Bianalytic polynomial approximations, Nevanlinna domains and univalent functions in model spaces

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Nevanlinna domains

Denote by $\mathbb{D}$ the open unit disc $\{z \in \mathbb{C}\colon |z| < 1\}$ and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. For an open set $U \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ let us denote by $H^\infty(U)$ the set of all bounded holomorphic functions on U.

Definition (J. Carmona, P. Paramonov, K.F., *Sb. Math., 2002*)

A bounded simply connected domain $G \subset \mathbb{C}$ is a Nevanlinna domain (N-domain for brevity) if $\exists\ u, v \in H^\infty(G)$, $v \not\equiv 0$, s. t. the equality

$$\overline{z} = \frac{u(z)}{v(z)} \qquad (\mathcal{N})$$

holds on ∂G almost everywhere in the sense of conformal mappings.

Property $(\mathcal{N})$ means the equality of angular boundary values

$$\overline{f(\zeta)} = \frac{(u \circ f)(\zeta)}{(v \circ f)(\zeta)} \qquad (\mathcal{N}')$$

for a.a. $\zeta \in \mathbb{T}$, where f is a conf. mapping from $\mathbb{D}$ onto G.

Put $ND = \{G\colon G \text{ is } N\text{-domain}\}$. See also [K.F., *Math. Notes, 1996*].

Nevanlinna domains

- the definition of a Nevanlinna domain does not depend on the choice of f;
- in view of the Luzin–Privalov boundary uniqueness theorem, the quotient u/v is uniquely defined in G (for a Nevanlinna domain G);
- for Jordan domains G with rectifiable boundaries $(\mathcal{N})$ may be understood directly as the equality of angular boundary values almost everywhere with respect to the length on ∂G;
- $(\mathcal{N})$ can be similarly understood on any rectif. Jordan arc $\gamma \subset \partial G$ such that each point $a \in \gamma$ is not a limit point for the set $\partial G \setminus \gamma$;
- every disc is N-domain, while every domain which is bounded by an ellipse which is not a circle, or by a polygonal line is not N-domain;
- yet another interesting example of a N-domain is the Neumann's oval, i.e. the domain bounded by the image of an ellipse (which is not a circle) with center at 0 under the mapping $z \mapsto 1/z$.
- Note that both functions u and v in the definition of N-domain are not assumed to have some extra regularity (e.g. continuity) in $\overline{G}$.

Why N-domains are of interest?

1. The role of N-domains in problems of polyanalytic polynomial approximation.

A function g is called polyanalytic of order n (for integer $n \geqslant 1$) or, in short, p.a. or n-analytic, on an open set $U \subset \mathbb{C}$ if it is of the form

$$g(z) = g_0(z) + \overline{z}g_1(z) + \cdots + \overline{z}^{n-1}g_{n-1}(z), \qquad (\mathcal{PAF})$$

where $g_0, \ldots, g_{n-1}$ are holomorphic functions in U.

By n-analytic polynomials and n-analytic rational functions we mean the functions of the form $(\mathcal{PAF})$, where $g_0, \ldots, g_{n-1}$ are polynomials and rational functions in the complex variable respectively.

The problems we are interested in is to describe the compact sets X such that every function f continuous on X and n-analytic on its interior can be approximated uniformly on X by n-analytic rational functions with no singularities in X, or by n-analytic polynomials.

These problems have attracted attention since the beginning of 1980s, but the main efforts were focused on approximations by polyanalytic rational functions, and there was no substantial progress in the problem of polyanalytic polynomial approximation until the middle of 1990s.

Why N-domains are of interest?

[K.F., *Math. Notes, 1996*]: description of rectifiable simple closed curves Γ in order that the system of n-analytic polynomials (for every integer $n \geq 2$) is dense in the space of continuous functions on Γ.

This description was stated in terms of N-domains, which were firstly appeared (with a slightly different definition) in [K.F., *Math. Notes, 1996*].

Why N-domains are of interest?

Later on, a criterion for the uniform approximation by p.a. polynomials on Carathéodory compact sets in $\mathbb{C}$ was obtained in terms of N-domains.

A compact set X is called a Carathéodory compact set if $\partial X = \partial \widehat{X}$, where $\widehat{X}$ is the union of X and all bounded connected components of $\mathbb{C} \setminus X$.

Theorem (J. Carmona, P. Paramonov, K.F., *Sb. Math., 2002*)

Let X be a Carathéodory compact set in $\mathbb{C}$, and $n \geq 2$ be an integer. In order that each function f which is continuous on X and n-analytic inside X can be approximated uniformly on X by n-analytic polynomials it is necessary and sufficient that every bounded connected component of the set $\mathbb{C} \setminus X$ is not a Nevanlinna domain.

Why N-domains are of interest?

2. Schwarz function of an analytic curve and its generalizations.

Let Γ be a simple closed analytic curve. It is well-known that in this case there exist an open set U, $\Gamma \subset U$, and a holomorphic function S in U, s.t.

$$\Gamma = \{z \in U \colon \overline{z} = S(z)\}.$$

The function S is called a Schwarz function of Γ.

- A Jordan domain G with analytic boundary is N-domain if and only if the Schwarz function S of ∂G is meromorphic in G.
- In this case S is meromorphic in G if and only if G is the image of the unit disc under conformal mapping by some univalent in $\mathbb{D}$ rational function R without poles on $\overline{\mathbb{D}}$.
- One rigidity property of N-domains

Theorem (J. Carmona, K.F., *Oper. Theor. Adv. Appl., 2005, vol. 158*)

Let G be a bounded simply conn. domain and L be an algebraic curve such that $G \in ND$ and $\exists$ an arc $\Lambda \subset \partial G \cap L$ with $\omega(z, \Lambda, G) > 0$ (harmonic measure, $z \in G$). Therefore, ∂G is analytic and $\partial G \subset L$.

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Let now G be a bounded domain possessing the following property: there exist a compact set $K \subset G$ and a function S holomorphic in $G \setminus K$, continuous up to ∂G, and such that $\overline{z} = S(z)$ on ∂G. In the latter case the aforesaid function S is called the one-sided Schwarz function of ∂G.

Theorem 5.2 in [M. Sakai, *Acta Math., 1991*] says that if ∂G admits the one-sided Schwarz function, then it consists of finitely many anal. curves.

In fact, the boundary of a quadrature domain admits the one-sided Schwarz function. Recall that a quadrature domain is a domain $G \subset \mathbb{C}$ such that there exists a distribution T, $\mathrm{Supp}(T) \subset G$, such that

$$\iint_G h(z)\,dxdy = T(f), \quad \forall h \text{ holomorphic/harmonic and integrable in } G.$$

Why N-domains are of interest?

2. Schwarz function of an analytic curve and its generalizations.

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In view of certain similarity of representations $\overline{z} = u(z)/v(z)$ and $\overline{z} = S(z)$, the next question seems to be quite meaningful and interesting: How far one can get from (piecewise) analytic curves considering the concept of a Nevanlinna domain as a generalization of a (one-sided) Schwarz function?

Two equivalent description of N-domains

a) Nevanlinna domains and pseudocontinuation.

Let G be a bounded simply connected domain and let φ be some conformal mapping from $\mathbb{D}$ onto G.

Proposition

$G \in ND$ if and only if φ admits a pseudocontinuation (p.c. in what follows), so that $\exists\ f_1, f_2 \in H^\infty(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$ such that $f_2 \not\equiv 0$ and for a.a. $\zeta \in \mathbb{T}$ the equality $\varphi(\zeta) = f_1(\zeta)/f_2(\zeta)$ holds, where $f_1(\zeta)$ and $f_2(\zeta)$ are the angular boundary values of f_1 and f_2, resp.

- If $G \in ND$ and g is a univalent in G rational function with poles outside $\overline{G}$, then $g(G) \in ND$.
- Nevanlinna domains have the following "density" property: any neighborhood of an arbitrary simple close curve contains an analytic Nevanlinna contour (i.e. the boundary of some Jordan Nevanlinna domain).

Two equivalent description of N-domains

b) Nevanlinna domains and model spaces.

$\Theta \in H^\infty = H^\infty(\mathbb{D})$ is an inner function if $|\Theta(\zeta)| = 1$ for a.a. $\zeta \in \mathbb{T}$.
For an inner function Θ we put (H^2 is the standard Hardy space)

$$K_\Theta := (\Theta H^2)^\perp = H^2 \ominus \Theta H^2.$$

- the spaces $K_\Theta \subset H^2$ are exactly the invariant subspaces of the backward shift operator $f \mapsto (f(z) - f(0))/z$ (Beurling);
- the spaces K_Θ are usually called model (sub)spaces: this terminology was suggested by N. Nikolski in view of the remarkable role these spaces play in the functional model of Sz.-Nagy and Foiaş.

Theorem (K.F. *Proc. Steklov Inst. Math., 2006*)

1) If $G \in ND$, then there exists an inner function Θ such that $\varphi \in K_\Theta$.

2) Let Θ be an inner function, and f be a bounded univalent function from K_Θ. Then $f(\mathbb{D}) \in ND$.

Thus, the class of all bounded univalent functions in $\bigcup_\Theta K_\Theta$ coincides with the class of conformal mappings from $\mathbb{D}$ onto N-domains.

Three problems on univalent functions in K_Θ

It is interesting and important to answer the following questions:

i) To describe inner functions Θ, such that the model space K_Θ contains bounded univalent functions.

ii) Let $f \in K_\Theta$ be bounded univalent function. How large (e.g. in the sense of dimension) can be the (accessible) boundary of $f(\mathbb{D})$?

iii) Let R be a univ. in $\mathbb{D}$ rational function of a degree n having no poles on $\overline{\mathbb{D}}$. How fast the quantity $\int_{\mathbb{T}} |R'(z)|\,|dz|$ may grow as $n \to \infty$?

This can be regarded as a quantitative version of the question on whether N-domains with unrectifiable boundaries exist or not?

What K_Θ contain bounded univalent functions?

The description of inner functions Θ such that the corresponding space K_Θ contains bounded univalent functions was recently obtained in [A. Baranov, Yu. Belov, A. Borichev, K.F., *arXiv:1705.05930*] and [Yu. Belov, K.F., *Russian Math. Surveys, 2018*].

Recall that every inner function Θ can be expressed in the form $\Theta(z) = e^{ic}B(z)S(z)$, where $c > 0$,

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \frac{z - a_n}{\overline{a}_n z - 1},$$

is a Blaschke product, and

$$S(z) = \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\mu_S(\zeta) \right),$$

is a singular inner function. Here $(a_n)_{n=1}^{\infty}$, $a_n \in \mathbb{D}$ is a sequence of points such that $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$ (this sequence is called Blaschke sequence), while μ_S is some finite positive singular (with respect to the arc length) measure on $\mathbb{T}$.

What K_Θ contain bounded univalent functions?

Theorem (A. Baranov, A. Borichev, K.F., *arXiv:1705.05930*; short proof: Yu. Belov, K.F., *Russian Math. Surveys, 2018*)

Let Θ be an inner funct. in $\mathbb{D}$. The space K_Θ contains bounded univ. functions if and only if one of the following two conditions is satisfied:

1. *Θ has a zero in $\mathbb{D}$;*
2. *$\Theta = S$ is a singular inner function and the measure μ_S is such that $\mu_S(E) > 0$ for some Beurling–Carleson set $E \subset \mathbb{T}$.*

Recall, that $E \subset \mathbb{T}$ is a Beurling–Carleson set, if

$$\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E)\,|d\zeta| > -\infty.$$

Beurling–Carleson sets first appeared as boundary zero sets of analytic functions in the disc which are smooth up to the boundary.

It is worth to mention that property (2) in this Theorem is also a necessary and sufficient condition for the space K_S to contain mildly smooth functions (e.g., from the standard Dirichlet space in $\mathbb{D}$).

Nevanlinna domains with "bad" boundaries

Question: How "bad" (more accurate, how non-regular in one sense or another) the boundary of a Nevanlinna domain could be.

To answer we need to be able to construct bounded univalent functions belonging to K_Θ (for specially chosen Θ) and possessing some prescribed boundary behavior.

It seems appropriate to study this question separately in two distinct cases: (i) $\Theta = B$ is a Blaschke product, and (ii) $\Theta = S$ is a singular inner function (it may be readily verified that $K_{BS} = K_B \oplus BK_S$).

- The first example of a Nevanlinna domain with nowhere analytic boundary was constructed in [M. Mazalov, *Math. Notes, 1997*]. The respective domain was constructed as the (conformal) image of $\mathbb{D}$ under a mapping f of the form

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{1 - \overline{a}_n z}, \qquad (*)$$

where $(a_n)_{n \geq 1}$ is some (infinite) Blaschke sequence satisfying the Carleson condition (interpolation Blaschke seq. in other terms):

Nevanlinna domains with "bad" boundaries

$$\inf_{n\in\mathbb{N}} \prod_{k=1,\ k\neq n}^{\infty} \left| \frac{a_n - a_k}{1 - a_n \bar{a}_k} \right| > 0,$$

and $(c_n)_{n\geq 1}$ is an appropriately chosen sequence of coefficients. Such Blaschke sequences are called interpolating.

For any interp. Blaschke seq. $(a_n)_{n\geq 1}$ the sequence of functions

$$\left\{ \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z} \right\}$$

forms a Riesz basis in the corresponding space K_B.

- The first example of a N-domain with boundary in the class C^1 but not $C^{1,\alpha}$ was given in [K.F., *Proc. Steklov Inst. Math., 2006*]
- In [A. Baranov, K.F., *Sb. Math., 2011*] it was proved that $\forall \alpha \in (0,1)$ and for every closed subset $E \subseteq \mathbb{T}$ there exists an int. Blaschke sequence $(a_n)_{n\geq 1}$ such that the set of its limit points is equal to E, and the corresponding space K_B contains a univalent function f of the form $(*)$ which maps $\mathbb{D}$ conf. onto a N-domain $f(\mathbb{D})$ with boundary in the class C^1 but not in the class $C^{1,\alpha}$.

Nevanlinna domains with "bad" boundaries

- Furthermore, [A. Baranov, K.F., *Sb. Math., 2011*], there is a construction of a function f of the form $(*)$ such that f is univalent in $\mathbb{D}$ but $f' \notin H^p$ for any $p > 1$. It means that the boundary $\partial f(\mathbb{D})$of a Nevanlinna domain $f(\mathbb{D})$ is "almost" non-rectifiable.
- The first example of a Jordan N-domain with non-rectifiable boundary was constructed in [M. Mazalov, *St. Petersburg Math. J., 2016*]. The corresponding domain is also $f(\mathbb{D})$, for some function f of the form $(*)$ univalent in the unit disc.
- In [M. Mazalov, *St. Petersburg Math. J., 2018*] an example of a N-domain G such that $\dim_H(\partial G) = \log_2 3$ was constructed. As before, $G = f(\mathbb{D})$ for a suitable function f of the form $(*)$.
- Let S be a singular inner function. If μ_S has atoms, then K_S contains bounded univ. functions. In particular, this is the case for $S(z) = \exp\left(\frac{z+1}{z-1}\right)$. Equivalently, the Paley–Wiener space $\mathcal{P}W_{[0,1]}$ (the Fourier image of $L^2[0,1]$, considered as a space of anal. funct. in $\mathbb{C}_+$), contains bounded univ. functions. Until recently only a few explicit examples of bounded univ. functions in $\mathcal{P}W_{[0,1]}$ were known, and all them map $\mathbb{C}_+$ onto domains with quite regular boundaries.

Nevanlinna domains with large boundaries

Let $D(a, r)$ stand for the open disc with center at the point $a \in \mathbb{C}$ and with radius $r > 0$. For a bounded set $E \subset \mathbb{C}$ its s-dimensional Hausdorff measure $\mathcal{H}^s(E)$ is defined as follows:

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf_{\{D_j\}} \sum_j r_j^s,$$

where the infimum is taken over all coverings of E by families of discs $\{D_j\}$, $D_j = D(z_j, r_j)$, of radius at most δ. By definition, the Hausdorff dimension $\dim_H(E)$ is the unique number such that $\mathcal{H}^s(E) = \infty$ for every $s < \dim_H(E)$, while $\mathcal{H}^t(E) = 0$ for every $t > \dim_H(E)$.

Given a bounded simply connected domain G we consider the set $\partial_a G \subset \partial G$ which consists of all points of ∂G being accessible from G by some curve. We have

$$\partial_a G = \{f(\zeta) \colon \zeta \in \mathcal{F}(f)\},$$

where f is some conformal mapping from the unit disc $\mathbb{D}$ onto G and $\mathcal{F}(f)$ is its Fatou set, that is the set of all points $\zeta \in \mathbb{T}$, where the angular boundary values $f(\zeta)$ exist.

Nevanlinna domains with large boundaries

The definition of N-domains (see $(\mathcal{N})$ and its interpretation $(\mathcal{N}')$), imposes conditions only on the accessible part $\partial_a G$ of their boundaries. By this reason it seems more accurate and adequate to pose the question about the existence of N-domains with large *accessible* boundaries.

Theorem (Yu. Belov, A. Borichev, K.F., *J. Funct. Anal., 2019*)

For every $\beta \in [1,2]$ there exists a function f of the form $(*)$ univ. in $\mathbb{D}$ and s.t. the N-domain $G = f(\mathbb{D})$ satisfies the property $\dim_H(\partial_a G) = \beta$.

Note that the function f from this theorem belongs to the space K_B for some appropriately chosen Blaschke product B.

Theorem (Yu. Belov, A. Borichev, K.F., *J. Funct. Anal., 2019*)

For every $\beta \in [1,2]$ there exists a univ. function $f \in \mathcal{P}W^{\infty}_{[0,1]}$ s.t. the N-domain $G = f(\mathbb{C}_+)$ satisfies the property $\dim_H(\partial G) = \beta$.

Thus we can get far away from domains with piecewise analytic boundaries (and, therefore, from quadr. dom.) if we consider N-domains instead of ones whose boundaries admit the one-sided Schwarz funct.

Nevanlinna domains with large boundaries

Nevanlinna 'needle':

For $b \ll 1$ and $N \gg 1$ we construct $F(z) = z + \sum_{k=1}^{N} \frac{c_k}{z - w_k}$, s.t.

- $c_k > 0$, $w_k \in [1 + bN^{-1}, 1 + b]$,
- F is univalent in $\mathbb{D}$, and F maps $\mathbb{D}$ onto N-domain having the form of a disk with very narrow and long 'needle' growing at one point.
- we have good 'control' of F' in $\mathbb{D}$ and, more accurate, in $D(1, b)$, so that we are able 'to iterate' the construction.

Using N-needles one can construct N-domain with non-rectifiable boundary. We need to "grow" from the disk N-needles of length $\approx 1/k$ on points $\zeta_k \in \mathbb{T}$ accumulates to 1. Such construction (its looks like a hedgehog) was used in [M. Mazalov, *St. Petersburg Math. J., 2016*].

Making N-needles with much more accurate control of F' in $D(1, b)$ we obtained in [Yu. Belov, A. Borichev, K.F., *J. Funct. Anal., 2019*] the method to grow new N-needles from points near to the "needle tip" of the previous one. Using these "needles on needles" we have "imitated" the construction of H-tree.