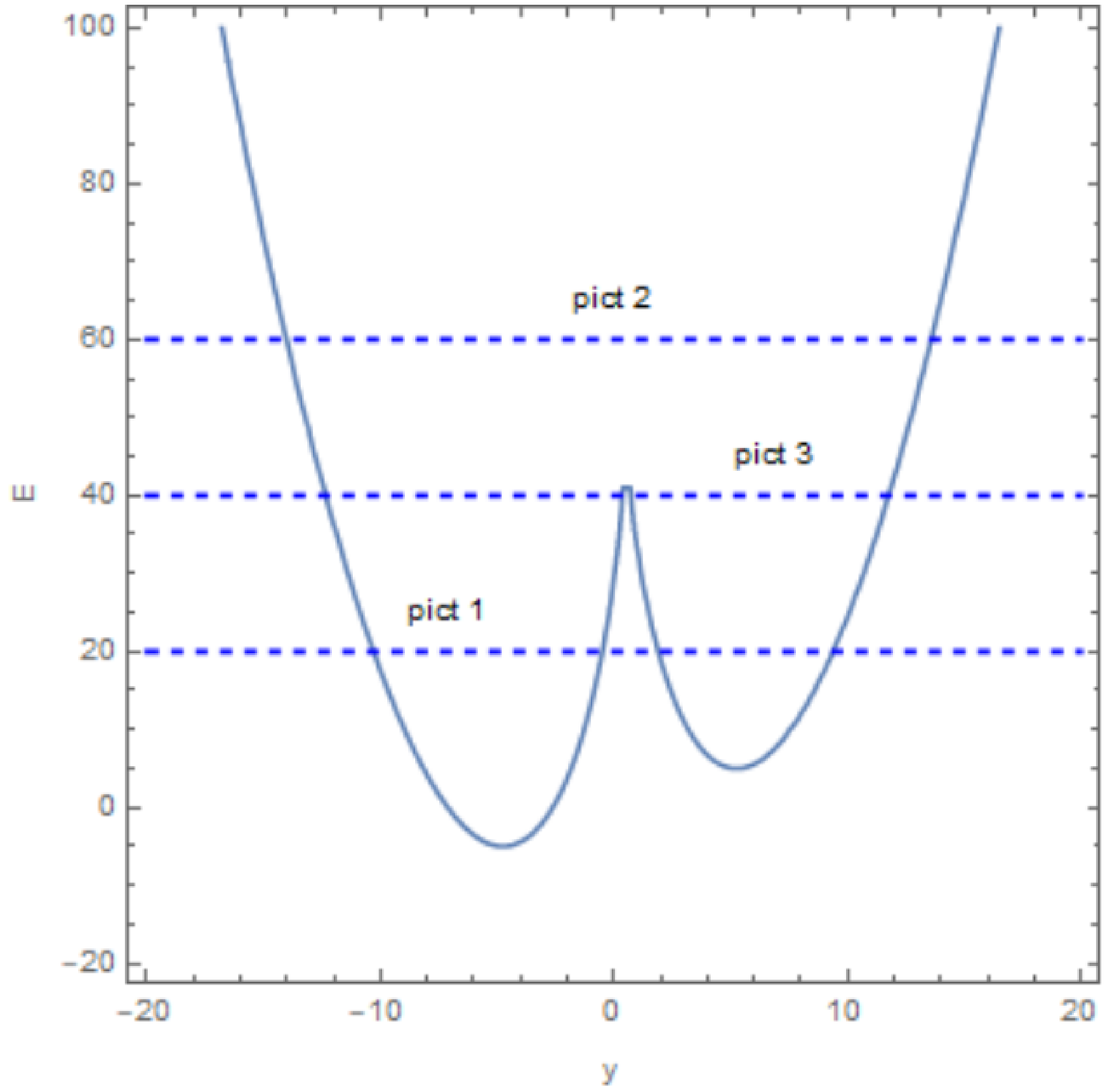
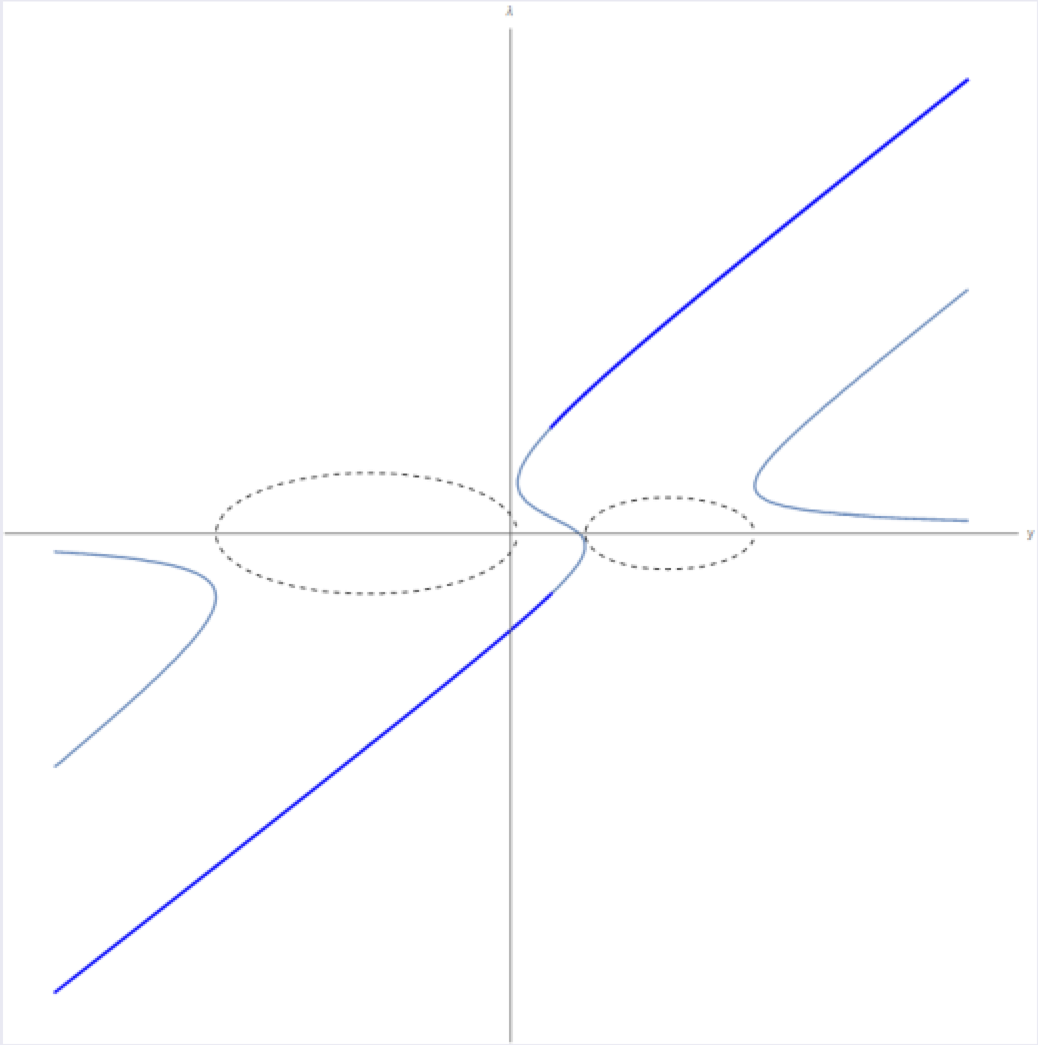


Figure: The zero line of the discriminant of the characteristic polynomial on the plane (y, E)

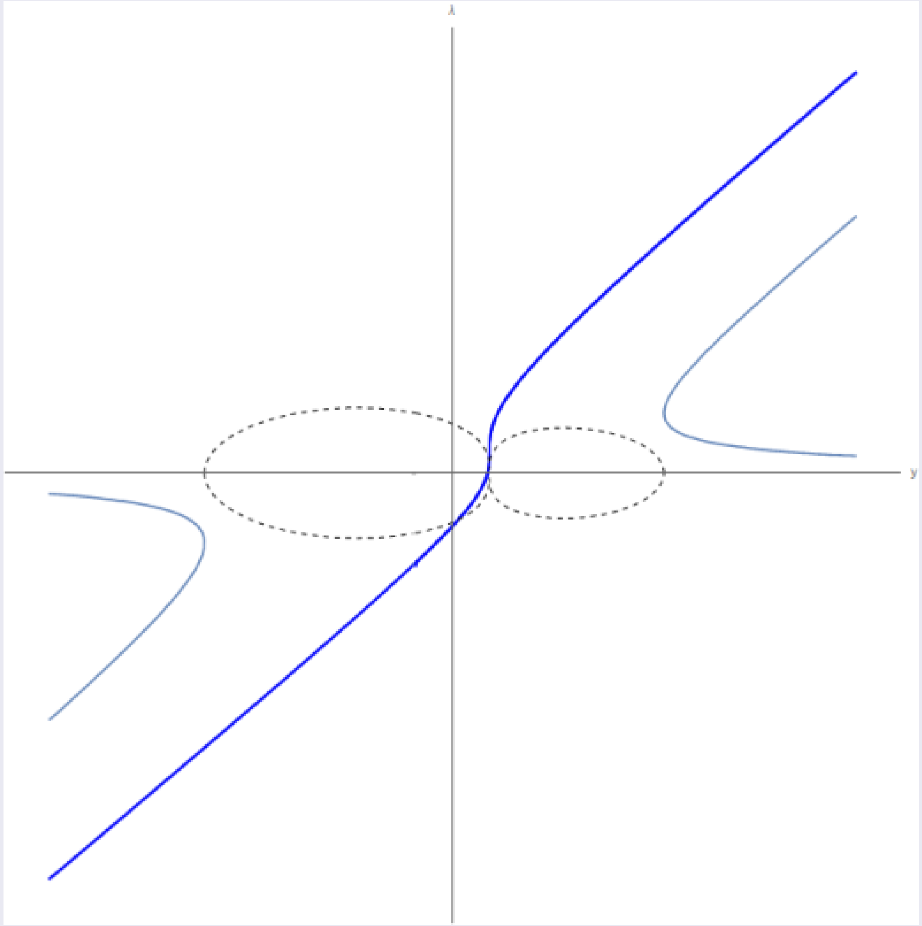
Remark

In the case when the root λ_0 has a break, we can take any point in the interval between the corresponding focal points as the break point, since the coefficients for the asymptotics to the left and to the right of the discontinuity point are determined independently. We take the midpoint at this interval.



Remark

We do not consider the case $54a^2I^2 = (E - 2a^2)(E + 16a^2)^2$, because in this case there is a singularity that corresponds to the Pearcey function.



Remark

Further we will make sure that the solution corresponding to the root λ_0 does not contribute to the asymptotics.

Splitting the equation into two

$$\hat{\mathcal{L}}\psi(y,a) = \Big(-i\hat{p}^3 + 4y\hat{p}^2 + (4(y^2-a^2)+2E-2h)i\hat{p} - 4(yE-aI)\Big)\psi(y,a) = 0. \tag{58}$$

To obtain the equation

$$\hat{L}^0\psi_0 = 0, \quad L^0 = ip - \lambda_0 + O(h), \tag{63}$$

apply the operator $\hat{R}_0 = (-\hat{p}^2 + ig_1\hat{p} + g_2)^{-1}$ to (2).

To obtain the equation

$$\hat{L}^1\psi_1 = 0, \quad L^1 = -p^2 + ig_1p + g_2 + O(h), \tag{64}$$

apply the operator $\hat{R}_1 = (i\hat{p} - \lambda_0)^{-1}$ to (2).

The symbol of the operator $\hat{L}^0 = (-\hat{p}^2 + ig_1\hat{p} + g_2)^{-1}\hat{\mathcal{L}}$ has the form

$$L^0(y,p) = ip - \lambda_0 + hL_1^0(y,ip) + O(h^2), \tag{9}$$

$$L_1^0(y,p) = -\left(\frac{\lambda_0'(y)(-2ip - g_1)}{((ip)^2 + g_1 ip + g_2)} + \frac{2ip}{((ip)^2 + g_1 ip + g_2)} - \frac{(-2ip - g_1)(-4(ip)^2 + 8yip - 4E)}{((ip)^2 + g_1 ip + g_2)^2}\right). \tag{10}$$

The symbol of the operator $\hat{L}^1 = (i\hat{p} - \lambda_0)^{-1}\hat{\mathcal{L}}$ has the form

$$L^1(y,p) = (ip)^2 + g_1 ip + g_2 + hL_1^1(y,p) + O(h^2), \tag{11}$$

$$L_1^1(y,p) = \left(\frac{-2ip}{ip - \lambda_0} - \frac{(ip)^2 + g_1 ip + g_2}{(ip - \lambda_0)^2}\frac{\partial\lambda_0}{\partial y} - \frac{-4(ip)^2 + 8yip - 4E}{(ip - \lambda_0)^2}\right). \tag{12}$$

Asymptotics of the solution of the equation $\hat{L}^0\psi_0 = 0$

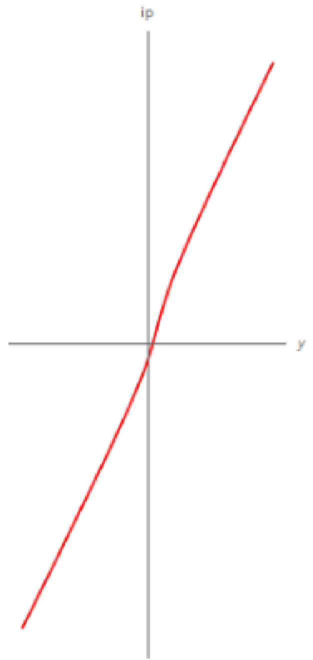
Asymptotics of the solution ψ_0 of the equation

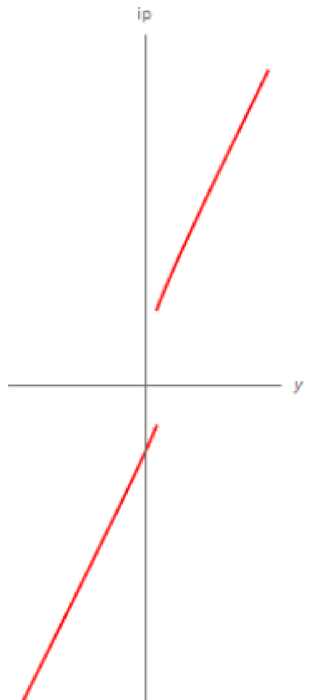
$$\hat{L}^0\psi_0 = 0, \quad L_0 = ip - \lambda_0 + hL_1^0 + O(h^2),$$

where L_1^0 is defined by the expression (10), can be obtained in the form

$$\psi_0 \approx \psi_0^{as} = e^{\frac{1}{h}\int_{y_0}^{y} \lambda_0(\xi)d\xi} A_0(y), \quad A_0(y) = e^{-\int_{y_0}^{y} L_1^0\left(\xi, -i\lambda_0(\xi)\right)d\xi}, \quad (13)$$

y_0 is a constant. Its choice affects only the phase factor.





Asymptotics of the solution of the equation $\hat{L}^1\psi_1 = 0$

Let us seek the asymptotics of the solution ψ_1 of the equation

$$\hat{L}^1\psi_1 = 0, \quad L^1 = -p^2 + ipg_1 + g_2 + hL_1^1(y,p) + O(h^2)$$

in the form

$$e^{\frac{\theta_0(y)}{h}+\theta_1(y)}\phi(y), \quad \theta(y) := \theta_0 + h\theta_1, \tag{14}$$

where phases θ_0, θ_1 are chosen in such a way that the function ϕ satisfies the " perturbed " Schrödinger equation

$$L(y,\hat{p})\phi = 0, \quad L(y,p) = -p^2 + V_0(y) + hV_1(y) + O(h^2), \tag{15}$$

$V_0(y)$, $V_1(y)$ are real functions.

Substitute (14) into (8) and "drag" the exponent through the operator $\hat{L}^1$, what gives the equation for ϕ

$$L^1(x,[\hat{p} - i\theta'])\phi = 0, \quad \theta' = \frac{d\theta}{dy} = \frac{\theta_0'}{h} + \theta_1'.$$

The symbol $L^1(x,[\hat{p}-i\theta'])$ is

$$-p^2+2ip\theta_0'+ipg_1+g_1\theta_0'+g_2+(\theta_0')^2$$

$$h\left(2i\theta_1'(p-i\theta_0')+g_1\theta_1'+\theta_0''+L_1^1(y,p-i\theta_0')\right)+O(h^2).$$

Choose θ_0 in such a way that

$$2\theta_0'+g_1=0.$$

Then symbol takes the form

$$-p^2-\frac{g_1^2}{4}+g_2+h\left(2ip\theta_1'-\frac{g_1'}{2}+L_1^1(y,p-i\theta_0')\right)+O(h^2).$$

Note that

$$L_1^1(y, p - i\theta_0') = -\frac{g_1\lambda_0 + 2g_2}{\lambda_0^2 + g_1\lambda_0 + g_2} + \frac{1}{(\lambda_0^2 + g_1\lambda_0 + g_2)^2}\cdot$$

$$\big(2Eg_1^2 - 4Eg_2 + 4g_2^2 + 4Eg_1\lambda_0 + 4g_1g_2\lambda_0 + 4E\lambda_0^2 + 2g_1^2\lambda_0^2 - 4g_2\lambda_0^2 +$$

$$4g_1g_2y + 16g_2\lambda_0y + 4g_1\lambda_0^2y\big) + ip\frac{1}{(\lambda_0^2 + g_1\lambda_0 + g_2)^2}\cdot$$

$$\cdot\left(4Eg_1 + 8E\lambda_0 - 6g_2\lambda_0 - 2g_1\lambda_0^2 + 2\lambda_0^3 + 8g_2y - 8\lambda_0^2y\right).$$

Using $p^2 = -g_1^2/4 + g_2$, we obtain the equation for θ_1'

$$\theta_1'(y) = \frac{1}{2}\left(-\frac{4Eg_1 + 8E\lambda_0 - 6g_2\lambda_0 - 2g_1\lambda_0^2 + 2\lambda_0^3 + 8g_2y - 8\lambda_0^2y}{(\lambda_0^2 + g_1\lambda_0 + g_2)^2}\right),$$

The Shrödinger equation

Thus,

$$\psi_1^{as}(y) = e^{\frac{\theta_0}{h}+\theta_1}\phi^{as}(y),$$

where

$$\theta_0(y) = -\int_{y^*}^{y}\frac{g_1(\xi)}{2}d\xi, \quad \theta_1(y) = -\frac{1}{2}\int_{y^*}^{y}\frac{1}{(\lambda_0^2(\xi)+g_1(\xi)\lambda_0(\xi)+g_2(\xi))^2}\cdot \tag{16}$$

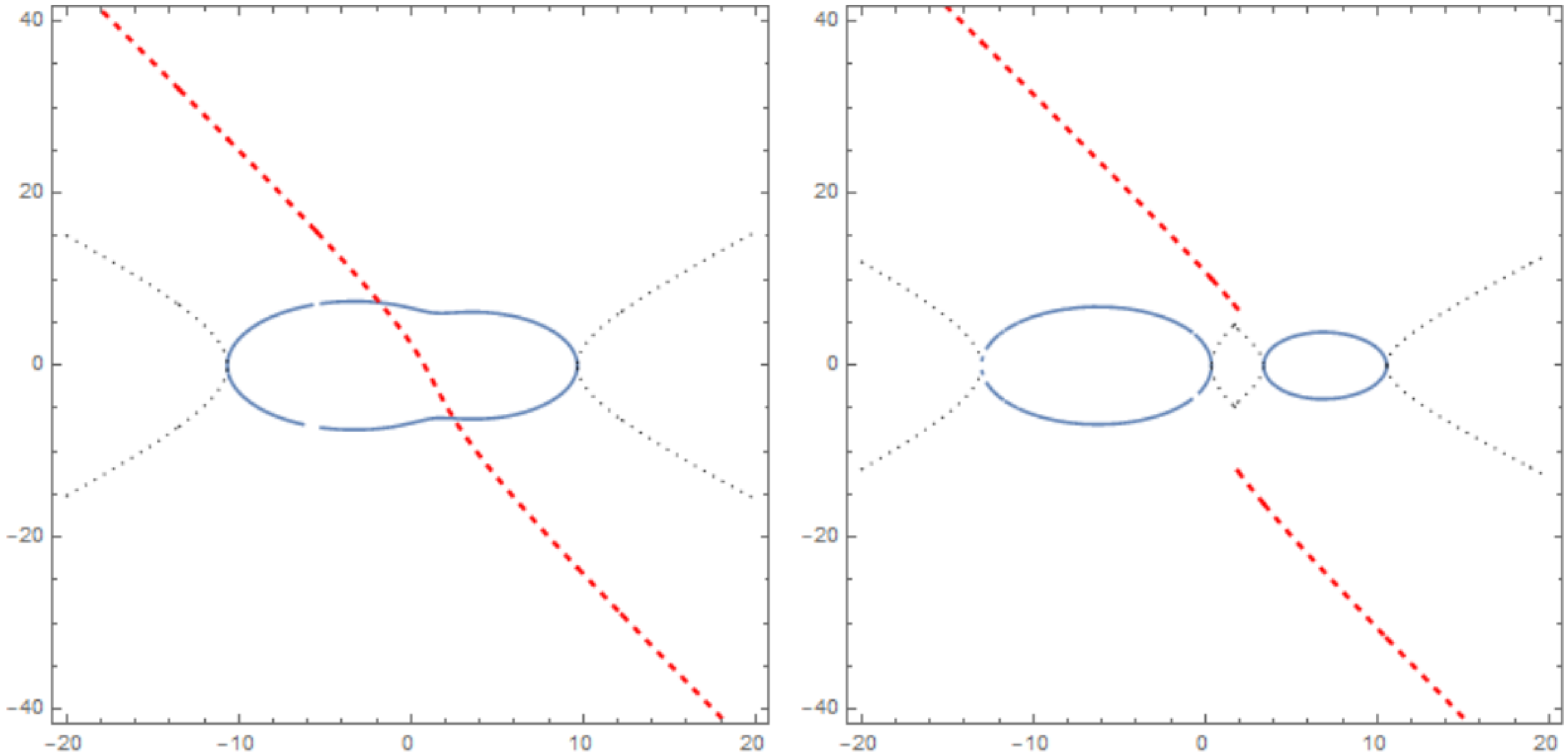
$$\left(4Eg_1(\xi)+8E\lambda_0(\xi)-6g_2(\xi)\lambda_0(\xi)-2g_1(\xi)\lambda_0^2(\xi)+2\lambda_0^3(\xi)+8g_2(\xi)\xi-8\lambda_0^2(\xi)\xi\right)d\xi, \tag{17}$$

and ϕ^{as} is asymptotics of the solution of the Shrödinger equation with the symbol

$$-p^2+V_0(y)+hV_1(y)+O(h^2), \tag{18}$$

$$V_0(y) = -\frac{g_1^2}{4}+g_2, \quad V_1(y) = -\frac{g_1'(y)}{2}+\frac{1}{(\lambda_0^2+g_1\lambda_0+g_2)^2}\cdot \tag{19}$$

$$\cdot\left(-g_1\lambda_0^3+(4E+g_1^2-6g_2+4g_1y)\lambda_0^2+(4Eg_1+g_1g_2+16g_2y)\lambda_0+\right.$$
$$\left.2Eg_1^2-4Eg_2+2g_2^2+4g_1g_2y\right). \tag{20}$$



a) two focal points b) four focal points

Figure: Solid closed lines - lines of the zero level of the Hamiltonian $-p^2 + V_0(y) = 0$ (Lagrangian manifolds). Black dashed lines – the imaginary part of the momentum corresponding to the solution ψ_1. Red dashed line – imaginary part of the momentum corresponding to the solution ψ_0 (i.e. $-\lambda_0$)

In the form of the Airy function

$$\phi^{as} = \left(\frac{w_1(y)}{h^{1/6}} \left(\frac{|\Phi(y)|}{|V_0(y)|} \right)^{1/4} \mathrm{Ai} \left(\pm \mathrm{sign}(y - y_*) \frac{\Phi(y)}{h^{2/3}} \right) \pm \qquad (21)$$

$$\pm w_2(y) h^{1/6} \left(\frac{1}{|\Phi(y)||V_0(y)|} \right)^{1/4} \mathrm{Ai}' \left(\pm \mathrm{sign}(y - y_*) \frac{\Phi(y)}{h^{2/3}} \right) \right), \qquad (22)$$

Here

$$\Phi(y) = \left(\frac{3}{2} \int_{y^*}^{y} \sqrt{|V_0(\xi)|} d\xi \right)^{2/3}, \quad w(y) = \left(\int_{y_\pm}^{y} \mp \frac{V_1(\xi)}{2\sqrt{|V_0(\xi)|}} d\xi \right). \qquad (23)$$

$$w_1(y) = \begin{cases} 2\cos w(y), \ \pm(y - y^*) < 0 \\ 2\cosh w(y), \ \pm(y - y^*) > 0 \end{cases}, \quad w_2(y) = \begin{cases} -2\sin w(y), \ \pm(y - y^*) < 0 \\ 2\sinh w(y), \ \pm(y - y^*) > 0 \end{cases} \qquad (24)$$

Coefficients C_0 and C_1

Thus,

$$\psi \approx C_0 \psi_0^{as} + C_1 \psi_1^{as}.$$

Let us obtain C_0 and C_1.

In the region where the polynomial has three real roots, taking into account the asymptotic behavior of the Airy function

$$\mathrm{Ai}(\eta) \approx \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\eta^{1/4}}, \ \mathrm{Ai}'(\eta) \approx -\eta^{1/4}\frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}}$$

as $\eta \to +\infty$, the solution is presented in the WKB form

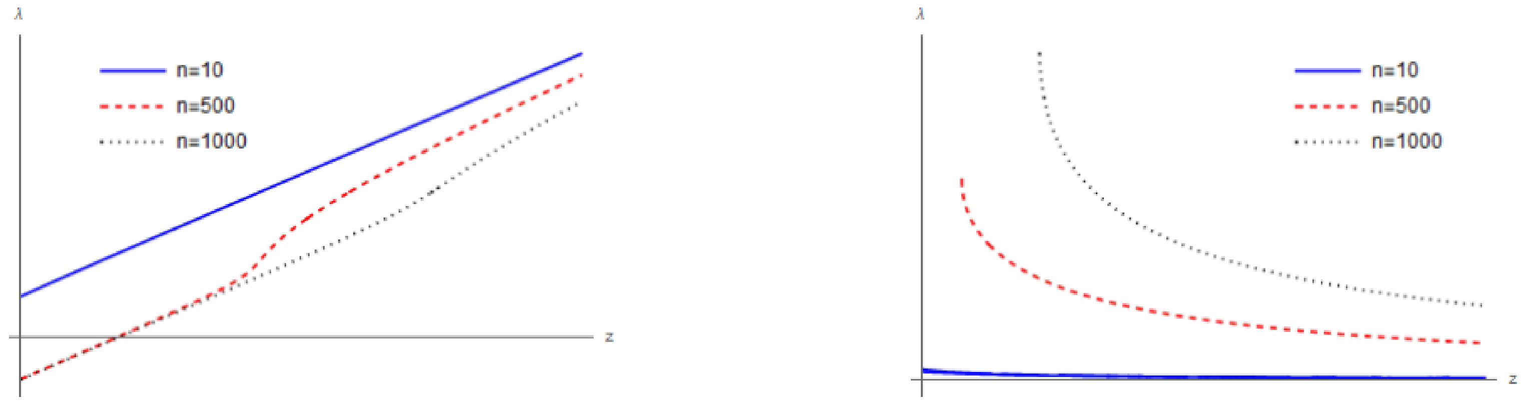
$$H_{n_1,n_2} \asymp C_0 e^{\int_{z_*}^{z} \lambda_0(\xi;\alpha,m,k)d\xi} A_0(z) + C_1 e^{\int_{z_*}^{z} \lambda_-(\xi;\alpha,m,k)d\xi} A_1(z), \ A_1(z) = e^{\theta_1(z) - w(z)}, \tag{25}$$

where $\lambda_- = -\frac{g_1}{2} - \sqrt{\frac{g_1^2}{4} - g_2}$ is the root of the equation $\lambda^2 + g_1\lambda + g_2 = 0$ (hence, the root of the characteristic equation (4)).

In the case $I = 0$ ($\Leftrightarrow n := n_1 = n_2$) the root λ_0 decreases with increasing n and λ_- grows with increasing n has the following asymptotics as $n \to \infty$

$$\lambda_0 \sim 2y + \frac{2a^2 y}{n}, \quad \lambda_- \sim \frac{2n}{y}.$$

A similar situation can be observed in an arbitrary case (when $0 \neq |I| < E$)



a) $\lambda_0(z; 9, n+10, n-10)$ b) $\lambda_-(z; 9, n+10, n-10)$

Figure: Root behavior with increasing indices

So the phase $\int_{z^*}^{z} \lambda_-(\xi; m, k)d\xi$ contributes more to the asymptotics than $\int_{z^*}^{z} \lambda_0(\xi; m, k)d\xi$. Thus, the coefficient C_0 can be set equal to zero.

To obtain the coefficient C_1 we find the value of the Hermite polynomial in points $z = \alpha$ и $z = -\alpha$.

In the case of four focal points (when the root λ_0 is discontinuous) for $z < (z_2 + z_3)/2$ we define coefficient

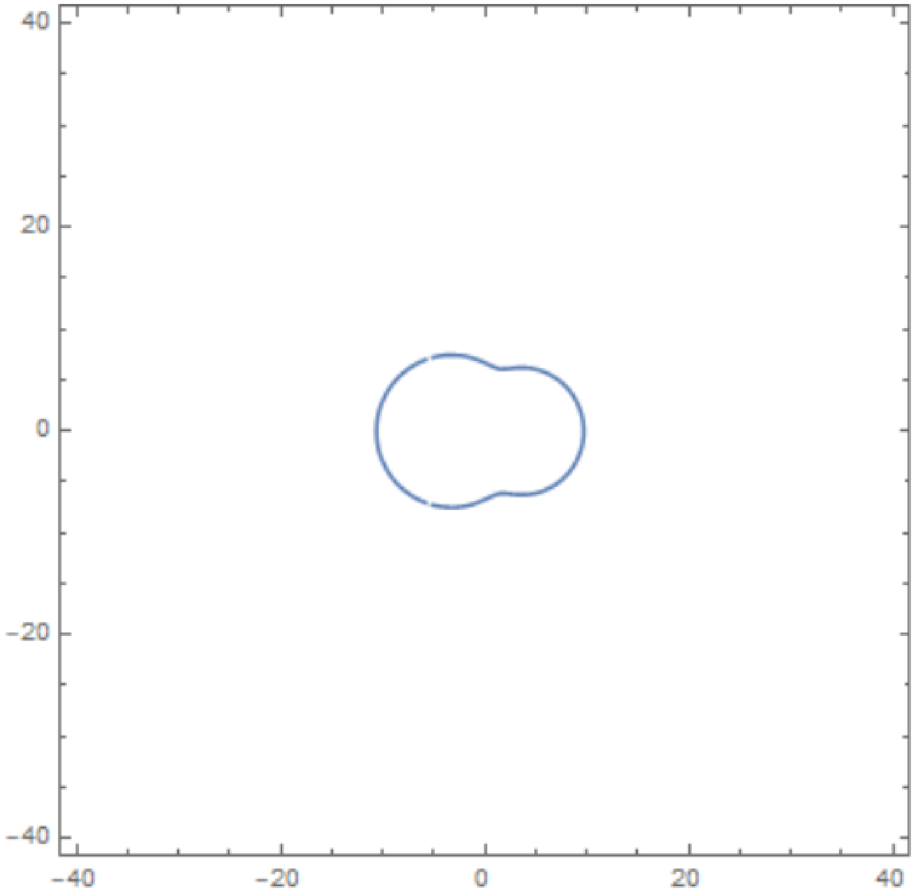
$$C_1^- = \frac{H_{n_1,n_2}(-\alpha, \alpha)}{\psi^{as}(-\alpha, \alpha)},$$

using the value in the point $z = -\alpha$, and for $z > (z_2 + z_3)/2$ – the coefficient

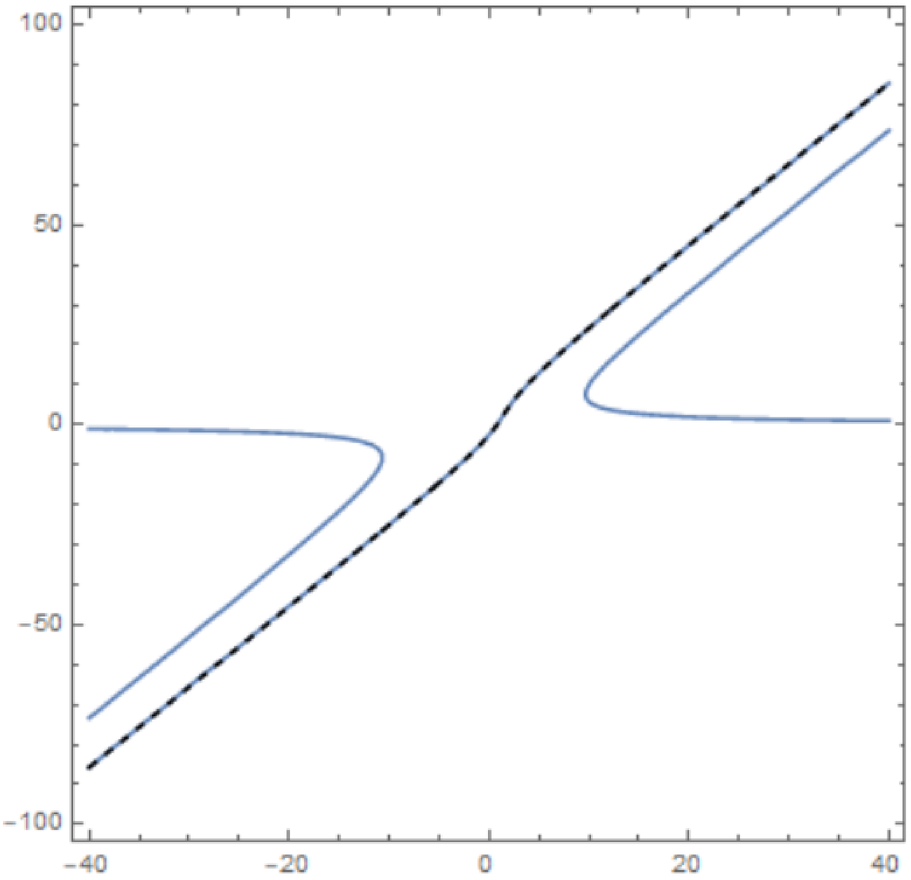
$$C_1^+ = \frac{H_{n_1,n_2}(\alpha, \alpha)}{\psi^{as}(\alpha, \alpha)},$$

using the value in the point $z = \alpha$.

Examples: $n_1 \neq n_2$ and two focal points



The Lagrangian manifolds



The roots of the characteristic polynomial. Black dashed line is the root λ_0

Figure: The case $n_1 \neq n_2$ and two focal points ($n_1 = 24$, $n_2 = 14$, $a = 3$)

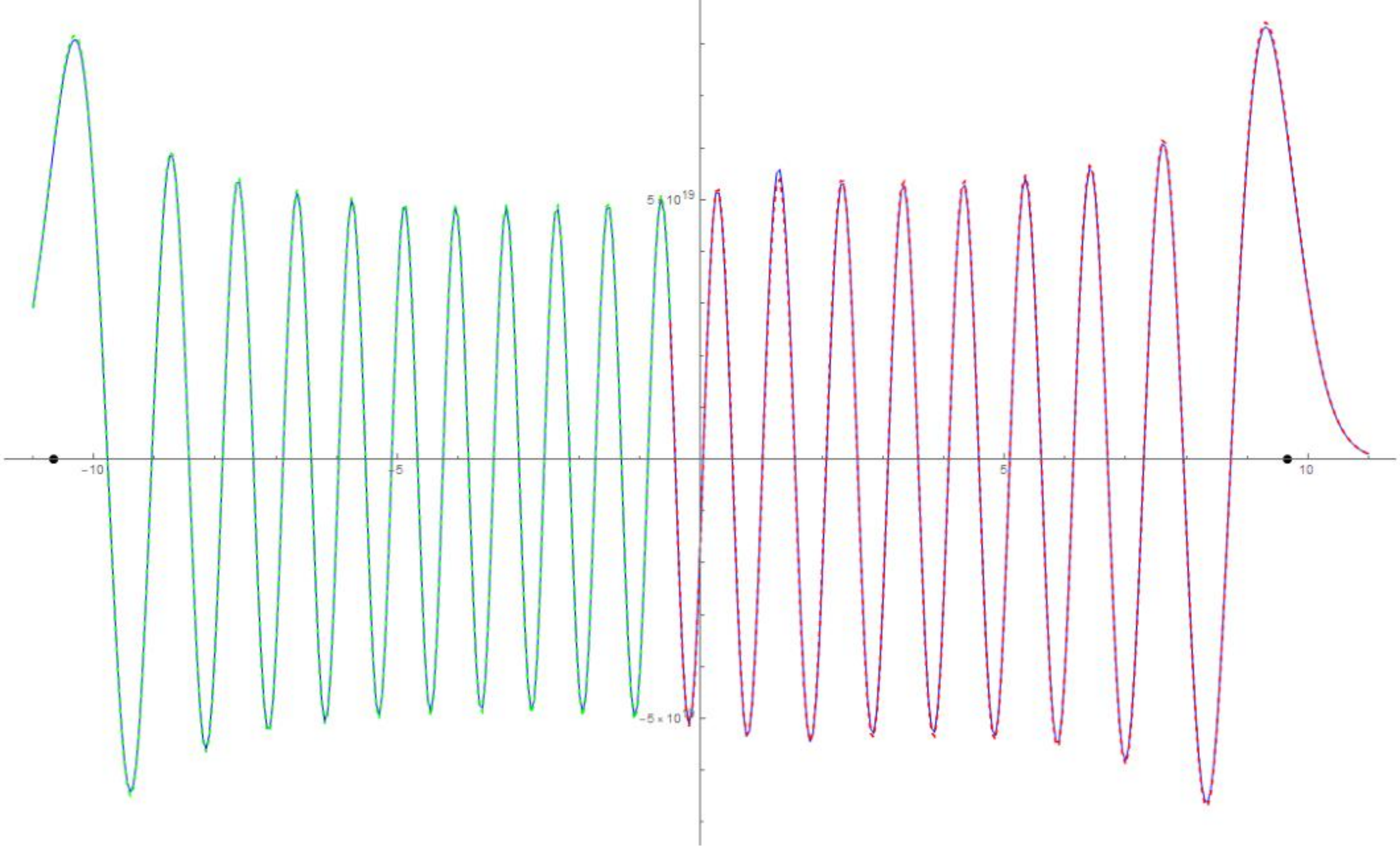
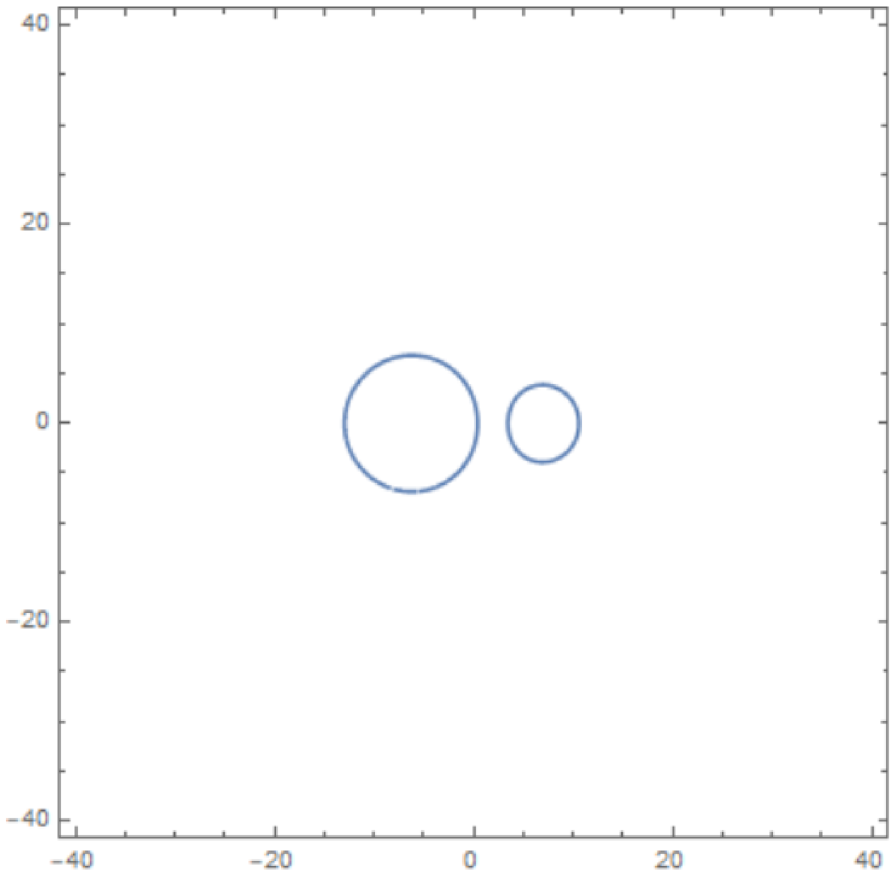
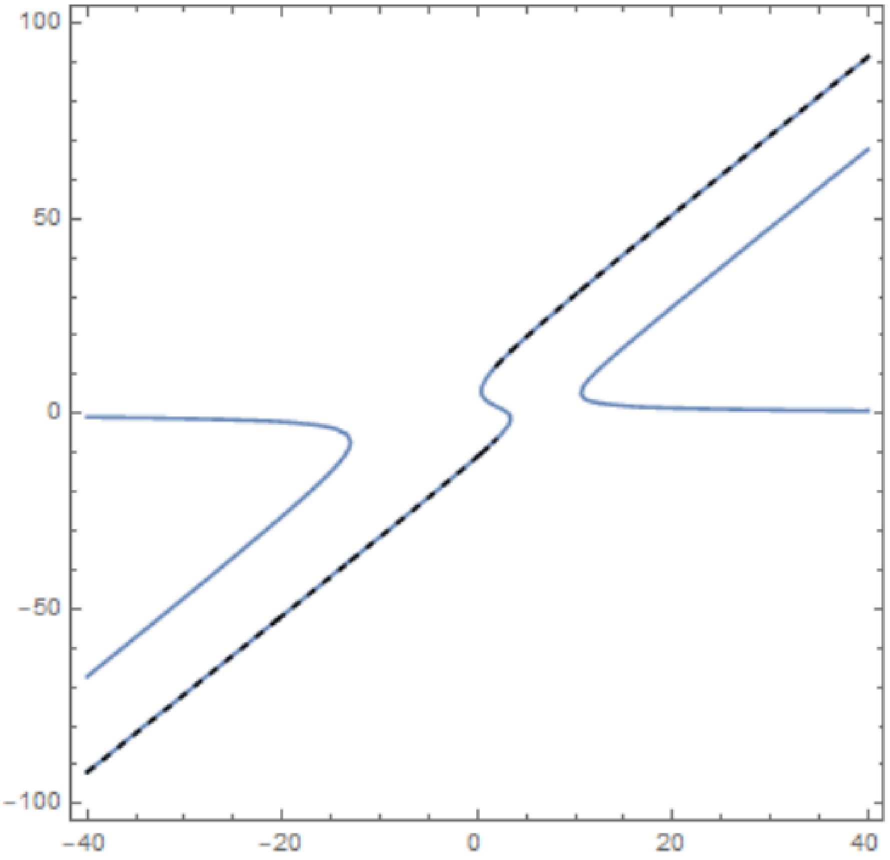


Figure: The blue line is the plot of $H_{24,14}(z,3)e^{-\theta_0(z)-\theta_1(z)}$, the red dashed line is the asymptotics in the neighbourhood of $z_2^* = 9.66189 > 0$, the green dashed line is the asymptotics in the neighbourhood $z_1^* = -10.6625 < 0$. The coefficient C_1 is taken in the point $z = \alpha = 3$

Examples: $n_1 \neq n_2$ and four focal points



The Lagrangian manifold



The roots of the characteristic polynomial. Black dashed line is the root λ_0

Figure: The case $n_1 \neq n_2$ and four focal points ($n_1 = 23$, $n_2 = 7$, $a = 6$)

Examples: $n_1 \neq n_2$ and four focal points

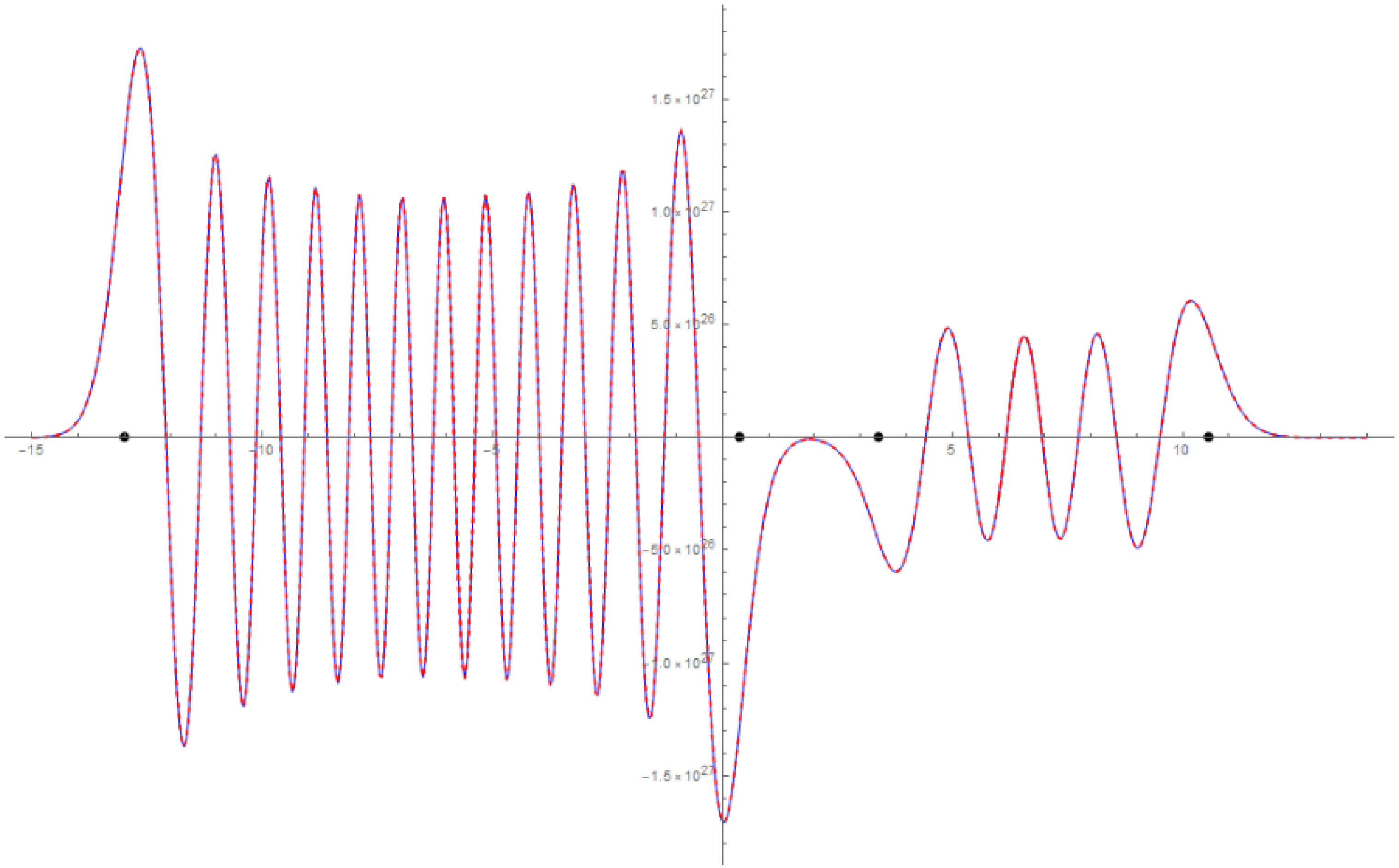


Figure: The blue line is the plot of $H_{23,7}(z,6)e^{-\theta_0(z)-\theta_1(z)}$, the red dashed line is the asymptotics. For $z < \frac{z_2^*+z_3^*}{2}$ we takes the coefficient C_1 in the point $z=-\alpha$, and for $z > \frac{z_2^*+z_3^*}{2}$ – in the point $z=\alpha$

The neighbourhood of the break point

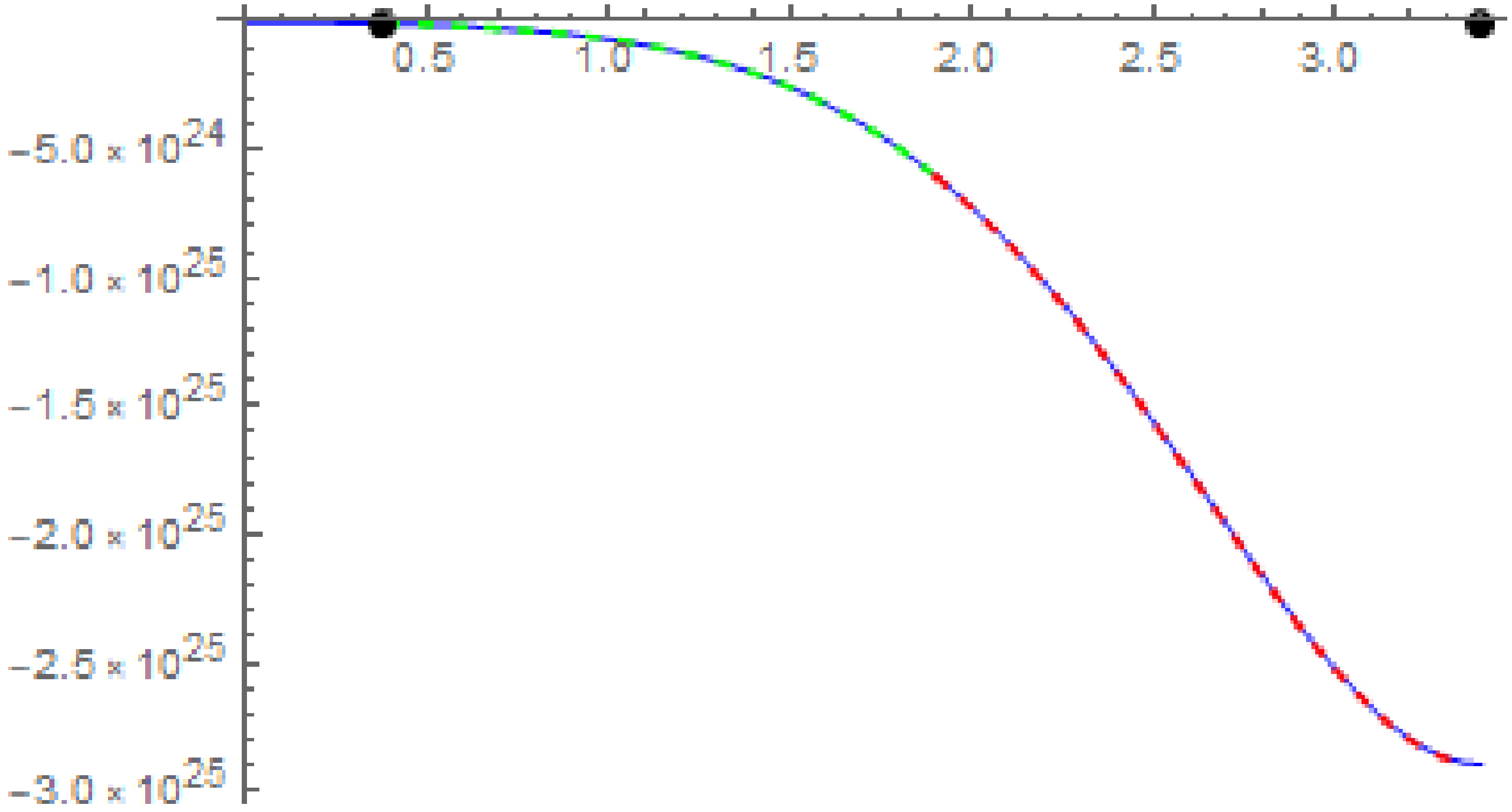


Figure: Blue solid line – the plot of polynomial $H_{23,7}(z, 6)$. Red dashed line is the asymptotics with the coefficient at $z = \alpha$, green dashed line – with the coefficient at $z = -\alpha$.

[1] A. I. Aptekarev, S. Yu. Dobrokhotov, D. N. Tulyakov, A. V. Tsvetkova, Plancherel-Rotach type asymptotics for multiple orthogonal Hermite polynomials and recurrence relations, Izvestiya: Mathematics 86:1, 32-91

[2] S. Yu. Dobrokhotov and A. V. Tsvetkova, An Approach to Finding the Asymptotics of Polynomials Given by Recurrence Relations, Russian Journal of Mathematical Physics, Vol. 28, No. 2, 2021, pp. 198-223

[3] S. Yu. Dobrokhotov and A. V. Tsvetkova, Asymptotics of multiple orthogonal Hermite polynomials $H_{n_1,n_2}(z,\alpha)$ determined by a third-order differential equation, Russian Journal of Mathematical Physics, Vol. 28, No. 4, 2021, pp. 439-454

THANK YOU FOR YOUR ATTENTION!