CONVOLUTION EQUATIONS ON LIE GROUPS, GENERIC BESSEL POTENTIAL SPACES AND FUNDAMENTAL SOLUTIONS

Roland Duduchava

V. Kupradze Institute of Mathematics, University of Georgia

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1. CONVOLUTIONS ON LIE GROUPS

There is given a Lie group G: a manifold with a binary operation $x \circ y \ : \ G \times G \to G$, the neutral element e, $x \circ e = e \circ x = x$, $x \in G$, and each $x \in G$ has the inverse $x^{-1}x = xx^{-1} = e$. We concentrate here on groups which are diffeomorphic with the Euclidean space $\omega \ : \ G \to \mathbb{R}^m$.

Then we have:

- **The Haar measure $d_G\mu$ defined uniquely;**
- **The Fourier transformation (see the next slide for definitions)**

$$\mathscr{F}_G \ : \ \mathbb{S}(G) \to \mathbb{S}(\mathbb{R}^m), \qquad \mathscr{F}_G \ : \ \mathbb{S}'(G) \to \mathbb{S}'(\mathbb{R}^m)$$

  with its inverse Forier transformation $\mathscr{F}_G^{-1} \ : \ \mathbb{S}(\mathbb{R}^m) \to \mathbb{S}(G)$, $\mathscr{F}_G^{-1} \ : \ \mathbb{S}'(\mathbb{R}^m) \to \mathbb{S}'(G)$ and the corresponding Plancherel theorem $\|\mathscr{F}_G\varphi\|_2 = (2\pi)^{n/2}\|\varphi\|_2$.
- **The generic differential operators $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$, generated by vector fields from the corresponding Lie algebra.**

- **Convolution operators, defined as follows**

$$W^0_{a,G} := \mathscr{F}_G^{-1} a \mathscr{F}_G \;:\; \mathbb{S}(G) \to \mathbb{S}'(G), \qquad (0.1)$$

where the symbol $a(\xi)$ **is a distribution on the dual group** $a \in \mathbb{S}'(\mathbb{R}^m)$**.**

The Schwartz space $\mathbb{S}(G)$ **and its dual space of distributions** $\mathbb{S}'(G)$ **represent the pull-back spaces of fast decaying infinitely smooth functions** $\mathbb{S}(\mathbb{R}^m)$ **and its dual space of distributions** $\mathbb{S}'(\mathbb{R}^m)$**:**

$$\begin{aligned} \omega_* &: \mathbb{S}(\mathbb{R}^m) \to \mathbb{S}(G), \quad \textbf{where} \quad \omega_*\varphi(x) := \varphi(\omega(x)), \quad \varphi \in \mathbb{S}(\mathbb{R}^m), \\ \omega_* &: \mathbb{S}'(\mathbb{R}^m) \to \mathbb{S}'(G), \quad \omega \;:\; G \to \mathbb{R}^m, \quad x \in G. \end{aligned}$$

Note, that as in the classical case the Fourier transformation and its inverse are bounded operators in the Schwartz spaces

$$\begin{aligned} \mathscr{F}_G^{\pm 1} &: \mathbb{S}(G) \to \mathbb{S}(G), \\ &: \mathbb{S}'(G) \to \mathbb{S}'(G). \end{aligned}$$

1. CONVOLUTIONS ON LIE GROUPS

What is important: all generic differential operators are convolutions $\mathfrak{D}_k = W^0_{d_k,G}$ with polynomial symbols $d_k(\xi) = -i\xi_k$, $k = 1, \ldots, m$.

Thus, we can solve easily the following integro-differential convolution equations and find their precise essential spectra

$$\sum_{|\alpha|+|\beta|,|\gamma|\leqslant n} \left[c_{\alpha,\beta}\mathfrak{D}^\gamma\varphi(x) + \mathfrak{D}^\alpha \int_G k_{\alpha,\beta}(x \circ y^{-1})\mathfrak{D}^\beta\varphi(y)\boldsymbol{d}_G\mu(y) \right] = f(y). \quad (1.1)$$

The symbol has polynomial growth

$$a(\xi) := \sum_{|\alpha|+|\beta|,|\gamma|\leqslant n} \left[c_{\alpha,\beta}(-i\xi)^\gamma + (-i\xi)^{\alpha+\beta}\mathscr{F}_G k_{\alpha,\beta}(\xi) \right], \quad \xi \in \mathbb{R}^m. \quad (1.2)$$

1. CONVOLUTIONS ON LIE GROUPS

By $\mathfrak{M}_p(\mathbb{R}^m)$ we denote the algebra of functions ($\mathbb{L}_p$-multipliers) such that the corresponding convolution operator is bounded in the space $\mathbb{L}_p(G, \boldsymbol{d}_G\mu)$:

$$\boldsymbol{W}^0_{a,G} \ : \ \mathbb{L}_p(G, \boldsymbol{d}_G\mu) \to \mathbb{L}_p(G, \boldsymbol{d}_G\mu), \qquad 1 < p < \infty.$$

Some informationn about the multipliers classes $\mathfrak{M}_p(\mathbb{R}^m)$:

- **Due to the Plancherel theorem, $\mathfrak{M}_2(\mathbb{R}^m) = \mathbb{L}_\infty(\mathbb{R}^m)$, i.e., for $a \in \mathbb{L}_\infty(\mathbb{R}^m)$ the operator $W^0_{a,G}$ is bounded in $\mathbb{L}_2(G, \boldsymbol{d}_G\mu)$.**
- **Functions of the Wiener class $a(\xi) = c_0 + \mathscr{F}_G k(\xi)$, where $k \in \mathbb{L}_1(G, d_G\mu)$, belong to $\mathfrak{M}_p(\mathbb{R}^m)$ for all $1 \leqslant p \leqslant \infty$.**
- **in 1-dimensional case all functions of bounded variation belong to $\mathfrak{M}_p(\mathbb{R})$ for all $1 < p < \infty$.**

1. CONVOLUTIONS ON LIE GROUPS

- **All piecewise-constant functions PC$(\boldsymbol{R}^m)$ on politops (intersections of finite number of hypersurfaces) belong to $\mathfrak{M}_p(\mathbb{R}^n)$ for all $1 < p < \infty$.**

PC$_p(\boldsymbol{R}^m)$ denotes the closure of PC$(\boldsymbol{R}^m)$ in the algebra $\mathfrak{M}_p(\mathbb{R}^m)$.

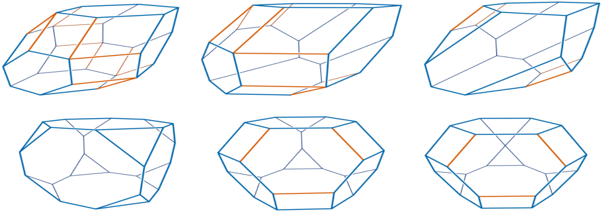

Fig. 1

Arises a natural question: which spaces are most relevant to consider solvability of the convolution integro-differential equation (1.2)?

2. GENERIC BESSEL POTENTIAL SPACES

For this purpose we define the generic Bessel potential spaces $\mathbb{GH}^s_p(G, \boldsymbol{d}_G\mu)$ on the Lie group G, endowed with the norm

$$\|\psi \,|\, \mathbb{GH}^s_p(G, \boldsymbol{d}_G\mu)\| := \|W^0_{\langle\xi\rangle^s}\psi \,|\, \mathbb{L}_p(G, \boldsymbol{d}_G\mu)\|$$
$$= \|\mathscr{F}_G^{-1}\langle\xi\rangle^s\mathscr{F}_G\psi \,|\, \mathbb{L}_p(G, \boldsymbol{d}_G\mu)\| \quad \langle\xi\rangle^s := (1 + |\xi|^2)^s, \quad s \in \mathbb{R}. \qquad (2.1)$$

For an integer $s = n = 1, 2, \ldots$ the space $\mathbb{GH}^n_p(G, \boldsymbol{d}_G\mu)$ is isomorphic to the generic Sobolev space $\mathbb{GW}^n_p(G, \boldsymbol{d}_G\mu)$, defined with the help of generic differential operators, endowed with the norm

$$\|\varphi \,|\, \mathbb{GW}^n_p(G, \boldsymbol{d}_G\mu)\| := \left[\sum_{|\alpha|\leqslant n} \|\mathfrak{D}^\alpha\varphi \,|\, \mathbb{L}_p(G, \boldsymbol{d}_G\mu)\|^p\right]^{1/p}.$$

Theorem

Let $1 < p < \infty$, $s \in \mathbb{R}$. The multiplier class for the space $\mathbb{GH}^s_p(G, \boldsymbol{d}_G\mu)$ is independent of s and coincides with the multiplier class $\mathfrak{M}_p(\mathbb{R}^m)$

2. GENERIC BESSEL POTENTIAL SPACES

Let $\mathfrak{M}^r_p(\mathbb{R}^m)$, $1 < p < \infty$, $-\infty < r < \infty$ denote those functions (r-multipliers) for which the convolution operator of order r is bounded in the setting

$$W^0_{a,G} \ : \ \mathbb{GW}^s_p(G, \boldsymbol{d}_G\mu) \to \mathbb{GW}^{s-r}_p(G, \boldsymbol{d}_G\mu).$$

The r-multipliers class $\mathfrak{M}^r_p(\mathbb{R}^m)$ is, naturally, independent of the smoothness parameter s. Moreover, there holds the following:

$$\mathfrak{M}^r_p(\mathbb{R}^m) := \{\langle\xi\rangle^r a(\xi) \ : \ a \in \mathfrak{M}_p(\mathbb{R}^m)\}, \qquad \langle\xi\rangle^r = (1 + |xi|^2)^{1/2},$$

$$\mathbf{PC}^r_p(\mathbb{R}^m) := \{\langle\xi\rangle^r a(\xi) \ : \ a \in \mathbf{PC}_p(\mathbb{R}^m)\}, \qquad r \in \mathbb{R}.$$

Theorem

Let $1 < p < \infty$**,** $s \in \mathbb{R}$**. A convolution integro-differential equation**

$$\sum_{|\alpha|+|\beta|,|\gamma|\leqslant n} \left[c_{\alpha,\beta}\mathfrak{D}^{\gamma}\varphi(x) + \mathfrak{D}^{\alpha}\int_G k_{\alpha,\beta}(x \circ y^{-1})\mathfrak{D}^{\beta}\varphi(y)\boldsymbol{d}_G\mu(y) \right] = f(y). \quad (2.2)$$

with the symbol $a \in \mathfrak{M}_p^n(\mathbb{R}^m)$**, (see** (1.2)**) is Fredholm in the setting**

$$\varphi \in \mathbb{GH}_p^s(G, \boldsymbol{d}_G\mu), \qquad f \in \mathbb{GH}_p^{s-n}(G, \boldsymbol{d}_G\mu) \quad (2.3)$$

if only the symbol is elliptic

$$\inf_{\xi\in\mathbb{R}^m} |\langle\xi\rangle^{-n} a(\xi)| > 0. \quad (2.4)$$

Theorem continued

Moreover, if $p = 2$ or $p \neq 2$ but $a \in \mathbf{PC}^r_p(\mathbb{R}^m)$, the ellipticity condition (2.4) is sufficient for equation (3.2) to have a unique solution $\varphi = \boldsymbol{W}^0_{G,a^{-1}} f \in \mathbb{GH}^s_p(G, \boldsymbol{d}_G\mu)$ for arbitrary $f \in \mathbb{GH}^{s-n}_p(G, \boldsymbol{d}_G\mu)$ (i.e., the operator $W^0_{G,a}$ is invertible).

The operator has only the essential spectrum and

$$\mathbf{Sp}\, W^0_{a,G} := \{z \in \mathbb{C}\ :\ z = a(\xi)\ :\ -\infty < \xi, \infty\}.$$

Remark

Note, that due to the "Igari Paradox" the ellipticity condition (2.4) can not be sufficient for the Fredholm property of equation (1.1): For $p \neq 2$ there exist even continuous elliptic multipliers $a \in \mathfrak{M}_p(\mathbb{R}^m) \cap C(\mathbb{R}^m)$ such that $W^0_{a,G}$ is not bounded in $\mathbb{GH}^s_p(G, \boldsymbol{d}_G\mu)$.

3. FUNDAMENTAL SOLUTION

Theorem

Let $1 < p < \infty$**. For an integro-differential convolution operator** $W^0_{G,a}$ **with the symbol** $a \in \boldsymbol{PC}^m_p(G)$ **there exists a kernel** $\mathscr{K}_a$ **(a distribution in general) such that the operator is written as a group convolution with this kernel**

$$W^0_{G,a}\varphi(x) := \mathscr{K}_a \star_G \varphi(x) = \int_G \mathscr{K}_a(x \circ y^{-1})\varphi(y) d_G\mu, \quad \varphi \in \mathbb{S}(G). \tag{3.1}$$

Note, that the G**-convolution** $\mathscr{K}_a \star_G \varphi$ **of a distribution** $\mathscr{K}_a \in \mathbb{S}'(G)$ **with a test function** $\varphi \in \mathbb{S}(G)$ **is a correctly defined operation.**

L. Hörmander proved the above theorem for the case $\{G, \circ\} = \{\mathbb{R}^n, +\}$ **(cf. [Hr60]). In our case the Theorem remains valid due to homeomorphism of** G **and** $\mathbb{R}^n$**.**

3. FUNDAMENTAL SOLUTION

Theorem

Distributional Hörmander's kernel $\mathscr{K}_{\boldsymbol{A}^{-1}}$ of the inverse to a generic differential operator with constant coefficients

$$\boldsymbol{A}(\mathfrak{D})\varphi(x) = \sum_{|\alpha|\leqslant n} c_\alpha \mathfrak{D}^\alpha \varphi(x) \tag{3.2}$$

and elliptic symbol $\inf_{\xi\in\mathbb{R}^n} \det|\mathscr{A}(\xi)| > 0$**, represents the Fundamental solution** $\boldsymbol{F}_{\boldsymbol{A}}(x) = \mathscr{K}_{\boldsymbol{A}^{-1}}$ **for the operator** $\boldsymbol{A}(fD)$**, i.e.** $\boldsymbol{A}\boldsymbol{F}_{\boldsymbol{A}^{-1}} = \delta$ **and coincides with the Hórmander's kernel of the inverse operator.**

The fundamental solution represents the inverse Fourier transform of the inverse symbol $\boldsymbol{F}_{\boldsymbol{A}} = \mathscr{F}_G^{-1}\mathscr{A}^{(-1)}$**.**

3. FUNDAMENTAL SOLUTION

Indeed, the symbol of the operator $A(\mathfrak{D})$ has polynomial growth

$$\mathscr{A}(\xi) := \sum_{|\alpha|\leqslant n} c_\alpha(-i\xi)^\alpha, \qquad \xi \in \mathbb{R}^m \tag{3.3}$$

and if the symbol is elliptic, the inverse operator is a convolution with the symbol $\mathscr{A}^{-1}(\xi)$, i.e.,

$$\boldsymbol{A}^{-1}(\mathfrak{D})\psi(x) = W^0_{\mathscr{A}^{-1}}\psi(x) = (\boldsymbol{F}_A *_G \psi)(x) = \int_G \boldsymbol{F}_{\boldsymbol{A}}(x \circ y^{-1})\psi(y)d_G y,$$

where $\boldsymbol{F}_{\boldsymbol{A}}(x)$ is the distributional Hörmander's kernel.

$\boldsymbol{F}_{\boldsymbol{A}}(x)$ is the Fundamental solution for $\boldsymbol{A}$ because $\psi = \boldsymbol{A}\boldsymbol{A}^{-1}\psi = \boldsymbol{A}\boldsymbol{F}_{\boldsymbol{A}} *_G \psi$ and, consequently, $\boldsymbol{A}\boldsymbol{F}_{\boldsymbol{A}} = \delta$.

The formula $\boldsymbol{F}_{\boldsymbol{A}} = \mathscr{F}_G^{-1}\mathscr{A}^{(-1)}$ follows from the formula $\mathscr{A}^{(-1)} = \mathscr{F}\mathscr{K}_{\boldsymbol{A}^{-1}}$.

3. FUNDAMENTAL SOLUTION

With the fundamental solution F_A for the generic differential operator A at hand we can define the Newtons, Single layer and double layer potential operators for a domain $\Omega \subset G$

$$\begin{aligned} \boldsymbol{N}_\Omega\psi(x) &= \int_\Omega \boldsymbol{F}_{\boldsymbol{A}}(x-y)\psi(y)\boldsymbol{d}_G\mu(y), \\ \boldsymbol{V}_\Omega\varphi(x) &= \int_{\partial\Omega} \boldsymbol{F}_{\boldsymbol{A}}(x-\tau)\varphi(\tau)\boldsymbol{d}_{\partial\Omega}\mu(\tau), \\ \boldsymbol{W}_\Omega\varphi(x) &= \int_{\partial\Omega} \partial_{\boldsymbol{\nu}}(x)\boldsymbol{F}_{\boldsymbol{A}}(x-y)\varphi(\tau)\boldsymbol{d}_{\partial\Omega}\mu(\tau), \end{aligned} \tag{3.4}$$

where $\boldsymbol{d}_{\partial\Omega}\mu$ is the induced measure on the boundary $\partial\Omega$ of Ω. There holds the following Gauss formula for the domain

$$\int_\Omega \mathscr{D}_j\psi(y)\boldsymbol{d}_G\mu(y) = \int_{\partial\Omega} \nu_j(\tau)\psi(\tau)\boldsymbol{d}_{\partial\Omega}\mu(\tau), \qquad j = 1, \dots, n, \tag{3.5}$$

where $\nu_j(t)$ is the component of the outer normal vector field $\boldsymbol{\nu}(t) = (\nu_1(t), \dots, \nu_n(t))^\top$ on the boundary $\partial\Omega$.

From the Gauss formula (3.5) **there follow I and II Green's formulae for the generic Laplace operator** $\Delta_G := \mathscr{D}_1^2 + \cdots + \mathscr{D}_n^2$**:**

$$\int_\Omega (\Delta_G \varphi)(y)\psi(y)\boldsymbol{d}_G\mu(y) = \int_{\partial\Omega} (\mathscr{D}_{\boldsymbol{\nu}}\varphi)(\tau)\psi(\tau)\boldsymbol{d}_{\partial\Omega}\mu(\tau)$$
$$- \int_\Omega (\mathscr{D}_j(\tau)\varphi)(\tau)(\mathscr{D}_j(\tau)\psi)(\tau)\boldsymbol{d}_{\partial\Omega}\mu(\tau),$$
$$\int_\Omega (\Delta_G \varphi)(y)\psi(y)\boldsymbol{d}_G\mu(y) - \int_\Omega \varphi(y)(\Delta_G\psi)(y)\boldsymbol{d}_G\mu(y)$$
$$= \int_{\partial\Omega} (\mathscr{D}_{\boldsymbol{\nu}}\varphi)(\tau)\psi(\tau)\boldsymbol{d}_{\partial\Omega}\mu(\tau) - \int_{\partial\Omega} \varphi(\tau)(\mathscr{D}_{\boldsymbol{\nu}}\psi)(\tau)\boldsymbol{d}_{\partial\Omega}\mu(\tau),$$
$$(\mathscr{D}_{\boldsymbol{\nu}}\varphi)(t) := \nu_1(t)(\mathscr{D}_1\varphi)(t) + \cdots + \nu_n(t)(\mathscr{D}_n\varphi)(t)$$

Green's I and II formulas can be derived for any other high order differential operator (see [Du01] where the classical cases are considered).

3. FUNDAMENTAL SOLUTION

For a mixed type boundary value problem (BVP) for arbitrary elliptic second order equation

$$\begin{cases} (\boldsymbol{A}u)(x) := \sum_{|\alpha|\leqslant 2}(\mathscr{D}^\alpha u)(x) = f(t,x), & x\in\Omega, t\in\mathbb{R}^+ = (0,\infty), \\ u(t,\omega) = g(t,\omega), \quad (\partial_{\boldsymbol{\nu}} u)^+(t,\omega) = h(t,\omega), \omega\in\partial\Omega, \end{cases} \tag{3.6}$$

we have the following representatio formula for a solution

$$u(x) = \boldsymbol{N}_\Omega f(x) + (\boldsymbol{W}_\Omega u^+)(x) - (\boldsymbol{V}_\Omega(\partial_{\boldsymbol{\nu}} u)^+)(x), \qquad x\in\Omega,$$

which can be used to derive boundary pseudodifferential equation for the BVP (3.6)**.**

EXAMPLE 1 (cf. [Du25]). The direct product of positive half axes $\mathbb{R}^n_+ = (0,\infty)^n$ is a Lie group with the group operation $x \circ y = (x_1 y_1, \ldots, x_n y_n)$, with the dual $\widehat{\mathbb{R}^n_+} = \mathbb{R}^n$ and the Haar measure $\dfrac{dx_1}{x_1} \cdots \dfrac{dx_n}{x_n}$. The group Fourier transformation on $\mathbb{R}^n_+$ coincides with the Mellin transformation (see below Example 4) and the generic differential operators in $\mathbb{R}^n_+$ are

$$\mathfrak{D}_k = x_k \partial_k, \qquad k = 1, 2, \ldots, n.$$

The fundamental solution for the Generic Laplacian $\Delta_{\mathbb{R}^n_+} = \mathscr{D}_1^2 + \cdots + \mathscr{D}_n^2$ is

$$\mathscr{K}_{\Delta_{\mathbb{R}^n_+}}(x) := \begin{cases} \dfrac{1}{4\pi} \ln(\ln^2 x_1 + \ln^2 x_2), & \text{if} \quad n = 2, \\ \dfrac{\Gamma(n/2)}{2(2-n)\pi^{n/2}} (\ln^2 x_1 + \cdots + \ln^2 x_n)^{\frac{2-n}{2}} & \text{if} \quad n > 2. \end{cases} \tag{4.1}$$

EXAMPLE 2 (cf. [Du25]). The direct product of intervals $I^n = (-1,1)^n$ is a Lie group with the group operation

$$x \circ y = \left(\frac{x_1 + y_1}{1 + x_1 y_1}, \dots, \frac{x_1 + y_1}{1 + x_1 y_1} \right), \qquad x, y \in I^n$$

with the Haar measure $\dfrac{\boldsymbol{d} x_1}{1 - x_1^2} \cdots \dfrac{\boldsymbol{d} x_n}{1 - x_n^2}$ **and the generic differential operators:**

$$\mathfrak{D}_k = (1 - x_k^2)\partial_k, \qquad k = 1, 2, \dots, n.$$

Then the fundamental solution for the Generic Laplacian $\Delta_{\mathbb{R}^n_+} = \mathscr{D}_1^2 + \cdots + \mathscr{D}_n^2$ **is**

$$\mathscr{K}_{\Delta_{\mathbb{R}^n_+}}(x) := \begin{cases} \dfrac{1}{4\pi} \ln \dfrac{1}{4} \left[\ln^2 \dfrac{1 + x_1}{1 - x_1} + \ln^2 \dfrac{1 + x_2}{1 - x_2} \right], & \text{if } n = 2, \\ \dfrac{2^{n/2-2}\Gamma(n/2)}{(2-n)\pi^{n/2}} \left[\ln^2 \dfrac{1 + x_1}{1 - x_1} + \cdots + \ln^2 \dfrac{1 + x_n}{1 - x_n} \right]^{\frac{2-n}{2}} & \text{if } n > 2. \end{cases} \tag{4.2}$$

EXAMPLE 3: Euclidean space $\mathbb{R}^m$

The most trivial example of the Lie group is the Euclidean space $\mathbb{R}^m$ with the sum as the binary operation

$$x \circ y = x + y, \qquad e = 0, \qquad x^{-1} = -x, \qquad x, y \in \mathbb{R}^m.$$

The Haar measure is the Lebesgue measure, the Fourier transform is the classical Fourier transform $\mathscr{F}$, generic differential operators are coordinate derivatives

$$\partial_k := \frac{\partial}{\partial x_k}, \qquad k = 1, \dots, m$$

and convolutions are classical

$$W_a^0 \varphi(x) = c_0 \varphi(x) + \int_{\mathbb{R}^n} k(x-y)\varphi(y)dy, \qquad a(\xi) = c_0 + (\mathscr{F}k)(\xi).$$

The generic Bessel Potential Space is the classical BPS.

5. EXAMPLES OF LIE GROUPS

EXAMPLE 4: Half line $\mathbb{R}^+$: is a Lie group with the group operation as the usual multiplication $x \circ y = xy$, with the Haar measure $\dfrac{dx}{x}$.
The group Fourier transformation on $\mathbb{R}_+$ is the Mellin transformation

$$\mathscr{M}_\beta \psi(\xi) := \int_{\mathbb{R}^+} t^{i\xi-\beta}\psi(t)\,\frac{dt}{t}, \quad \xi \in \mathbb{R},$$

The corresponding convolution equation is ($g(\xi) = c_0 + \mathscr{M}_\beta k(\xi)$)

$$\mathfrak{M}^0_g \varphi(t) = c_0\varphi(t) + \int_{\mathbb{R}^+} k\left(\frac{t}{\tau}\right)\varphi(\tau)\,\frac{d\tau}{\tau} = f(t), \quad t \in \mathbb{R}^+.$$

The generic differential operator is

$$\mathfrak{D} = t\frac{d}{dt} = \boldsymbol{W}^0_{-i\xi}.$$

Mellin convolution equations have many applications in Mathematical Physics, Mechanics etc.

5. EXAMPLES OF LIE GROUPS

EXAMPLE 5: The interval $G := (-1, 1)$: If we endow the interval $G := (-1, 1)$ with the binary operation

$$x \circ y := \frac{x + y}{1 + xy}, \qquad x, y \in G,$$

which is an automorphism of $G = (-1, 1)$, it makes G a Lie group with the neutral element $e = 0$ and $-x$ the inverse to $x \in G$. The invariant Haar measure on G is $d_G x := \dfrac{dx}{1 - x^2}$ and the invariance means

$$d_G(x \circ y) = d_G x, \qquad \forall\, x, y \in G.$$

The fourier transformation on G is:

$$(\mathscr{F}_G v)(\xi) := \int_{-1}^{1} \left(\frac{1 - y}{1 + y}\right)^{i\xi} v(y) d_G y, \qquad \xi \in \mathbb{R}. \qquad (5.1)$$

Thus, the dual group is isomorpic to the set of real numbers $\widehat{G} = \mathbb{R}$.

5. EXAMPLES OF LIE GROUPS

The inverse Fourier transformation writes

$$(\mathscr{F}_G^{-1}\psi)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-x}{1+x}\right)^{-i\xi}\psi\left(\frac{\xi}{2}\right)d\xi, \qquad x \in G. \tag{5.2}$$

Convolution equation on the Lie group $G = (-1, 1)$ has the form

$$\boldsymbol{A}u(x) := c_0 u(x) + \int_{-1}^{1} k\left(\frac{x-y}{1-xy}\right)\frac{v(y)dy}{1-y^2} = h(x), \tag{5.3}$$
$$x \in G := (-1, 1)$$

and the symbol $(\mathscr{F}_G\boldsymbol{A}u)(\xi) = [c_0 + (\mathscr{F}_G k)(\xi)]\,(\mathscr{F}_G u)(\xi) = (\mathscr{F}_G h)(\xi)$ **governs its solvability.**
The generic differential operator is

$$\mathfrak{D} = (1-x^2)\frac{d}{dx}.$$

6. PRANDTL EQUATION

The above results were applied to the celebrated Prandtl equation

$$\boldsymbol{P}u(x) = \frac{cu(x)}{1-x^2} + \frac{d}{\pi i}\int_{-1}^{1}\frac{u'(y)dy}{y-x} = f(x), \qquad x \in G. \qquad (6.1)$$

This equation has an ample applications in Mechanics and Mathematical Physics (called also Airfoil equation) and was investigated by many authors, but solved only numerically. The symbol of the equation is

$$\mathscr{P}(\xi) := c + 2di\xi\coth(\pi\xi), \qquad \xi \in \mathbb{R}. \qquad (6.2)$$

Since the equation is considered in the setting

$$u \in \mathbb{GH}_p^s(G, \boldsymbol{d}_G\mu), \qquad f \in \mathbb{GH}_p^{s-1}(G, \boldsymbol{d}_G\mu),$$

the symbol responsible for the solvability is

$$\mathscr{P}_1(\xi) := \frac{c + 2di\xi\coth(\pi\xi)}{\sqrt{1+\xi^2}}, \qquad \xi \in \mathbb{R}.$$

6. PRANDTL EQUATION

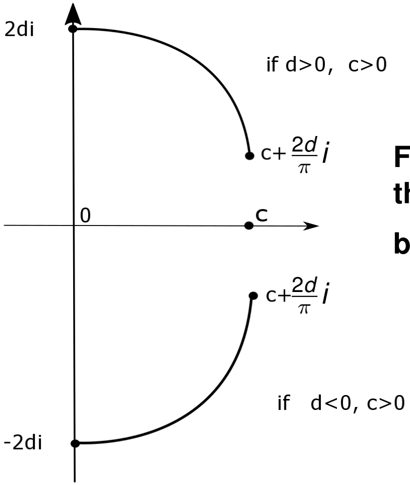

Fig. 2. The essential epectrum of the operator P: Plot of the symbol $\mathscr{P}_1(\xi) = \dfrac{c + 2di\xi \coth(\pi\xi)}{\sqrt{1+\xi^2}}$.

7. TRICOMI EQUATION

Another equation which turned out to be of type (5.3) **is the famous** **Singular Tricomi equation**

$$\boldsymbol{T}v(x) = cv(x) + \frac{d}{\pi i}\int_{-1}^{1}\frac{v(y)dy}{y-x} + \frac{e}{\pi i}\int_{-1}^{1}\frac{v(y)dy}{1-xy} = g(x), \qquad x \in G := (0,1), \tag{7.1}$$

which arises in boundary value problems for the partial differential equations which change type (elliptic to hyperbolic). The symbol of the equation is

$$\boldsymbol{T}(\xi) := c - id\tanh(\pi\xi) + \frac{e}{\cosh(\pi\xi)}, \quad \xi \in \mathbb{R}. \tag{7.2}$$

7. TRICOMI EQUATION

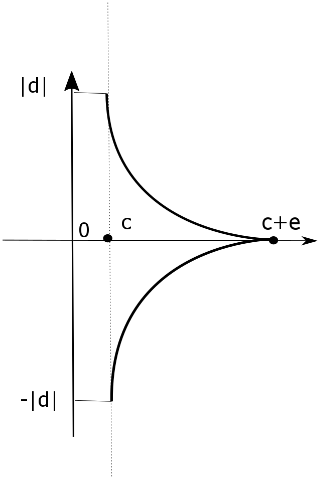

Fig. 3. The essential epectrum of the operator *P*: Plot of the symbol

$c - id\tanh(\pi\xi) + \dfrac{e}{\cosh(\pi\xi)}$.

8. LAVRENTJEV-BITSADZE EQUATION

The third equation from the class is the Lavrent'ev-Bitsadze equation

$$\boldsymbol{LB}\varphi(t) = c\varphi(t) + \frac{d}{\pi}\int_0^1 \left[\frac{1}{\tau - t} + \frac{1-2\tau}{t+\tau-2t\tau}\right]\varphi(\tau)d\tau = h(t), \quad (8.1)$$
$$x \in G := (0,1).$$

This equation also emerges in the investigation of boundary value problems for the partial differential equations which change type. The symbol of the equation is

$$\mathscr{L}\mathscr{B}(\xi) := c - id\tanh\frac{\pi\xi}{2}, \quad \xi \in \mathbb{R}. \quad (8.2)$$

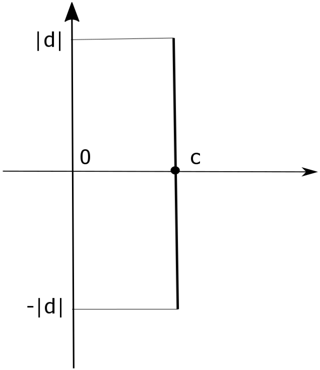

Fig. 4. The essential epectrum of the operator P: Plot of the symbol $c - id\,\tanh\frac{\pi\xi}{2}$.

9. REMARKS TO THE LIE GROUP $G = (-1, 1)$

My interest to equations was attracted by the paper of V. Petrov and V. Petrov & T. Suslina who applied the diffeomorphism

$$t(x) = \frac{1}{2} \ln \frac{1-x}{1+x}, \qquad x(t) = -\tanh\, t, \qquad x \in G = (-1, 1), \quad t \in \mathbb{R}$$

and transformed the Prandtl, Tricomi and Lavrentjev-Bitsadze equation on $G = (-1, 1)$ into a classical Fourier convolution equation on the real axes $\mathbb{R}$. Equations were solved in spaceless setting and, later, also in the Sobolev space $\mathbb{W}^1$-setting. Nugzar Shavlakadze (Tbilisi) investigated the Prandtl equation with a similar approach independently.
I have noted, that equation (5.3) belongs to a class of convolution equations on the Lie group $G = (-1, 1)$. Thia approach has many advantages and allowed to investigate convolution equations for other Lie groups and get deeper results: multiplier theory, general space setting $\mathbb{H}^s_p$, $\mathbb{GH}^s_p$ etc.

10.BVP FOR THE LAPLACE-BELTRAMI EQUATION ON HYPERSURFACE WITH LIPSCHITZ BOUNDARY

Let $\mathscr{C}$ be a smooth hypersurface in $\mathbb{R}^3$ with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$. We assume a bit more: the boundary Γ is piecewise-smooth, i.e., the tangent vector to Γ has jumps only at the finite number of knots

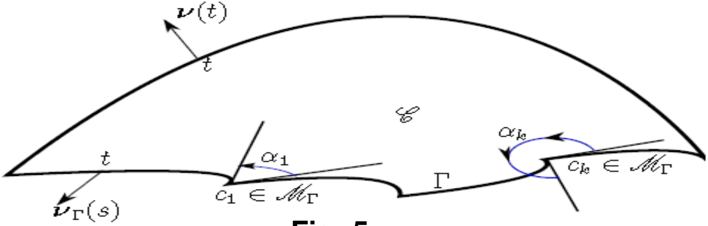

Fig. 5

$\mathfrak{M}_\Gamma := \{c_1, \ldots, c_n\} \subset \Gamma$ **(see Fig. 5). The inner angle α_j between the tangent lines at the knot c_j satisfies the inequality $0 < \alpha_j < 2\pi$.**

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Let $\boldsymbol{\nu} := (\nu_1, \nu_2, \nu_3)^\top$ be the normal vector field on the surface $\mathscr{C}$ and $\boldsymbol{\nu}_\Gamma := (\nu_{\Gamma,1}, \nu_{\Gamma,2}, \nu_{\Gamma,3})^\top$ be the normal vector field on the boundary Γ, tangential to the surface $\mathscr{C}$.
The boundary Γ is decomposed in two parts $\partial\mathscr{C} = \Gamma = \Gamma_D \cup \Gamma_N$ and we study the following mixed boundary value problem

$$\begin{cases} \Delta_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(s) = g(s), & \text{on} \quad \Gamma_D, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = h(s), & \text{on} \quad \Gamma_N. \end{cases} \tag{10.1}$$

Here $\Delta_{\mathscr{C}}$ is the Laplace-Beltrami operator

$$\Delta_{\mathscr{C}} := \mathscr{D}_1^2 + \mathscr{D}_2^2 + \mathscr{D}_3^2, \qquad \mathscr{D}_j := \partial_j - \nu_j \partial_{\boldsymbol{\nu}}, \quad j = 1, 2, 3$$

$\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3$ are Günter's tangential derivatives on the surface and

10.BVP FOR THE LAPLACE-BELTRAMI EQUATION ON HYPERSURFACE WITH LIPSCHITZ BOUNDARY

$$\partial_{\nu_\Gamma} := \nu_{\Gamma,1}\mathscr{D}_1 + \nu_{\Gamma,2}\mathscr{D}_2 + \nu_{\Gamma,3}\mathscr{D}_3$$

is the normal derivative on the boundary Γ, tangential to the surface $\mathscr{C}$.

The pure Dirichlet and pure Neumann problems are particular cases of the BVP (10.1) when, respectively, $\Gamma_N = \emptyset$ and $\Gamma_D = \emptyset$. The next theorem was proved in R. Duduchava, M. Tsaava T. Tsutsunava in [DTT14].

Theorem

The Mixed BVP (10.1) and the pure Dirichlet BVP (when $\Gamma_N = \emptyset$) have unique solutions in the classical weak setting:

$$\begin{aligned} &u \in \mathbb{H}^1(\mathscr{C}), \qquad &f \in \widetilde{\mathbb{H}}^{-1}(\mathscr{C}), \\ &g \in \mathbb{H}^{1/2}(\Gamma_D), \qquad &h \in \mathbb{H}^{-1/2}(\Gamma_N). \end{aligned} \tag{10.2}$$

10.BVP FOR THE LAPLACE-BELTRAMI EQUATION ON HYPERSURFACE WITH LIPSCHITZ BOUNDARY

Theorem continued

The pure Neumann BVP (10.1)**, when $\Gamma_D = \emptyset$, has a unique solution in the classical weak setting** (10.2) **only for those data which satisfy the following necessary and sufficient compatibility condition $(f, 1)_{\mathscr{C}} - (h, 1)_\Gamma = 0$.**

If f and h are regular integrable functions, the solvability condition acquires the form:

$$\int_{\mathscr{C}} f(y)\, d\sigma - \oint_\Gamma h(s) ds = 0.$$

To get more information about solutions to the BVP (10.1) **we consider it in the following non-classical setting:**

$$u \in \mathbb{GH}^s_p(\mathscr{C}, \rho), \quad f \in \mathbb{GH}^{s-2}_p(\mathscr{C}, \rho), \quad g \in \mathbb{GH}^{s-1/p}_p(\Gamma, \rho), \qquad (10.3)$$

$$h \in \mathbb{GH}^{s-1-1/p}_p(\Gamma, \rho), \quad 1 < p < \infty, \quad s > \frac{1}{p}, \quad \rho(t) = \prod_{j=1}^{n} |t - c_j|^{\gamma_j}.$$

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The weighted spaces $\mathbb{GH}^s_p(\mathscr{C},\rho)$ and $\mathbb{GH}^s_p(\Gamma,\rho)$ are defined, as usual, locally.

To the set of knots $\mathfrak{M}_\Gamma$ we add all those smoothness points on Γ where the Dirichlet and Neumann boundary conditions collide and the angle there is $\alpha_j=\pi$.

Let $\mathfrak{M}_\Gamma=\mathfrak{M}_{DD}\bigcup\mathfrak{M}_{NN}\bigcup\mathfrak{M}_{DN}$, where:

- **$\mathfrak{M}_{DD}$ consists of those knots c_j where the Dirichlet boundary conditions collide;**
- **$\mathfrak{M}_{NN}$ consists of those knots c_j where the Neumann boundary conditions collide;**
- **$\mathfrak{M}_{DN}$ consists of those knots c_j where the DIrichlet and Neumann boundary conditions collide.**

10.BVP FOR THE LAPLACE-BELTRAMI EQUATION ON HYPERSURFACE WITH LIPSCHITZ BOUNDARY

To each knot c_j we assocoate a model domain-angle of the same magnitude α_j (see Fig. 6).

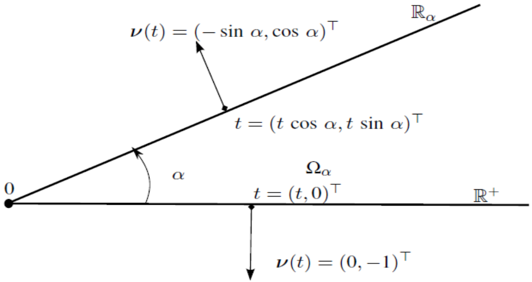

Fig. 6. Angular domain Ω_α.

10.BVP FOR THE LAPLACE-BELTRAMI EQUATION ON HYPERSURFACE WITH LIPSCHITZ BOUNDARY

In the model domain consider the following model BVPs:

DD. **At a knot $c_j \in \mathfrak{M}_{DD}$ the Dirichlet BVP for the Laplace equationin in the setting:**

$$\begin{cases} \Delta u(x) = f(x), & x \in \Omega_{\alpha_j}, \\ u^+(t) = g(t), & \text{on} \quad \Gamma_{\alpha_j} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_j}, \end{cases} \tag{10.4}$$

$$u \in \mathbb{GH}^s_p(\Omega_{\alpha_j}, t^{\beta_j}), \quad f \in \mathbb{GH}^{s-2}_p(\Omega_{\alpha_j}, t^{\beta_j}), \quad g \in \mathbb{GW}^{s-1/p}_p(\Gamma_{\alpha_j}, t^{\beta_j}).$$

NN. **At a knot $c_j \in \mathfrak{M}_{NN}$ the Neumann BVP for the Laplace equationin in the setting:**

$$\begin{cases} \Delta u(x) = f(x), & x \in \Omega_{\alpha_j}, \\ (\partial_{x_2} u)^+(t) = h(t), & \text{on} \quad \Gamma_{\alpha_j}. \end{cases} \tag{10.5}$$

$$u \in \mathbb{GH}^s_p(\Omega_{\alpha_j}, t^{\beta_j}), \quad f \in \mathbb{GH}^{s-2}_p(\Omega_{\alpha_j}, t^{\beta_j}), \quad h \in \mathbb{GW}^{s-1-1/p}_p(\Gamma_{\alpha_j}, t^{\beta_j}).$$

DN. **At a knot $c_j \in \mathfrak{M}_{DN}$ the Mixed Dirichlet-Neumann BVP in the setting:**

$$\begin{cases} \Delta u(x) = f(x), & x \in \Omega_{\alpha_j}, \\ u^+(t) = g(t), & \text{on} \quad \mathbb{R}^+, \\ (\partial_{x_2} u)^+(t) = h(t), & \text{on} \quad \mathbb{R}_{\alpha_j}. \end{cases} \tag{10.6}$$

$$u \in \mathbb{GH}^s_p(\Omega_{\alpha_j}, t^{\beta_j}), \quad f \in \mathbb{GH}^{s-2}_p(\Omega_{\alpha_j}, t^{\beta_j}),$$

$$g \in \mathbb{GW}^{s-1/p}_p(\Gamma_{\alpha_j}, t^{\beta_j}), \quad h \in \mathbb{GW}^{s-1-1/p}_p(\Gamma_{\alpha_j}, t^{\beta_j}).$$

10.BVP FOR THE LAPLACE-BELTRAMI EQUATION ON HYPERSURFACE WITH LIPSCHITZ BOUNDARY

Theorem

(Local Principle). The initial mixed boundary value problem (10.1) **in the non-classical setting** (10.3) **is Fredholm if and only if all model BVPs** (10.4), (10.5)**and** (10.6) **are Fredholm.**

The model BVPs (10.4), (10.5) **and** (10.6) **have at most one solution in the space** $\mathbb{H}^1(\Omega_\alpha)$ **in the classical setting.**

10.BVP FOR THE LAPLACE-BELTRAMI EQUATION ON HYPERSURFACE WITH LIPSCHITZ BOUNDARY

Theorem

Let $\alpha_j \in (0, 2\pi)$**,** $\beta_j = \dfrac{1+\alpha_j}{p}$**,** $j = 1, 2 \ldots, n$**. The mixed BVP** (10.1) **is Fredholm in the setting** (10.3) **if and only if:**

$$\beta_j \neq (\pi - \alpha_j)/(2\pi - \alpha_j) \qquad \textbf{for all} \quad c_j \in \mathfrak{M}_{DD}, \tag{10.7}$$

$$\beta_j \neq \pi(2\pi - \alpha_j), \qquad \textbf{for all} \quad c_j \in \mathfrak{M}_{NN}, \tag{10.8}$$

$$\beta_j \neq (3\pi - 2\alpha_j)/(4\pi - 2\alpha_j),\ (\pi \quad -\alpha_j)/(2\pi - \alpha_j),\ (2\alpha_j - \pi)/(2\alpha_j) \qquad \textbf{for all} \quad c_j \in \mathfrak{M}_{DN}. \tag{10.9}$$

Outline of the proof: We apply the standard potential method and derive the equivalent Boundary Integral Equations (BIEs) for our three model BVPs. Here we expose only one:

10.BVP FOR THE LAPLACE-BELTRAMI EQUATION ON HYPERSURFACE WITH LIPSCHITZ BOUNDARY

DD. **At a point $c_j \in \mathfrak{M}_{DD}$ the equivalent 2×2 system of BIE to the model Dirichlet BVP is:**

$$\boldsymbol{A}_{DD}\Psi := \begin{bmatrix} I & \frac{1}{2}\left[\boldsymbol{K}^1_{e^{i\alpha_j}} + \boldsymbol{K}^1_{e^{i(2\pi-\alpha_j)}}\right] \\ \frac{1}{2}\left[\boldsymbol{K}^1_{e^{i\alpha_j}} + \boldsymbol{K}^1_{e^{i(2\pi-\alpha_j)}}\right] & I \end{bmatrix} \Psi = G, \qquad (10.10)$$

where $\Psi = (\Psi_1, \Psi_2)^\top,\ G = (G_1, G_2)^\top \in \mathbb{GW}^{s-1/p}_p(\mathbb{R}^+, t^{\beta_j})$ **and**

$$\boldsymbol{K}^1_{e^{\pm i\alpha}}\psi(t) := \frac{1}{\pi}\int_0^\infty \frac{\psi(\tau)d\tau}{t - e^{\pm i\alpha}\tau} = \frac{1}{\pi}\int_0^\infty \frac{\psi(\tau)}{\frac{t}{\tau} - e^{\pm i\alpha}}\frac{d\tau}{\tau}, \quad 0 < \alpha < \pi.$$

is a Mellin convolution and its symbol (the Mellin transform $\mathfrak{M}_{\beta_j}$ of the kernel) is:

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$$\mathscr{K}^1_{e^{i\omega}}(\xi) = \frac{e^{i(\omega-\pi)(\beta_j+i\xi-1)}}{\sin\pi(\beta_j+i\xi)}, \qquad \beta_j = \frac{\gamma_j+1}{p}, \quad 0<\omega<\pi.$$

Then the symbol of A_{DD} is

$$\mathscr{A}_{DD}(\xi) = \begin{bmatrix} 1 & \dfrac{\cos(\pi-\alpha_j)(\beta_j+i\xi-1)}{\sin\pi(\beta_j+i\xi)} \\ \dfrac{\cos(\pi-\alpha_j)(\beta_j+i\xi-1)}{\sin\pi(\beta_j+i\xi)} & 1 \end{bmatrix}$$

and

$$\det\mathscr{A}_{DD}(\xi) = \frac{\cos(2\pi-\alpha_j)(\beta_j+i\xi-1)\cos\alpha_j(\beta_j+i\xi-1)}{\sin^2\pi(\beta_j+i\xi)} \neq 0.$$

is necessary and sufficient condition for the invertibility.

Similarly are investigated the model Neumann-Neumann and Dirichlet-Neumann BVPs.

11. GLOBAL PSEUDO-DIFFERENTIAL OPERATORS ON THE LIE GROUP $G = (-1, 1)^n$

With Duván Cardona, Arne Hendrickx & Michael Ruzhansky

In the recent work (submitted for publication) we characterise the Hörmander classes on the open manifold $G = (-1, 1)^n$. We show that by endowing the open manifold $G = (-1, 1)^n$ with a group structure, the corresponding global Fourier analysis on the group allows to define a global notion of symbol on the phase space $G \times \mathbb{R}^n$. We study the class of pseudo-differential operators $\Psi^m_{\rho,\delta}(G \times \mathbb{R}^n)$ associated with the global Hörmander classes $\mathbb{S}^m_{\rho,\delta}(G \times \mathbb{R}^n)$. Are proved, in particular, L^p-Fefferman type estimates and Calderón-Vaillancourt theorems, formulas for the composition, Gohberg lemma on compactness, new theorems on boundedness of ΨDOs with non-classical symbols, having jump-type discontinuities on $\mathbb{R}^n$. We prove also Atyah-Singer-Fedosov Index formula for elliptic ΨDOs with elliptic matrix symbols from the Hörmander classes $\mathbb{S}^m_{\rho,\delta}(G \times \mathbb{R}^n)$.

REFERENCES:

- **BDKT13** - T. Buchukuri, R. Duduchava, D. Kapanadze & M. Tsaava, Localization of a Helmholtz boundary value problem in a domain with piecewise-smooth boundary, *Proc. A. Razmadze Math. Inst*, 162, 37-44, 2013.
- **CDHR23** - D. Cardona, R. Duduchava, A. Hendrickx, M. Ruzhansky, Global pseudo-differential operators on the Lie group $G = (-1, 1)^n$. https://arxiv.org/abs/2209.09751, 34 pages, 2022.
- **DD16** - V. Didenko, R.Duduchava, Mellin convolution operators in the Bessel potential spaces. Accepted in *Journal of Analysis and Applications*. 443 (2016) 707-731.
- **Du79** - R. Duduchava, *Integral equations with fixed singularities*. Teubner, Leipzig, 1979.

REFERENCES:

- **Du13** - R. Duduchava, Mellin convolution operators in Bessel potential spaces with admissible meromorphic kernels, *Mem. Differ. Equ. Math. Physics* 60 (2013), 135–177. Corrected and extended version: http://arxiv.org/pdf/1502.02756.pdf
- **Du01** - R.Duduchava, *The Green formula and layer potentials* Integral Equations and Operator Theory 41, 2, 127-178, 2001.
- **Du21** - R. Duduchava, Mixed type boundary value problems for Laplace-Beltrami equation on a surface with the Lipschitz boundary. Georgian Mathematical Journal v. 28, 2, 219-232, 2021.
- **Du23** - R. Duduchava, Convolution equations on the Lie group G=(-1,1), Georgian Mathematical Journal, 2023. https://doi.org/10.1515/gmj-2023-2035. https://arxiv.org/abs/2208.08765, 26 pages, 2022.

REFERENCES:

- **Du25** - R. Duduchava, Convolutions on Lie groups and fundamental solutions, Memorial volume to Lars Eric Prersson's 80-th birth- day anniversary.
- **DMM06** - R.Duduchava, D.Mitrea, M.Mitrea, Differential operators and boundary value problems on surfaces. *Mathematische Nachrichten* 279, No. 9-10 (2006), 996-1023.
- **DT16** - R. Duduchava, M. Tsaava, Mixed boundary value problems for the Laplace-Beltrami equation. *Complex Variables and Elliptic Equations* 63, 10, 2018, 1468-1496
- **DT20** - R. Duduchava, & M. Tsaava, Mixed boundary value problems for the Helmholtz equation in a model 2D angular domain. *Georgian Mathematical Journal* 27, 2, 2020, 211–231.
- **DTT14** - R. Duduchava, M. Tsaava, T. Tsutsunava, Mixed boundary value problem on hypersurfaces. *International Journal of Differential Equations*, Hindawi Publishing Corporation, Volume 2014, Article ID 245350, 8 pages.

REFERENCES:

- **Hr60** - L. HÖRMANDER, *Estimates for translation invariant operators on L^p-spaces*, **Acta Mathematica 104, 93-140, 1960.**
- **Pe06a - Petrov V. E., Integral transform on a segment, *Problemy Matem. Analiza* 31 (2005), 67–95; English transl. *J. Math. Sci.* 132, no. 4, 2006, 451-481.**
- **Pe06b - Petrov V. E., The generalized singular Tricomi equation as a convolution equation, *Dokl. Ros. Akad. Nauk* 411 (2006), no. 2, 1–5; English transl., *Dokl. Math.* 74, no. 3, 2006, 901–905.**
- **SP21 - Suslina T.A., Petrov V.E., Regularity of the solution of the Prandtl equation, Math Zametki 110 (2021), no. 4, 550–568. https://doi.org/10.1134/S0001434621090248**

Thank you!
Questions?