Superposition operators acting on spaces of analytic functions

Daniel Girela
Universidad de Málaga, Spain

International biweekly online Seminar on Analysis, Differential Equations and Mathematical Physics

May 4, 2023

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$\mathrm{Hol}(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.

Composition operators

Let φ be analytic in $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. The operator $C_\varphi : \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$ defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in \mathrm{Hol}(\mathbb{D}),$$

is called the composition operator with symbol φ.

C_φ is a linear operator.

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Question

If X and Y are two (Banach) subspaces of $\mathrm{Hol}(\mathbb{D})$, for which entire functions φ does the operator S_φ map (continuously) X into Y?

Remark

In general S_φ is not linear.
In fact, it is easy to see that S_φ is linear if and only if φ is of the form $\varphi(z) = \lambda z$, λ being a constant.

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In fact, it is easy to see that S_φ is linear if and only if φ is of the form $\varphi(z) = \lambda z$, λ being a constant.

It is clear that if Y contains the constant functions, and if φ is constant then we have that S_φ maps X into Y.

If $\varphi(z) = z$, then $S_\varphi(f) = f$ for every $f \in \mathrm{Hol}(\mathbb{D})$, and, hence,

$$S_\varphi(X) \subset Y \Leftrightarrow X \subset Y.$$

Informally, we can say that if $X \subset Y$, the answer to our question tells us 'how small is X compared with Y'.
If there are a lot of φ's for which $S_\varphi(X) \subset Y$, X is 'small compared Y'.

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X and Y subspaces of $\mathrm{Hol}(\mathbb{D})$. We want to characterize those entire functions φ which act from X to Y by superposition.

It turns out that a number of distinct ideas can be used to deal with this problem depending on the spaces under consideration.

In this talk, I am going to try to present some of these ideas with a number of works I have made over the last few years in collaboration with
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Hardy spaces

If $0 < p \leq \infty$, the Hardy space H^p consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty$.

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{\infty} |f(re^{it})|^p \, dt \right)^{1/p}.$$

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

Bergman and Dirichlet spaces

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A^p_α consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1) \int_{\mathbb{D}} (1-|z|^2)^\alpha |f(z)|^p \, dA(z) \right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p.

The space of Dirichlet type $\mathcal{D}^p_\alpha$ consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $f' \in A^p_\alpha$.

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BMOA and the Bloch space

The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values function has bounded mean oscillation, that is, lies in $BMO(\mathbb{T})$.

The Bloch space $\mathcal{B}$ is the space of all functions $f \in \mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

We have the inclusions

$$BMOA \subset \mathcal{B}, \qquad H^\infty \subset BMOA \subset \bigcap_{0<p<\infty} H^p.$$

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Let φ be an entire function.

For $0 < p, q < \infty$, the superposition operator S_φ maps H^p into H^q, or A^p into A^q, if and only φ is a polynomial of degree less than or equal to p/q. [Cámera and Cámera and Giménez].

Buckley, Fernández and Vukotić studied superposition operators acting between the Dirichlet type spaces $\mathcal{D}_0^p$ and $\mathcal{D}_0^q$.

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- φ acts from A^p into $\mathcal{B}$ by superposition if and only if φ is constant.
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Some results previous to our work

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The results we have stated and, actually, all the results we know in this setting have the following in common:
If φ acts from X to Y by superposition then so does φ'.

Question

Is this always true?
Or, at least ... find a general theorem in this line ...

In collaboration with S. Domínguez (2019) we have found some classes of spaces Y with the property that if φ is an entire function which acts from a certain space X into Y by superposition then so does φ'.

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Spaces of analytic functions with restricted growth

A weight v on $\mathbb{D}$ will be a positive and continuous function defined on $\mathbb{D}$ which is radial, i. e. $v(z) = v(|z|)$, for all $z \in \mathbb{D}$, and satisfying that $v(r)$ is strictly decreasing in $[0, 1)$ and that $\lim_{r\to 1} v(r) = 0$. For such a weight, the weighted Banach space H_v^∞ is defined by

$$H_v^\infty = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_v \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\}.$$

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Theorem

Let v be weight on $\mathbb{D}$ and let $(X, \|\cdot\|)$ be a Banach space of analytic function in $\mathbb{D}$. Let φ be an entire function. If the superposition operator S_φ is a bounded operator from X into H_v^∞, then $S_{\varphi'}$ maps X into H_v^∞.

Basic steps in the proof

Suppose S_φ is a bounded operator form X into H_v^∞. Take $f \in X$, then we prove:

- If $|f(z)| \leq 1$ then $\left|S_{\varphi'}(f)(z)\right| \leq \frac{Av(0)}{v(z)}$ with $A = \sup_{|\xi|\leq 1} |\varphi'(\xi)|$.
- If $|f(z)| \geq 1$ then

$$\left|S_{\varphi'}(f)(z)\right| \leq \frac{1}{2} |S_\varphi(g)(z)|,$$

  for a certain $g \in X$ (which depends on z) and satisfies $\|g\| = \|f\|$.
- Putting these two things together we find $C > 0$ such that $\left|S_{\varphi'}(f)(z)\right| \leq \frac{C}{v(z)}$. This gives $S_{\varphi'}(f) \in H_v^\infty$.

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If v is a weight on $\mathbb{D}$, we define DH_v^∞ as follows

$$DH_v^\infty = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : f' \in H_v^\infty \right\}.$$

The space DH_v^∞ is a Banach space with the norm $\| \cdot \|_{D,v}$ defined by

$$\|f\|_{D,v} = |f(0)| + \|f'\|_v.$$

Theorem

Let v be a weight on $\mathbb{D}$ and let $(X, \| \cdot \|)$ be a Banach space of analytic function in $\mathbb{D}$. Let φ be an entire function. If the superposition operator S_φ is a bounded operator from X into DH_v^∞, then $S_{\varphi'}$ maps X into DH_v^∞.

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If we take $v(z) = (1 - |z|)$, the space DH_v^∞ reduces to the Bloch space. Hence, as a particular case we obtain.

Corollary

Let $(X, \|\cdot\|)$ be a Banach space of analytic function in $\mathbb{D}$. Let φ be an entire function. If the superposition operator S_φ is a bounded operator from X into the Bloch space $\mathcal{B}$, then $S_{\varphi'}$ maps X into $\mathcal{B}$.

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Sets of zeros as a tool

Let us recall one the results I mentioned before.

Theorem (AMV). Suppose $0 < p < \infty$ and φ is an entire function. If φ acts from A^p into $\mathcal{B}$ under superposition, then φ is constant.

The original proof uses a number of results on conformal mappings.
We have used different ideas to prove the following extension.

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Theorem (D-G)

Suppose that $0 < p < \infty$, $\alpha > -1$ and let φ be an entire function. If $S_\varphi(A^p_\alpha) \subset \mathcal{B}$ then φ is constant.

The ingredient in our proof is that there exists 'a zero sequence of A^p_α' which is not 'a zero sequence of $\mathcal{B}$'.
Indeed, it is known (G-Nowak-Waniurski-2000) that if $\{z_k\}$ is the sequence of zeros of a function $f \in \mathcal{B}$ with $f(0) \neq 0$ then

$$\prod_{k=1}^{n} \frac{1}{|z_k|} = O\left((\log n)^{1/2}\right) .(*)$$

While (Horowitz-1974) proved that for any given $\varepsilon > 0$, there exist $g \in A^p_\alpha$ with $g(0) \neq 0$ whose sequence of zeros $\{z_k\}$ satisfies

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Theorem (D-G)

Suppose that $0 < p < \infty$, $\alpha > -1$ and let φ be an entire function. If $S_\varphi(A^p_\alpha) \subset \mathcal{B}$ then φ is constant.

The ingredient in our proof is that there exists 'a zero sequence of A^p_α' which is not 'a zero sequence of $\mathcal{B}$'.
Indeed, it is known (G-Nowak-Waniurski-2000) that if $\{z_k\}$ is the sequence of zeros of a function $f \in \mathcal{B}$ with $f(0) \neq 0$ then

$$\prod_{k=1}^{n} \frac{1}{|z_k|} = O\left((\log n)^{1/2} \right). (*)$$

While (Horowitz-1974) proved that for any given $\varepsilon > 0$, there exist $g \in A^p_\alpha$ with $g(0) \neq 0$ whose sequence of zeros $\{z_k\}$ satisfies

$$\prod_{k=1}^{n} \frac{1}{|z_k|} \neq O\left(n^{(1+\alpha)/(p(1+\varepsilon))} \right). (**)$$

Suppose φ is not constant and $S_\varphi(A^p_\alpha) \subset \mathcal{B}$.
Take $g \in A^p_\alpha$, $g \not\equiv 0$ whose sequence of zeros satisfies (**) for some $\epsilon > 0$.
We have that $S_\varphi(g) = \varphi \circ g \in \mathcal{B}$ and $\varphi \circ g$ is not constant.
Set $F = S_\varphi(g) - \varphi(0)$. We have that

$$F = S_\varphi(g) - \varphi(0) = \varphi \circ g - \varphi(0) \in \mathcal{B}, \quad \text{and} \quad F \not\equiv 0.$$

Now, all the zeros of g are zeros of F. In other words, the sequence $\{z_k\}$ is contained in the sequence $\{\xi_k\}$ of zeros of F. Since $\{z_k\}$ satisfies (**), $\{\xi_k\}$ does not satisfies (*). This contradicts the fact that $F \in \mathcal{B}$.

Let X and Y be two spaces of analytic functions in $\mathbb{D}$ satisfying the following conditions:

(i) X contains the constants.

(ii) There exists a function $f \in X$ with $f(0) \neq 0$ whose sequence of zeros $\{z_k\}$ is not a subsequence of a sequence of zeros of Y.

Let φ be an entire funtion. Then φ acts from X into Y by superposition if and only if φ is constant. .

Hardy space, $\mathcal{D}^p_{p-1}$-spaces, *BMOA*, $\mathcal{B}$

Among all the spaces of Dirichlet type $\mathcal{D}^p_\alpha$, the spaces $\mathcal{D}^p_{p-1}$ are the closest ones to Hardy spaces.

$$\mathcal{D}^p_{p-1} \subset H^p \subset A^{2p}, \quad 0 < p \le 2.$$

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Similarities

- For $0 < p \leq 2$, the Carleson measures for H^p and those for $\mathcal{D}^p_{p-1}$ are the same.
- The univalent functions in H^p and $\mathcal{D}^p_{p-1}$ $(0 < p < \infty)$ are the same (BGP2004).
- A number of operators are bounded on H^p iff and only if they are bounded for $\mathcal{D}^p_{p-1}$.
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- $H^\infty \not\subset \mathcal{D}^p_{p-1}$,if $0 < p < 2$.
  There are Blaschke products not belonging to any of the $\mathcal{D}^p_{p-1}$-spaces, $0 < p < 2$.
- For $p > 2$, there are functions in $\mathcal{D}^p_{p-1}$ without radial limits.
- There is no inclusion relation between $\mathcal{D}^p_{p-1}$ and $\mathcal{D}^q_{q-1}$, $p \neq q$.
- The zero sequence of a $\mathcal{D}^p_{p-1}$-function, $p > 2$, may not satisfy the Blaschke condition.
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Objective

Studying similarities and differences between Hardy spaces and $\mathcal{D}^p_{p-1}$-spaces regarding superposition operators.

Let us consider superposition operators between the Hardy spaces and the spaces *BMOA* and the Bloch space $\mathcal{B}$, and compare them with those between the $\mathcal{D}^p_{p-1}$-spaces and *BMOA* or $\mathcal{B}$.

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Superposition operators between *BMOA* spaces and Hardy spaces

In a work in collaboration with M. A. Márquez we proved the following.

Theorem

Let φ be an entire function. Then

(a) For $0 < p < \infty$, $S_\varphi(\mathcal{B}) \subset H^p$ if and only if φ is constant.

(b) For $0 < p < \infty$, $S_\varphi(BMOA) \subset H^p$ if and only if φ is of order less than one, or of order one and type zero.

(a) can be proved using the idea on the zero sequences.
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Superposition operators from the Bloch space into $\mathcal{D}^p_{p-1}$-spaces

We know that for any p

$$S_\varphi(\mathcal{B}) \subset H^p \iff \varphi \text{ is constant.}$$

Since $\mathcal{D}^p_{p-1} \subset H^p$ for $p \leq 2$, this implies:

If $0 < p \leq 2$, then $S_\varphi(\mathcal{B}) \subset \mathcal{D}^p_{p-1}$ if and only if φ is constant.

What about the case $2 < p < \infty$?

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There are function in $\mathcal{D}^p_{p-1}$ ($p > 2$) whose sequence of zeros do not satisfy the Blaschke condition. So...

More precise results about the zero sequences of Bloch functions and $\mathcal{D}^p_{p-1}$-functions

Suppose $f \in \mathcal{B}$ with $f(0) \neq 0$ and let $\{a_n\}$ be the (ordered) sequence of the zeros of f. Then

$$\prod_{n=1}^{N} \frac{1}{|a_n|} = \mathrm{O}\left((\log N)^{1/2}\right), \quad \text{as } N \to \infty.$$

This is sharp: There exists $f \in \mathcal{B}$ for which

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Suppose $2 < p < \infty$, and $S_\varphi(\mathcal{B}) \subset \mathcal{D}^p_{p-1}$.

Take $f \in \mathcal{B}$ with $f(0) \neq 0$ whose sequence of zeros $\{a_n\}$ satisfies

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Set $g(z) = \varphi \circ f(z) - \varphi(0)$.

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Recall: $S_\varphi(BMOA) \subset H^p$ if and only if φ is of order less than one, or of order one and type zero.
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For these values of p the φ's for which $S_\varphi(BMOA) \subset \mathcal{D}^p_{p-1}$ do not coincide with those for which $S_\varphi(BMOA) \subset H^p$.
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Suppose that $0 < p < 2$ and let φ be an entire function. Then

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Suppose $0 < p < 2$, φ is non-constant, and $S_\varphi(BMOA) \subset \mathcal{D}^p_{p-1}$. Take $f \in H^\infty \subset BMOA$ such that

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$\varphi' \not\equiv 0$ and then

$$\varphi' \circ f \in H^\infty \text{ and } \varphi' \circ f \not\equiv 0.$$

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THANK YOU!