Quasi-Bessel equations: existence and hyper-dimensionality

P. B. Dubovski with L. Boyadzhiev and J. A. Slepoi

INTERNATIONAL BIWEEKLY ONLINE SEMINAR ON ANALYSIS, DIFFERENTIAL EQUATIONS AND MATHEMATICAL PHYSICS

April 20, 2023

Classical Bessel equation

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0$$

Okrasiński and Plociniczak (2013):

$$x^\alpha D^\alpha (x^\beta D^\beta y) = (x^{2\mu} + \nu^{2\mu})y, \quad 0 < \alpha, \beta, \mu < 1,$$

and sought for the solution in the form

$$u(x) = \sum_{n=0}^{\infty} c_n x^{\gamma+\mu n}.$$

The approach works because $D^\alpha x^b = k \cdot x^{b-\alpha}$.

Rodrigues, Viera and Yakubovich (2014):

$$x^{2\alpha} D^{2\alpha} u(x) + x^\alpha D^\alpha u(x) + (x^{2\alpha} - \nu^2)u(x) = 0, \quad \alpha \in (0, 1]$$

sought for a solution in a form of series and applied Mellin integral transform.

Authors (2021): The multi-term fractional Bessel equation

$$\sum_{i=1}^{m_1} d_i x^{\alpha_i} D^{\alpha_i} u(x) + (x^{\beta} - \nu^2) u(x) = 0, \quad \alpha_i > 0,\ \beta > 0 \tag{1}$$

constructed solutions in the form of series

$$u(x) = \sum_{n=0}^{\infty} c_n x^{\gamma + \beta n} \tag{2}$$

with coefficients c_n found as

$$c_n = \frac{(-1)^n}{\prod_{k=1}^{n} \left(\sum_{i=1}^{m_1} \frac{d_i \cdot \Gamma(1+\gamma+\beta k)}{\Gamma(1+\gamma+\beta k - \alpha_i)} - \nu^2 \right)}. \tag{3}$$

The following characteristic equation allows to find the values of γ

$$\sum_{i=1}^{m_1} \frac{d_i \cdot \Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha_i)} - \nu^2 = 0. \tag{4}$$

Goal: to introduce and analyze the next generalization of multi-term Bessel equations – the quasi-Bessel fractional equations

$$\sum_{i=1}^{m} d_i x^{\alpha_i+p_i} D^{\alpha_i} u(x) + (x^{\beta} - \nu^2) u(x) = 0, \tag{5}$$

where $\alpha_1 = \max_{1\leq i\leq m} \{\alpha_i\}$ and $p_1 = 0$. Unlike the multi-Bessel equations, only the highest derivative D^{α_1} must coincide with the power of x^{α_1}.

Quasi-Bessel equations also generalize Cauchy-Euler and constant-coefficient equations.

1 Construction of fractional series solution

Thus, we consider equation

$$d_1 x^{\alpha_1} D^{\alpha_1} u(x) + \sum_{i=2}^{m} d_i x^{\alpha_i + p_i} D^{\alpha_i} u(x) + (x^\beta - \nu^2) u(x) = 0. \tag{6}$$

Definition. Equation (6) is called quasi-Bessel equation provided that $\alpha_1 = \max\{\alpha_i\}$, $\alpha_i, \beta \in \mathbb{R}^+ = [0, \infty)$, $i \geq 0$.

Particular cases at $\nu = 0$ are quasi-Euler equations

$$d_1 x^{\alpha_1} D^{\alpha_1} u(x) + \sum_{i=2}^{m} d_i x^{\alpha_i + p_i} D^{\alpha_i} u(x) + x^\beta u(x) = 0.$$

We assume that the shifting indices $p_i \geq 0$ and search for the solution to equation (6)

$$u(x) = \sum_{n=0}^{\infty} c_n x^{\gamma + sn}. \tag{7}$$

In the multi-term Bessel equation it was possible to use $s = \beta$ as a step (the increase of powers of x). For essentially more general equation (6) $s = \beta$ fails.

In addition to $p_1 = 0$, several other terms could also have $p_i = 0$. Let us call them the pure Bessel terms. For these terms the power of the factor x^α matches the order of the derivative $D^\alpha u(x)$. Let m_1 be the number of pure Bessel terms in (6). From the definition of quasi-Bessel equations, $p_1 = 0$, which implies $m_1 \geq 1$. The terms with $i = m_1 + 1, ..., m$ have strictly positive shifted powers $p_i > 0$.

We consider both Caputo and Riemann-Liouville derivatives. The only difference is the condition on acceptable γ in the characteristic equation (11) needed to generate a true solution. For Riemann-Liouville derivative $\gamma > -1$ and for Caputo case $\gamma > \lceil \alpha_1 \rceil - 1$ we need to assure the existence of derivatives.

Our nearest goal is to determine the acceptable value of step s. By plugging expression (7) into equation (6), we obtain

$$\sum_{n=0}^{\infty} c_n x^{\gamma+sn} \left(\sum_{i=1}^{m_1} d_i Q(ns, \alpha_i) - \nu^2 + \sum_{i=m_1+1}^{m} x^{p_i} d_i Q(ns + p_i, \alpha_i) + x^{\beta} \right) = 0.$$

Here

$$Q(r, p) = \frac{\Gamma(1 + \gamma + r)}{\Gamma(1 + \gamma + r - p)}. \tag{8}$$

If we choose step s is such that $\dfrac{p_i}{s} = n_{p_i} \in \mathbb{N}$ and $\dfrac{\beta}{s} = n_{\beta} \in \mathbb{N}$, then

$$\begin{aligned} \sum_{n=0}^{\infty} c_n x^{\gamma+sn} \Bigg(& \sum_{i=1}^{m_1} d_i Q(ns, \alpha_i) - \nu^2 \\ & + \sum_{i=m_1+1}^{m} x^{sn_{p_i}} d_i Q(ns + p_i, \alpha_i) + x^{sn_{\beta}} \Bigg) = 0. \end{aligned}$$

Step s should be such that any powers of x are included in the set $\gamma + sn$.

- If β, p_i are rational, then we represent them as irreducible fractions $p_i = \frac{a_i}{b_i}$, $a_i, b_i \in \mathbb{N}$. For $p_i = 0, i > 1$, we set $a_i = 0$, $b_i = 1$.
- Find the lowest common denominator: $N_{lcd} = \text{LCD}\{b_i\}$.
- Calculate the acceptable step and corresponding shifts for β and p_i:

$$s^0 = \frac{1}{N_{lcd}}; \; n_\beta^0 = \frac{\beta^0}{s^0} \in \mathbb{N}; \; n_{p_i}^0 = \frac{p_i^0}{s^0} \in \mathbb{N}, \; m_1 < i \leq m. \tag{9}$$

- The identified parameters β^0, p_i^0, $m_1 < i \leq m$ in (9) can still have common factors. To maximize step s we need to identify their greatest common factor (N_{gcf}), adjust step s and each parameter. Then, finally, we obtain:

$$s = s^0 \cdot N_{gcf}; \; n_\beta = \frac{n_\beta^0}{N_{gcf}}; \; n_{p_i} = \frac{n_{p_i}^0}{N_{gcf}}, \; m_1 < i \leq m. \tag{10}$$

The equation can be re-written as follows:

$$\sum_{n=0}^{\infty} c_n x^{\gamma+sn} \left(\sum_{i=1}^{m_1} d_i Q(ns, \alpha_i) - \nu^2 \right)$$
$$+\sum_{i=m_1+1}^{m} \left(\sum_{n=n_{p_i}}^{\infty} c_n x^{\gamma+sn} d_i Q(ns + p_i, \alpha_i) \right) + \sum_{n=n_\beta}^{\infty} c_n x^{\gamma+sn} = 0.$$

The coefficients for different powers of x must be zeroed. The coefficient for x^γ, i.e. for $n = 0$, must be equal to zero. Then we arrive at the characteristic equation

$$G(\gamma) = \sum_{i=1}^{m_1} \frac{d_i \Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha_i)} - \nu^2 = 0 \tag{11}$$

where $p_i = 0$ for $1 \leq i \leq m_1$. As we see, for quasi-Bessel equations the characteristic equation and its roots γ are determined by the non-deviating terms with $p_i = 0$. This is an unusual and unexpected behavior.

Example. For equation

$$2x^{2.4}D^{2.4}u(x) - 3x^{1.8}D^{1.5}u(x) + xD^{0.4}u(x) + (x^3 - \nu^2)u(x) = 0$$

we have $d_1 = 2$, $d_2 = -3$, $d_3 = 1$, $\alpha_1 = 2.4$, $\alpha_2 = 1.5$, $\alpha_3 = 0.4$. Then $\beta = 3$, $p_2 = 0.3$, $p_3 = 0.6 = \frac{3}{5}$, and we obtain $b_1 = 1$, $b_2 = 10$, $b_3 = 5$, their $N_{lcd} = 10$. Thus,

$$s^0 = \frac{1}{N_{lcd}} = 0.1,\ n^0_\beta = \frac{\beta}{s^0} = 30,\ n^0_{p_2} = \frac{p_2}{s^0} = 3,\ n^0_{p_3} = \frac{p_3}{s^0} = 6.$$

Since $N_{gsf} = \mathrm{GCF}(30, 3, 6) = 3$, then finally,

$$s = s^0 \cdot N_{gsf} = 0.3,\ n_\beta = \frac{30}{3} = 10,\ n_{p_2} = 1,\ n_{p_3} = 2. \qquad \square$$

Back to the characteristic equation (11). In order to satisfy it, coefficient c_n needs to be split into $c_n^{p_i}$ for $m_1 < i \leq m$, and c_n^{β}. They should compensate like terms, the terms which are n_{p_i} steps before the term with the coefficient c_n together with the term which is n_β steps before the term with the same coefficient c_n. These coefficients can be expressed as

$$c_n = -\frac{U(n-n_\beta)c_{n-n_\beta} + \sum_{i=m_1+1}^{m} U(n-n_{p_i})c_{n-n_{p_i}} \cdot d_i Q((n-n_{p_i})s, \alpha_i)}{\sum_{i=1}^{m_1} d_i Q(ns, \alpha_i) - \nu^2} \tag{12}$$

$$Q(r,p) = \frac{\Gamma(1+\gamma+r)}{\Gamma(1+\gamma+r-p)}, \quad n_{p_i} = \frac{p_i}{s}, \; n_\beta = \frac{\beta}{s}.$$

Constant coefficients

$$\sum_{i=1}^{m} d_i D^{\alpha_i} u(x) + u(x) = 0, \quad \alpha_1 > \alpha_i > 0,\ i = 2, ..., m. \tag{13}$$

We multiply each term by x^{α_1}. Then we obtain

$$d_1 x^{\alpha_1} D^{\alpha_1} u(x) + \sum_{i=2}^{m} d_i x^{\alpha_1} D^{\alpha_i} u(x) + (x^{\alpha_1} - 0)u(x) = 0, \tag{14}$$

which is the quasi-Bessel equation with $\nu = 0, \beta = \alpha_1$. Characteristic equation (11) becomes

$$\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha_1)} = 0, \tag{15}$$

and we arrive at roots $\gamma = \alpha_1 - k, k \geq 1$. Since only one term has matching power of x, then in this case the roots are independent of coefficients d_i. The solutions to these equations were previously identified by Kilbas, Srivastava, Trujillo (2006).

Quasi-Euler equations

We assume that β_1, the power of x at the highest derivative α_1, satisfies inequality $\beta_1 \leq \alpha_1$. We consider quasi-Euler equation

$$\sum_{i=1}^{m} d_i x^{\beta_i} D^{\alpha_i} u(x) + x^{\delta} u(x) = 0, \tag{16}$$
$$\alpha_1 > \alpha_i > 0, i = 2, ..., m,\ \ \alpha_1 \geq \beta_1,\ \ \alpha_1 - \beta_1 \geq \alpha_i - \beta_i.$$

We multiply each term by $x^{\alpha_1 - \beta_1}$ and obtain

$$d_1 x^{\alpha_1} D^{\alpha_1} u(x) + \sum_{i=2}^{m} d_i x^{\alpha_1 - \beta_1 + \beta_i} D^{\alpha_i} u(x) + (x^{\alpha_1 - \beta_1 + \delta}) u(x) = 0. \tag{17}$$

In this case $\nu = 0, \beta = \alpha_1 - \beta_1 + \delta$.

Theorem 1. Caputo derivatives
Let α_1 be fractional and m_1, $1 \le m_1 < m$, be the number of pure Bessel terms with $p_i = 0$.
Let ν satisfy the threshold inequality

$$\nu^2 \ge \nu_{\min} = \Gamma(\lceil \alpha_1 \rceil) \sum_{i=1}^{m_1} \frac{d_i}{\Gamma(\lceil \alpha_1 \rceil - \alpha_i)}. \tag{18}$$

Then there exists a unique series solution (7), (12) for fractional equation (6) with Caputo derivatives in any domain $x \in [0, b], b \in \mathbb{R}_+$.
If $\nu = 0$ and $\beta \ge \lceil \alpha_1 \rceil$ then at least one solution in the form of series can always be found.

Remark 1. *If there exists n such that $\gamma + sn$ is another root of* (11) *with step s defined in* (10)*, then γ does not generate solution* (7) *for equation* (6) *because in this case the series is divergent.*

It can happen when $\nu = 0$ and only $p_1 = 0$ among all p_i. Then the difference between the γ roots is exactly one. If step s is a fraction of one, the smaller γ root at some step falls onto a bigger γ root, and the series blows up.

Example 2. Equation

$$x^{1.5}D^{1.5}u(x) + x^{0.7}D^{0.5}u(x) + x^{1.2}u(x) = 0$$

generates characteristic equation

$$\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-1.5)} = 0.$$

In this case $\beta = \dfrac{6}{5}$, $p_2 = \frac{1}{5}$, $\nu = 0$. Then $N_{lcd} = 5, s = \dfrac{1}{5}, n_\beta = 6, n_{p_2} = 1$. The characteristic equation has roots $\gamma_1 = -0.5, \gamma_2 = 0.5$. Therefore $\gamma_2 = \gamma_1 + 5s$, which means that for $n = 5$

$$Q(5s, \alpha_1) = \frac{\Gamma(1+\gamma_1+5s)}{\Gamma(1+\gamma_1+5s-\alpha_1)} = \frac{\Gamma(1-0.5+1)}{\Gamma(1-0.5+1-1.5)} = \frac{\Gamma(1.5)}{\Gamma(0)} = 0.$$

Since $Q(5s, \alpha_1)$ is the denominator of c_5 in (12) and makes blow-up $c_5 = \infty$, then γ_1 does not generate a solution in the form of proposed series. □

Thus, if characteristic equation (11) has several roots γ, then all the roots, except for the largest root, need to be checked for validity.

Theorem 2. Series solution (7) with coefficients (12) of fractional quasi-Bessel equation (6) with $p_1 = 0$, $d_i > 0$, $1 \leq i \leq m_1$, converges and represents the solution to (6) provided that the threshold condition (18) for ν is satisfied in the equations with Caputo derivatives. No such threshold condition is required for the equations with Riemann-Liouville derivatives.

If $p_1 > 0$ but for some $i > 1$ there exists at least one $p_i = 0$, then the series diverges and a series solution in form (7) does not exist for equations with both Caputo and Riemann-Liouville derivatives. □

Remark 2. *For equations with Caputo derivative, based on conditions in Theorem 1*

- *If* $p_1 = 0$, $\nu > 0$ *and* α_1 *is fractional then the found series solution is unique up to a constant,*
- *It* $p_1 = 0$ *and* α_1 *is integer, equation* (6) *may have multiple solutions.*

Remark 3. *The root* γ *in the solution, which is calculated in* (11), *depends solely on the terms in equation* (6) *with* $p_i = 0$.

Theorem 3 (Uniqueness) We assume that the initial value problem for fractional equation (6) in domain $x \in [0, b]$ with Caputo derivatives and initial conditions $u^{(j)}(0) = u_0^{(j)}$, $j = 0, 1, ..., \lceil \alpha_1 - 1 \rceil$, has a continuous solution.
Let

$$\nu^2 > b^\beta + \sum_{i=1}^{m} q_i |d_i| b^{n_i + p_i}, \tag{19}$$

where

$$q_i = \begin{cases} \dfrac{1}{\Gamma(n_i - \alpha_i)(n_i - \alpha_i + 1)} & , \alpha_i < n_i \\ 1 & , \alpha_i = n_i \end{cases}. \tag{20}$$

The proof is close to that by Rodrigues, Viera, and Yakubovich (2013).

Example 3. (quasi-Bessel equation with Caputo derivatives).
Let us consider equation

$$1.5x^{1.5}D_C^{1.5}u(x) - 1.2x^{1.9}D_C^{1.1}u(x) + 3xD_C^{0.5}u(x) + (x^2 - \nu^2)u(x) = 0. \tag{21}$$

Here $\beta = 2, d_1 = 1.5, d_2 = -1.2, d_3 = 3, \alpha_1 = 1.5, \alpha_2 = 1.1, \alpha_3 = 0.5,$
$p_2 = 0.8, p_3 = 0.5.$

Characteristic equation becomes:

$$G(\gamma) = \frac{1.5\Gamma(1+\gamma)}{\Gamma(1+\gamma-1.5)} - \nu^2 = 0. \tag{22}$$

The graph of the expression on the left side of equation (22) is in Figure 1. It is the same for any ν except for the ν^2 shift down difference. To satisfy equation (22) for $\nu = 2$ we find $\gamma = 2.1995$; for $\nu = 3.5$ we have $\gamma = 4.3181$.

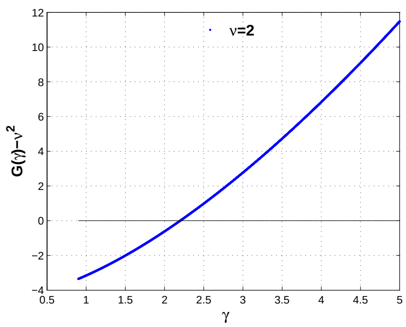

Figure 1: Function $G(\gamma) - \nu^2$ for equation (21) with $\nu = 2$.

Other parameters involved in the process as described before are:

- Since all $p_i, \beta \in \mathbb{Q}^+$, we get $p_2^0 = p_2 = 0.8 = \dfrac{4}{5}$;
  $p_3^0 = p_3 = 0.5 = \dfrac{1}{2}$; $\quad s^0 = \beta = 2 = \dfrac{2}{1}$.
- The lowest common denominator $N_{lcd} = \text{LCM}\{5, 2, 1\} = 10$.
- $s = \dfrac{1}{N_{lcd}} = \dfrac{1}{10} = 0.1, n_{p_2} = \dfrac{p_2}{s} = \dfrac{0.8}{0.1} = 8, n_{p_3} = \dfrac{p_3}{s} = \dfrac{0.5}{0.1} = 5,$
  $n_\beta = \dfrac{\beta}{s} = \dfrac{2}{0.1} = 20,\ N_{gcf} = 1.$

The solutions are represented in Figure 2. The red line is the recalculation of equation (21) by plugging in the calculated solution $u(x)$ into the equation, this line shows that the error is very close to zero.

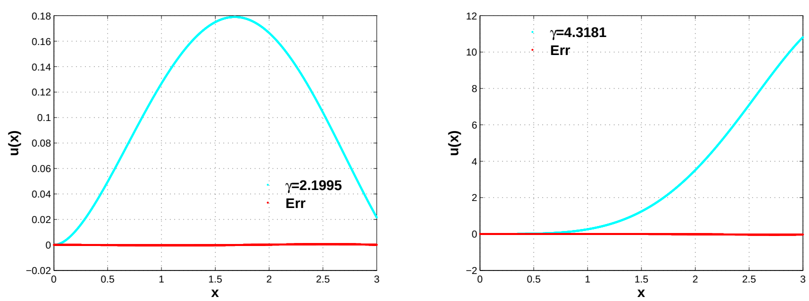

Figure 2: Solution for equation in Example 3. Red line close to zero is the check for the accuracy of the solution. Step $h = 0.001$.

It is important to point out that the closer ν is to the minimum threshold, the less accurate the result is due to the loss of accuracy in the calculation of fractional derivative.

Example 4. (constant coefficients, integer derivatives).

Equation $u' + u = 0$ is converted into $xu' + xu = 0$. The series solution has form $u(x) = \sum_{n=0}^{\infty} c_n x^{sn}$, characteristic equation

$$\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-1)} = 0$$

has root $\gamma = 0$. Based on (10), step $s = 1$. Therefore, we get

$$c_n = c_n^{\beta} = -\frac{c_{n-1}}{Q(n,1)} = -c_{n-1}\frac{\Gamma(1+n-1)}{\Gamma(1+n)} = -\frac{c_{n-1}}{n} = c_0\frac{(-1)^n}{n!},$$

and the solution as expected is

$$u(x) = \sum_{n=0}^{\infty} c_n x^{sn} = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = c_0 e^{-x}. \qquad \square$$

Example 5. Equation

$$d_1 D_R^{2.1} u(x) + d_2 D_R^{1.4} u(x) + d_3 D_R^{0.7} u(x) + u(x) = 0,$$

should be rewritten in the quasi-Bessel form as

$$d_1 x^{2.1} D_R^{2.1} u(x) + d_2 x^{1.4+0.7} D_R^{1.4} u(x) + d_3 x^{0.7+1.4} D_R^{0.7} u(x) \quad + (x^{2.1} - 0) u(x) = 0.$$

Then $\alpha_1 = 2.1$, $p_2 = 0.7$, $p_3 = 1.4$, $\beta = 2.1$ have the greatest common factor $s = 0.7$, which serves as the step in the fractional series. The corresponding characteristic equation has three roots: $\gamma_1 = -0.9, \gamma_2 = 0.1$ and $\gamma_3 = 1.1$. Since $\gamma_2 \neq \gamma_1 + sn, \gamma_3 \neq \gamma_1 + sn, \gamma_3 \neq \gamma_2 + sn$ for any $n \in \mathbb{N}$, then, based on these roots, we can construct three different solutions

$$u_1(x) = \sum_{n=0}^{\infty} c_n x^{-0.9+sn}; \quad u_2(x) = \sum_{n=0}^{\infty} c_n x^{0.1+sn}; \quad u_3(x) = \sum_{n=0}^{\infty} c_n x^{1.1+sn}. \tag{23}$$

In the case of a similar, almost the same constant-coefficient equation (1.5 instead of 1.4)

$$d_1 D_R^{2.1} u(x) + d_2 D_R^{1.5} u(x) + d_3 D_R^{0.7} u(x) + u(x) = 0, \tag{24}$$

which turns into

$$d_1 x^{2.1} D_R^{2.1} u(x) + d_2 x^{1.5+0.6} D_R^{1.5} u(x) + d_3 x^{0.7+1.4} D_R^{0.7} u(x) + x^{2.1} u(x) = 0 \tag{25}$$

with $\alpha_1 = 2.1, \quad p_2 = 0.6, \quad p_3 = 1.4, \quad \beta = 2.1$, we obtain the same three characteristic roots $\gamma_1 = -0.9, \gamma_2 = 0.1$ and $\gamma_3 = 1.1$. However, unlike the previous equation with $s = 0.7$, the greatest common factor of $\alpha_1 = 2.1, \quad p_2 = 0.6, \quad p_3 = 1.4, \quad \beta = 2.1$ is now equal to 0.1 and, thus, $s = 0.1$. In this case γ_1 tread upon γ_2 in the 10th step, γ_2 "set foot on" γ_3 in its 10th step: $\gamma_2 = \gamma_1 + 10s$, $\gamma_3 = \gamma_2 + 10s$, and we have two blow-ups thanks to $c_{10} = \infty$ in both cases. Consequently, neither γ_1 nor γ_2 represent a root which can be used to generate a series solution in the proposed form.

The highest root $\gamma_3 = \alpha_1 - 1$ ($\gamma_3 = 1.1$ in our example) does not generate a blow-up of the series.

Conclusions

Quasi-Bessel and quasi-Euler equations have the same orders of the highest derivative and the corresponding power function. The other terms $x^{\alpha+p}D^{\alpha}u(x)$, $p \geq 0$, cover a broad class of fractional differential equations. The powers of x may avoid matching the order of the corresponding fractional derivatives (except for the highest derivative), whereas such a match is usually required for the successful applications of integral transforms.

If the deviating (shifting) parameters p are non-negative and meet certain non-restrictive conditions, then we construct the existence theory for quasi-Bessel equations in the class of fractional series solutions. The presented theory is based on the unusual characteristic equation, whose roots γ, the power parameters in the series presentation of the solutions, are independent of the addends $x^{\xi}D^{\alpha}u$ with non-matching values $\alpha \neq \xi$.

If the power factors x have an order less than the order of the corresponding derivatives, then developing the theory for such quasi-Bessel equations still remains an open problem.

Dubovski and Slepoi, Construction and analysis of series solutions for fractional quasi-Bessel equations, Fract. Calc. Appl. Anal. 25, pages 1229–1249 (2022)