# A Semi-Discrete Convexification Method Combined with Deep Learning for Coefficient Inverse Problems for PDEs

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# Overview

- We develop a new **semi-discrete version** of the Carleman estimate-based convexification method.
- The method is used to provide a **starting point** for the training procedure of deep learning.
- A key feature of the continuous convexification method is its **global convergence**, which does not rely on a good initial guess.
- We introduce the notion of $h$-**strong convexity**, where $h$ is the grid step size.
- This leads to an **a priori accuracy estimate** for the starting point used in deep learning.

# Motivation

- We consider a broad class of **Coefficient Inverse Problems (CIPs)** for partial differential equations.
- These problems are well known to be **highly nonlinear** and computationally challenging.
- The convexification method, based on **Carleman estimates**, provides a globally convergent framework.
- However, convexification is typically **computationally slow**, especially on fine meshes.
- This motivates a hybrid strategy combining:
  - a robust first-stage global method, and
  - a fast second-stage data-driven refinement.

# Two-Stage Numerical Procedure

The rapid solution obtained by the semi-discrete convexification method on a coarse grid is quite blurry. Hence, to improve its accuracy, we use a second stage: deep learning.

More precisely, we use the solution computed by the semi-discrete version of the convexification method on the coarse grid as the starting point for the minimization of the loss function in the training procedure of deep learning.

The resulting two-stage numerical procedure works both rapidly and accurately.

# Four Significantly New Elements

- For the first time, the so-called "$h$-strong convexity property" is introduced. This property is then established for the semi-discrete analogue of the CWF-weighted Tikhonov-like objective functional.
- The result in item 1 makes it possible to estimate the accuracy of the approximation of the minimizer of the above continuous functional $F$ by its semi-discrete analogue.

# Four Significantly New Elements

- In the training procedure of deep learning, the solution obtained by the semi-discrete version of the convexification method on a coarse grid is used as the starting point for the iterations.
- Therefore, items 2 and 3 yield an *a priori* accuracy estimate for the starting point of the training step of the deep learning procedure. This estimate is

$$O\left(\sqrt{\alpha}+\sqrt{h}\right), \qquad \text{as } \sqrt{\alpha}+\sqrt{h}\to 0^{+},$$

  where $\alpha \in (0,1)$ is a small regularization parameter used in the semi-discrete convexification method.

# Statements of Forward and Inverse Problems

Below $\mathbf{x} = (x, y)$ denotes points in $\mathbb{R}^2$. We now construct such a domain $\Omega \subset \mathbb{R}^2$, which is convenient for our computational purpose. Let $B, D$ be two numbers such that $0 < B < D$. Let $a, b$ be two other numbers, where $a > 0$. Consider two concentric disks $P_D\left(a, b\right)$ and $P_B\left(a, b\right) \subset P_D\left(a, b\right)$ with the center at $\mathbf{x} = (a, b)$, and let $E_B\left(a, b\right) = \partial P_B\left(a, b\right)$ be the circle, which is the boundary of the disk $P_B\left(a, b\right)$,

$$
\begin{gathered}
P_D\left(a, b\right) = \{\mathbf{x} = (x, y) : (x - a)^2 + (y - b)^2 < D^2\}, \\
P_B\left(a, b\right) = \{\mathbf{x} = (x, y) : (x - a)^2 + (y - b)^2 < B^2\} \subset P_D\left(a, b\right), \\
E_B\left(a, b\right) = \{\mathbf{x} = (x, y) : (x - a)^2 + (y - b)^2 = B^2\}.
\end{gathered}
$$

Let $c \in (0, a)$ be a number. Define the square $\Omega$ as

$$\Omega = \{\mathbf{x} : x \in (a - c, a + c), y \in (b - c, b + c)\}.$$

We choose $c$ such that $\overline{\Omega} \subset P_B(a, b)$.

$$\overline{\Omega} \cap \{x = 0\} = \varnothing.$$

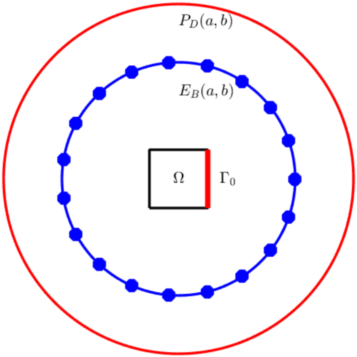


Figure 1: A schematic diagram of our measurements.

In our setting point sources are

$$\mathbf{x}_0 = \mathbf{x}_0(\theta) \in E_B\left(a, b\right), \theta \in [0, 2\pi]\,.$$

We model the point sources by a smooth, compactly supported function $g(\mathbf{x} - \mathbf{x}_0)$ which approximates the $\delta-$function,

$$g\left(\mathbf{x} - \mathbf{x}_0\right) = \begin{cases} C_\xi \exp\left(\frac{|\mathbf{x}-\mathbf{x}_0|^2}{|\mathbf{x}-\mathbf{x}_0|^2-\xi^2}\right), |\;\mathbf{x} - \mathbf{x}_0| < \xi, \\ 0, \;\; |\mathbf{x} - \mathbf{x}_0| \geq \xi \end{cases}.$$

Here, $\xi \in (0, 1)$ is a sufficiently small number such that

$$\{|\mathbf{x} - \mathbf{x}_0| < \xi\} \cap \overline{\Omega} = \varnothing, \;\; \{|\mathbf{x} - \mathbf{x}_0| < \xi\} \subset P_D\left(a, b\right), \;\forall \mathbf{x}_0 \in E_B\left(a, b\right),$$

and the number $C_\xi$ is chosen such that

$$\int\limits_{|\mathbf{x}-\mathbf{x}_0|<\xi} g\left(\mathbf{x} - \mathbf{x}_0\right) dx = 1, \;\forall \mathbf{x}_0 \in E_B\left(a, b\right).$$

Let $\sigma(\mathbf{x})$ denotes the electric conductivity coefficient. We assume that satisfies the conditions

$$\sigma \in C^2\left(\overline{P_D(a,b)}\right), \quad \sigma(\mathbf{x}) \geq 1 \text{ in } P_D(a,b),$$

$$\sigma(\mathbf{x}) = 1 \quad \text{for} \ \ \mathbf{x} \in P_B(a,b) \setminus \Omega.$$

For each $\mathbf{x}_0 \in E_B$,the underlying elliptic PDE is:

$$\nabla_{\mathbf{x}} \cdot \left(\sigma\left(\mathbf{x}\right) \nabla_{\mathbf{x}} v\left(\mathbf{x}, \mathbf{x}_0\right)\right) = -g\left(\mathbf{x} - \mathbf{x}_0\right), \ \mathbf{x} \in P_D\left(a, b\right),$$
$$v\left(\mathbf{x}, \mathbf{x}_0\right) = 0, \ \forall \mathbf{x} \in \partial P_D\left(a, b\right).$$

$v\left(\mathbf{x}, \mathbf{x}_0\right)$ is the voltage. For every $\gamma \in (0, 1)$ and for every $\mathbf{x}_0 \in E_B\left(a, b\right)$ there exists unique solution $v(\mathbf{x}, \mathbf{x}_0) \in C^{3+\gamma}\left(\overline{P_D}\left(a, b\right)\right)$ of problem. There exists a number $\beta > 0$ such that $v(\mathbf{x}, \mathbf{x}_0) \geq \beta, \ \forall \mathbf{x} \in \overline{\Omega}, \forall \mathbf{x}_0 \in E_B\left(a, b\right)$.
Let $\Gamma_0 \subset \partial\Omega$ be the right side of the square $\Omega$,

$$\Gamma_0 = \left\{\mathbf{x} = (x, y) : x = a + c, \ y \in (b - c, b + c)\right\},$$

# Coefficient Inverse Problem (CIP)

For each $\mathbf{x}_0 \in E_B\left(a,b\right)$ let $v(\mathbf{x},\mathbf{x}_0) \in C^{3+\alpha}\left(\overline{P_D}\left(a,b\right)\right)$ be the above solution of problem. Assume that the following functions $h_0(\mathbf{x},\mathbf{x}_0)$ and $h_1(\mathbf{x},\mathbf{x}_0)$ are given:

$$
\begin{gathered}
v(\mathbf{x},\mathbf{x}_0) = h_0(\mathbf{x},\mathbf{x}_0),\ \forall \mathbf{x} \in \partial\Omega, \\
v_x(\mathbf{x},\mathbf{x}_0) = h_1(\mathbf{x},\mathbf{x}_0),\ \forall \mathbf{x} \in \Gamma_0, \\
\forall x_0 \in E_B(a,b).
\end{gathered}
$$

Find the coefficient $\sigma\left(\mathbf{x}\right)$ in for $\mathbf{x} \in \Omega$.

# Transformation procedure

First, we change variables as $V(\mathbf{x},\mathbf{x}_0)=\sqrt{\sigma(\mathbf{x})}v(\mathbf{x},\mathbf{x}_0)$. We obtain

$$\Delta V+r(\mathbf{x})V=0,\ \text{in }\Omega,$$

$$V(\mathbf{x},\mathbf{x}_0)=h_0(\mathbf{x},\mathbf{x}_0),\ \mathbf{x}\in\partial\Omega,$$

$$V_x(\mathbf{x},\mathbf{x}_0)=h_1(\mathbf{x},\mathbf{x}_0),\ \forall\mathbf{x}=(a+c,y)\in\Gamma_0,\ y\in(b-c,b+c),$$

$$\forall\mathbf{x}_0\in E_B(a,b),$$

$$r(\mathbf{x})=\frac{\Delta\left(\sqrt{\sigma(\mathbf{x})}\right)}{\sqrt{\sigma(\mathbf{x})}}.$$

Let $\mathbf{x}_0=\mathbf{x}_0(\theta)$, where the angle $\theta\in[0,2\pi]$. Next, we set

$$\psi(\mathbf{x},\theta)=\ln V(\mathbf{x},\theta)$$

We obtain

$$\Delta\psi+(\nabla\psi)^2+r(\mathbf{x})=0,\ \mathbf{x}\in\Omega.$$

If the function $\psi(x,\theta)$ is found, then

$$r(\mathbf{x}) = -\frac{1}{2\pi}\int\limits_0^{2\pi}\left(\Delta\psi + (\nabla\psi)^2\right)(\mathbf{x},\theta)\,d\theta,\ x \in \Omega.$$

Given $r(x)$, the question of finding the target coefficient $\sigma(\mathbf{x})$ is addressed below.

The above differentiation with respect to $\theta$ leads to:

$$\Delta\psi_\theta + 2\nabla\psi_\theta\nabla\psi = 0,\ x \in \Omega.$$

To express $\psi$ via $\psi_\theta$, one might try to use

$$\psi\left(\mathbf{x},\theta\right) = \int\limits_{\theta_0}^{\theta} \psi_\theta\left(\mathbf{x},s\right) ds + \psi\left(\mathbf{x},\theta_0\right)$$

for a certain $\theta_0 \in (0, 2\pi)$ . However, the function $\psi\left(\mathbf{x},\theta_0\right)$ is unknown for any value of $\theta_0$. This causes the introduction of a viscosity term $-\varepsilon\Delta\psi$ with a small parameter $\varepsilon \in (0,1)$ . equation

$$-\varepsilon\Delta\psi + \Delta\psi_\theta + 2\nabla\psi_\theta\nabla\psi = 0,\ \mathbf{x}\in\Omega.$$

Denote

$$\phi = \psi_\theta - \varepsilon\psi.$$

Then they lead to the following system of coupled nonlinear PDEs for the functions $\psi_\theta$ and $\phi$,

$$F_1\left(\psi_\theta,\phi\right)(\theta) = \Delta\psi_\theta + 2\nabla\psi_\theta\nabla\left(\frac{\psi_\theta-\phi}{\varepsilon}\right) = 0,\ \mathbf{x}\in\Omega,\ \theta\in[0,2\pi]\,,$$

$$F_2\left(\psi_\theta,\phi\right)(\theta) = \Delta\phi + 2\nabla\psi_\theta\nabla\left(\frac{\psi_\theta-\phi}{\varepsilon}\right) = 0,\ \mathbf{x}\in\Omega,\ \theta\in[0,2\pi]\,.$$

This system is subject to the overdetermined boundary conditions derived from the data

$$\psi_\theta(\mathbf{x},\theta)\mid_{\mathbf{x}\in\partial\Omega} = \partial_\theta s_0(\mathbf{x},\theta), \ \ (\mathbf{x},\theta)\in\partial\Omega\times[0,2\pi],$$

$$\partial_x(\partial_\theta\psi(\mathbf{x},\theta))\mid_{\mathbf{x}\in\Gamma_0} = \partial_\theta s_1(\mathbf{x},\theta), \ \forall(\mathbf{x},\theta)\in\Gamma_0\times[0,2\pi],$$

$$\phi(\mathbf{x},\theta)\mid_{\mathbf{x}\in\partial\Omega} = \partial_\theta s_0(\mathbf{x},\theta) - \varepsilon s_0(\mathbf{x},\theta), \ \forall(\mathbf{x},\theta)\in\partial\Omega\times[0,2\pi],$$

$$\partial_x\phi(\mathbf{x},\theta) = \partial_\theta s_1(\mathbf{x},\theta) - \varepsilon s_1(\mathbf{x},\theta), \ \forall(\mathbf{x},\theta)\in\Gamma_0\times[0,2\pi].$$

Introduce a new notation for convenience

$$(q,p)(\mathbf{x},\theta) = (\psi_\theta,\phi)(\mathbf{x},\theta).$$

The transformation procedure is complete.

# The Carleman estimate

The construction of the globally convergent convexifiction method for our CIP is based on a Carleman estimate. Denote

$$H_0^2(\Omega) = \left\{ u \in H^2(\Omega) : u \mid_{\partial\Omega} = 0, \partial_x u \mid_{\Gamma_0} = 0 \right\}.$$

Carleman Weight Function: (CWF) as

$$W_\kappa(\mathbf{x}) = \exp\left(2\kappa x^2\right),$$

where $\kappa \geq 1$ is a parameter.

**Theorem 1**. *There exists a sufficiently large number* $\kappa_0 = \kappa_0(\Omega) \geq 1$ *and a number* $C = C(\Omega) > 0$, *both numbers depending only on* $\Omega$, *such that with the CWF* $W_\kappa(\mathbf{x})$ *the following Carleman estimate holds:*

$$\int\limits_{\Omega} (\Delta u)^2 W_\kappa(\mathbf{x})\, d\mathbf{x} \geq \frac{C}{\kappa} \int\limits_{\Omega} \left(u_{xx}^2 + u_{xy}^2 + u_{yy}^2\right) W_\kappa(\mathbf{x})\, d\mathbf{x} +$$

$$+ C\kappa \int\limits_{\Omega} (\nabla u)^2 W_\kappa(\mathbf{x})\, d\mathbf{x} + C\kappa^3 \int\limits_{\Omega} u^2 W_\kappa(\mathbf{x})\, d\mathbf{x},$$

$$\forall \kappa \geq \kappa_0,\ \forall u \in H_0^2(\Omega).$$

Everywhere below $C = C(\Omega) > 0$ denotes different numbers depending only on the domain $\Omega$.

# The globally strongly convex CWF-weighted Tikhonov-like functional

We define the space $H_0^6(\Omega)$ as

$$H_0^6(\Omega) = \left\{ u \in H^6(\Omega) : u \mid_{\partial\Omega} = 0, \partial_x u \mid_{\Gamma_0} = 0 \right\}.$$

Define spaces $H_{1,\theta}$ and $H_{1,2,\theta}$ depending on the parameter $\theta \in [0, 2\pi]$,

$$H_{1,\theta} = \left\{ \begin{array}{c} q(\mathbf{x},\theta) : q(\mathbf{x},\theta) \in H^6(\Omega), \\ \|q(\mathbf{x},\theta)\|_{H_{1,\theta}}^2 = \|q(\mathbf{x},\theta)\|_{H^6(\Omega)}^2 < \infty, \ \forall\theta \in [0, 2\pi], \end{array} \right\},$$
$$H_{1,\theta}^0 = \left\{ q(\mathbf{x},\theta) : q(\mathbf{x},\theta) \in H_0^6(\Omega) \right\},$$
$$H_{1,2,\theta} = H_{1,\theta} \times H_{1,\theta}, \ H_{1,2,\theta}^0 = H_{1,\theta}^0 \times H_{1,\theta}^0,$$
$$\|(q,p)(\mathbf{x},\theta)\|_{H_{1,2,\theta}}^2 = \|q(\mathbf{x},\theta)\|_{H_{1,\theta}}^2 + \|p(\mathbf{x},\theta)\|_{H_{1,\theta}}^2.$$

By Sobolev embedding theorem

$$
\begin{gathered}
(q,p)(\mathbf{x},\theta) \in C^4\left(\overline{\Omega}\right) \times C^4\left(\overline{\Omega}\right),\ \forall (q,p)(\mathbf{x},\theta) \in H_{1,2,\theta}, \\
\left\|(q,p)(\mathbf{x},\theta)\right\|_{C^4(\overline{\Omega})\times C^4(\overline{\Omega})} \le C\left\|(q,p)(\mathbf{x},\theta)\right\|_{H_{1,2,\theta}}, \\
\forall (q,p)(\mathbf{x},\theta) \in H_{1,2,\theta},\ \forall \theta \in [0,2\pi].
\end{gathered}
$$

Let $A > 0$ be an arbitrary number, which we fix. We define the set $G_\theta(A) \subset H_{1,2,\theta}$ as:

$$G_\theta(A) = \left\{ \begin{array}{c} (q,p)(\mathbf{x},\theta) \in H_{1,2,\theta} : \|(q,p)(\mathbf{x},\theta)\|_{H_{1,2,\theta}} < A, \\ q \text{ satisfies boundary conditions} \\ p \text{ satisfies boundary conditions} \\ \forall \theta \in [0,2\pi] \end{array} \right\}.$$

Hence,

$$\|(q,p)(\mathbf{x},\theta)\|_{C^4(\overline{\Omega}) \times C^4(\overline{\Omega})} \leq CA, \forall (q,p)(\mathbf{x},\theta) \in H_{1,2,\theta}, \forall \theta \in [0,2\pi].$$

# Minimization Problem

We define the functional

$$J_{\kappa,\alpha}:\overline{G_{\theta}\left(A\right)}\to\mathbb{R},\ \forall\theta\in[0,2\pi]$$

$$J_{\kappa,\alpha}(q,p)(\theta)=\sqrt{\varepsilon}\int\limits_{\Omega}\left[\left(F_1(q,p)(\mathbf{x},\theta)\right)^2+\left(F_2(q,p)(\mathbf{x},\theta)\right)^2\right]W_{\kappa}(\mathbf{x})d\mathbf{x}$$

$$+$$

$$+\alpha\left\|(q,p)\left(\mathbf{x},\theta\right)\right\|^2_{H^6(\Omega)\times H^6(\Omega)},\ \forall\theta\in[0,2\pi],$$

where $\alpha\in(0,1)$ is the regularization parameter.
Minimize the functional $J_{\kappa,\alpha}\left(q,p\right)\left(\theta\right)$ on the set $\overline{G_{\theta}\left(A\right)}$ for each $\theta\in[0,2\pi]$.

**Theorem 2** (Global strong convexity). *The following hold true:*

1. *For each $\kappa > 0$, for each $\theta \in [0, 2\pi]$, and for each pair $(q, p) \in \overline{G_\theta(A)}$, the functional $J_{\kappa,\alpha}(q, p)(\theta)$ has the Fréchet derivative*

$$J'_{\kappa,\alpha}(q, p)(\theta) \in H^0_{1,2,\theta}, \qquad \forall \theta \in [0, 2\pi].$$

2. *Let $\kappa_0 = \kappa_0(\Omega) \geq 1$ be the number of Theorem 1. There exists a sufficiently large number*

$$\kappa_1 = \kappa_1(A, \Omega, \varepsilon) \geq \kappa_0$$

*such that for each $\kappa \geq \kappa_1$ the functional $J_{\kappa,\alpha}(q, p)$ is strongly convex on the set $\overline{G_\theta(A)}$, i.e. there exists a number $C_1 = C_1(A, \Omega, \varepsilon) > 0$ such that*

$$\begin{aligned}
&J_{\kappa,\alpha}(q_2, p_2)(\theta) - J_{\kappa,\alpha}(q_1, p_1)(\theta) \\
&\quad - J'_{\kappa,\alpha}(q_1, p_1)(\theta)\big(q_2 - q_1,\, p_2 - p_1\big)(\mathbf{x}, \theta) \\
&\geq C_1 \exp\big(2\kappa(a - c_1)^2\big) \left\|\big(q_2 - q_1,\, p_2 - p_1\big)(\mathbf{x}, \theta)\right\|^2_{H^2(\Omega)\times H^2(\Omega)} \\
&\quad + \alpha \left\|\big(q_2 - q_1,\, p_2 - p_1\big)(\mathbf{x}, \theta)\right\|^2_{H_{1,2,\theta}}.
\end{aligned}$$

2. *The estimate above holds for all $\theta \in [0, 2\pi]$, for all $(q_1, p_1)(\mathbf{x}, \theta), (q_2, p_2)(\mathbf{x}, \theta) \in \overline{G_\theta(A)}$, and for all $\kappa \geq \kappa_1$. The numbers $\kappa_1(A, \Omega, \varepsilon)$ and $C_1(A, \Omega, \varepsilon)$ depend only on the listed parameters.*

3. *For every $\kappa \geq \kappa_1$ and for every $\theta \in [0, 2\pi]$ there exists a unique minimizer*
$$(q_{\min,\kappa,\alpha}, p_{\min,\kappa,\alpha})(\mathbf{x}, \theta) \in \overline{G_\theta(A)}$$
*of the functional $J_{\kappa,\alpha}(q, p)(\theta)$ on the set $\overline{G_\theta(A)}$. In addition,*
$$J'_{\kappa,\alpha}(q_{\min,\kappa,\alpha}, p_{\min,\kappa,\alpha})\,(\theta)\Big((q_{\min,\kappa,\alpha} - q,\ p_{\min,\kappa,\alpha} - p)(\mathbf{x}, \theta)\Big) \leq 0,$$
$$\forall \theta \in [0, 2\pi], \qquad \forall (q, p)(\mathbf{x}, \theta) \in \overline{G_\theta(A)}.$$

Everywhere below $C_1 = C(A, \Omega, \varepsilon) > 0$ denotes different numbers depending only on listed parameters. Suppose now that the minimizer $(q_{\min,\kappa,\alpha}, p_{\min,\kappa,\alpha})(\mathbf{x}, \theta)$ of the functional $J_{\kappa,\alpha}(q,p)(\theta)$ on the set $\overline{G_\theta(A)}$ is found.

$$\partial_\theta \psi_{\min,\kappa,\alpha}(\mathbf{x},\theta) = q_{\min,\kappa,\alpha}(\mathbf{x},\theta), \phi_{\min,\kappa,\alpha}(\mathbf{x},\theta) = p_{\min,\kappa,\alpha}(\mathbf{x},\theta),$$

$$\psi_{\min,\kappa,\alpha}(\mathbf{x},\theta) = \frac{\partial_\theta \psi_{\min,\kappa,\alpha}(\mathbf{x},\theta) - \phi_{\min,\kappa,\alpha}(\mathbf{x},\theta)}{\varepsilon},$$

$$r_{\kappa,\alpha}(\mathbf{x}) = -\frac{1}{2\pi} \int_0^{2\pi} \left(\Delta \psi_{\min,\kappa,\alpha} + (\nabla \psi_{\min,\kappa,\alpha})^2\right)(\mathbf{x},\theta)\, d\theta,\ \mathbf{x} \in \Omega.$$

Denote

$$w(\mathbf{x}) = \sqrt{\sigma(\mathbf{x})}.$$

We now need to solve the following elliptic equation:

$$\Delta w - r_{\kappa,\alpha}(\mathbf{x})\, w = 0,\ \ \mathbf{x} \in \Omega.$$

$$w \mid_{\partial\Omega} = 1, \partial_{\nu} w \mid_{\partial\Omega} = 0,$$

where $\partial_{\nu}$ is the normal derivative at $\partial\Omega$. Next, we assign the reconstructed function $\sigma_{\kappa,\alpha}(\mathbf{x})$ as:

$$\sigma_{\kappa,\alpha}(\mathbf{x}) = w^2(\mathbf{x}).$$

This problem has an overdetermination in the boundary conditions. Hence, this problem was solved in by the Quasi-Reversibility Method. More precisely, the following functional $I\left(w\right)$ was minimized on the set of functions $w\in H^{2}\left(\Omega\right)$ satisfying boundary conditions

$$I\left(w\right)=\int\limits_{\Omega}\left(\Delta w-r_{\kappa,\alpha}\left(\mathbf{x}\right)w\right)^{2}d\mathbf{x}.$$

# The accuracy of the minimizer

We assume the existence of the true solution $\sigma^*(\mathbf{x})$ of our CIP satisfying conditions with the noiseless data $h_0^*(\mathbf{x},\theta), h_1^*(\mathbf{x},\theta)$. An important question is the question of an estimate of the accuracy of the regularized solution, i.e. an estimate of a norm of the difference between $(q_{\min,\kappa,\alpha}, p_{\min,\kappa,\alpha})(\mathbf{x},\theta)$ and $(q^*, p^*)(\mathbf{x},\theta)$, where the last pair corresponds to $\sigma^*(\mathbf{x})$. We need to estimate of a certain norm of the difference $\sigma_{\kappa,\alpha}(\mathbf{x}) - \sigma^*(\mathbf{x})$.

We estimate here the difference $r_{\kappa,\alpha}(\mathbf{x}) - r^*(\mathbf{x})$, assuming that

$$h_0(\mathbf{x},\theta) = h_0^*(\mathbf{x},\theta), \;\; h_1(\mathbf{x},\theta) = h_1^*(\mathbf{x},\theta).$$

Since we now work in the framework of the viscosity solution, then we assume that the pair $(q^*, p^*)(\mathbf{x},\theta) = (\psi_\theta^*, \phi^*)(\mathbf{x},\theta)$ satisfies equations

$$F_1(q^*, p^*)(\mathbf{x},\theta) = F_2(q^*, p^*)(\mathbf{x},\theta) = 0, \;\; \forall \mathbf{x} \in \Omega, \;\; \forall \theta \in [0, 2\pi].$$

**Theorem 3.**

1. *Let the function* $\sigma^*(\mathbf{x})$ *be the exact solution of our CIP satisfying conditions and with the noiseless data* $h_0^*(\mathbf{x},\theta), h_1^*(\mathbf{x},\theta)$. *Assume that we have noiseless data.*

2. *Also, let the function* $r^*(\mathbf{x})$ *be linked with the function* $\sigma^*(\mathbf{x})$. *Let* $(q^*, p^*)(\mathbf{x}, \theta)$ *be the pair of functions generated by the function* $\sigma^*(\mathbf{x})$. *Let* $G_\theta(A)$ *be the set of vector functions defined. Assume that*

$$(q^*, p^*)(\mathbf{x}, \theta) \in G_\theta(A).$$

3. *Let* $\kappa_1$ *be the number of Theorem 2 and let* $\kappa \geq \kappa_1$*. Let* $(q_{\min,\kappa,\alpha}, p_{\min,\kappa,\alpha})(\mathbf{x}, \theta) \in \overline{G_\theta(A)}$ *be the minimizer of the functional* $J_{\kappa,\alpha}(q,p)(\theta)$ *on the set* $\overline{G_\theta(A)}$*, which was found in Theorem 2. Let the function* $r_{\kappa,\alpha}(x)$ *be the corresponding approximation for function* $r^*(\mathbf{x})$*. Then the following accuracy estimates are valid:*

$$\|(q_{\min,\kappa,\alpha} - q^*)(\mathbf{x},\theta)\|_{H^2(\Omega)} + \|(p_{\min,\kappa,\alpha} - p^*)(\mathbf{x},\theta)\|_{H^2(\Omega)} \leq$$

$$\leq C_1\sqrt{\alpha}\exp\left[-\kappa(a-c)^2\right], \ \forall\theta \in [0,2\pi],$$

$$\|r_{\kappa,\alpha} - r^*\|_{L_2(\Omega)} \leq C_1\sqrt{\alpha}\exp\left[-\kappa(a-c)^2\right].$$

# Discretization

Recall definition of the domain $\Omega$ and consider the following grid $\{(x_i, y_j)\}_{i,j=0}^{n+1} \subset \overline{\Omega}$ in $\overline{\Omega}$ with the grid step size $h \in (0,1)$ and $n > 4$

$$
\begin{aligned}
a - c = x_0 < x_1 < \ldots < x_n < x_{n+1} = a + c, \quad x_i - x_{i-1} = h, \\
b - c = y_0 < y_1 < \ldots < y_n < y_{n+1} = b + c, \quad y_j - y_{j-1} = h.
\end{aligned}
$$

Everywhere below $O\left(h^k\right)$ denotes different functions such that

$$
\left| O\left(h^k\right) \right| \leq C_1 h^k, k > 0.
$$

Denote

$$
\begin{gathered}
\Omega^h = \{(x_i, y_j)\}_{i,j=1}^{n}, \\
\overline{\Omega^h} = \{(x_i, y_j)\}_{i,j=0}^{n+1}, \\
\partial\Omega^h = \overline{\Omega^h} \diagdown \Omega^h, \\
\Gamma_0^h = \{(a + c, y_j), \ j = 0, ..., n+1\}.
\end{gathered}
$$

Denote the running point in $\overline{\Omega^h}$ as $\mathbf{x}_{i,j}^h = (x_i, y_j)$, $\forall i, j = 0, ..., n+1$. Any function $z(\mathbf{x}) \in C\left(\overline{\Omega}\right)$ generates the discrete function $z^h\left(\mathbf{x}_{i,j}^h\right)$. Hence,

$$z^h\left(\mathbf{x}_{i,j}^h\right) = z(x_i, y_j), i, j = 0, ..., n+1.$$

Let the function $f(\mathbf{x}) \in C^4\left(\overline{\Omega}\right)$. We define its first derivative with respect to $x$ in finite differences at the points $(x_i, y_j) \in \Omega^h$, $i, j = 1, ..., n$ as:

$$\partial_x^h f_{i,j} = \partial_x^h f^h(x_i, y_j) = \frac{f(x_{i+1}, y_j) - f(x_{i-1}, y_j)}{2h}.$$

Next, its second derivative with respect to $x$ in finite differences at the points $(x_i, y_j) \in \Omega^h$, $i, j = 1, ..., n$ is defined as:

$$\partial_x^{h,2} f_{i,j} = \partial_x^{h,2} f^h(x_i, y_j) = \frac{f(x_{i+1}, y_j) - 2f(x_i, y_j) + f(x_{i-1}, y_j)}{h^2}.$$

Because of the Neumann boundary conditions at $\Gamma_0$, we define its finite difference approximation at points $f\left(x_{n+1}, y_k\right) = f\left(a+c, y_k\right)$ as:

$$
\begin{gathered}
\partial_x^h f_{n+1,j} = \partial_x^h f^h\left(a+c, y_j\right) = \\
= \left[3f\left(a+c, y_j\right) - 4f\left(x_n, y_j\right) + f\left(x_{n-1}, y_j\right)\right]/\left(2h\right), \\
\left(a+c, y_j\right) \in \Gamma_0^h,\ j = 0, ..., n+1.
\end{gathered}
$$

Note that

$$
\begin{gathered}
\partial_x^h f_{i,j}^h = f_x\left(x_i, y_j\right) + O\left(h^2\right),\ \partial_x^{h,2} f_{i,j} = f_{xx}\left(x_i, y_j\right) + O\left(h^2\right), \\
i, j = 1, ..., n, \\
\partial_x^h f_{n+1,k} = \partial_x^h f^h\left(a+c, y_k\right) = f_x\left(a+c, y_k\right) + O\left(h^2\right), k = 0, ..., n+1, \\
\left|O\left(h^2\right)\right| \leq C\left\|f\right\|_{C^4\left(\overline{\Omega}\right)} h^2, \\
\forall f \in C^4\left(\overline{\Omega}\right).
\end{gathered}
$$

**Proposition.** *For each function $f \in C^4\left(\overline{\Omega}\right)$ the following accuracy estimates are valid for points $\mathbf{x}_{i,j}^h \in \Omega^h$ as $h \to 0^+$*

$$
\begin{gathered}
\partial_x^{h,2} f_{i,j} = f_{xx}\left(x_i, y_j\right) + O\left(h^2\right), \ \partial_y^{h,2} f_{i,j} = f_{yy}\left(x_i, y_j\right) + O\left(h^2\right), \\
\partial_{xy}^{h,2} f_{i,j}^h = f_{xy}\left(x_i, y_j\right) + O\left(h^2\right), \\
\nabla^h f_{i,j} = \nabla f\left(x_i, y_j\right) + O\left(h^2\right), \\
\partial_x f_{n+1,k}^h = f_x\left(a + c, x_{2,k}\right) + O\left(h^2\right), \\
\left|O\left(h^2\right)\right| \leq C \left\|f\right\|_{C^4\left(\overline{\Omega}\right)} h^2, \\
i, j = 1, ..., n; \ k = 0, ..., n+1.
\end{gathered}
$$

For any function $f \in C^1\left(\overline{\Omega}\right)$:

$$\int\limits_{\Omega} f\left(\mathbf{x}\right) dx = \int\limits_{\Omega, rt} f\left(\mathbf{x}_{ij}^h\right) d\mathbf{x}_{ij}^h + O\left(h\right), \ \ \left|O\left(h\right)\right| \leq C \left\|f\right\|_{C^1\left(\overline{\Omega}\right)} h,$$

where the symbol

$$\int\limits_{\Omega, rt} f\left(\mathbf{x}_{ij}^h\right) d\mathbf{x}_{ij}^h$$

$$\int\limits_{\Omega, rt} f\left(\mathbf{x}_{ij}^h\right) d\mathbf{x}_{ij}^h = h^2 \sum_{i,j=0}^{n+1} f^2\left(x_i, y_j\right), \ \forall f \in C^1\left(\overline{\Omega}\right).$$

# Some function spaces

We introduce some discrete analogs of function spaces.

$$L_2^h\left(\Omega^h\right) = \left\{ f^h : \left\| f^h \right\|_{L_2^h\left(\Omega^h\right)}^2 = \int\limits_{\Omega,rt} f^2\left(\mathbf{x}_{ij}^h\right) d\mathbf{x}_{ij}^h = h^2 \sum_{i,j=0}^{n+1} f^2\left(x_i, y_j\right) \right\}.$$

$$H^{2,h}\left(\Omega^h\right) = \left\{ \begin{array}{c} f^h : \left\| f^h \right\|_{H^{2,h}\left(\Omega^h\right)}^2 = \left\| f^h \right\|_{L_2^h\left(\Omega^h\right)}^2 + \\ + h^2 \sum\limits_{i,j=1}^{n} \left[ \left(\partial_x^h f_{i,j}\right)^2 + \left(\partial_y^h f_{i,j}\right)^2 \right] + \\ + h^2 \sum\limits_{i,j=1}^{n} \left[ \left(\partial_x^{h,2} f_{i,j}\right)^2 + \left(\partial_y^{h,2} f_{i,j}\right)^2 + \left(\partial_{yy}^2 f_{i,j}^h\right)^2 \right] \end{array} \right\}.$$

By Proposition

$$\left\|f^h\right\|^2_{L_2^h\left(\Omega^h\right)} = \|f\|^2_{L_2(\Omega)} + O\left(h\right) \text{ as } h \to 0^+, \forall f \in C^1\left(\overline{\Omega}\right),$$
$$|O\left(h\right)| \leq C\,\|f\|^2_{C^1\left(\overline{\Omega}\right)} h.$$

$$\left\|f^h\right\|^2_{H^{2,h}\left(\Omega^h\right)} = \|f\|^2_{H^2(\Omega)} + O\left(h\right) \text{ as } h \to 0^+,\ \forall f \in C^4\left(\overline{\Omega}\right),$$
$$|O\left(h\right)| \leq C\,\|f\|^2_{C^4\left(\overline{\Omega}\right)} h.$$

$$H_{1,2,\theta,fd} = \left\{ \begin{array}{c} \left(q^h, p^h\right)\left(\mathbf{x}^h_{i,j}, \theta\right) : q^h, p^h \in H^{2,h}\left(\Omega^h\right), \\ \left\|\left(q^h, p^h\right)\left(\mathbf{x}^h_{i,j}, \theta\right)\right\|^2_{H_{1,2,\theta,fd}} = \\ = \left\|q^h\left(\mathbf{x}^h_{i,j}, \theta\right)\right\|^2_{H^{2,h}\left(\Omega^h\right)} + \left\|p^h\left(\mathbf{x}^h_{i,j}, \theta\right)\right\|^2_{H^{2,h}\left(\Omega^h\right)} < \infty, \\ \forall \theta \in [0, 2\pi], \end{array} \right\},$$

$$H_{1,2,2,\theta,fd} = \left\{ \begin{array}{c} \left(q^h, p^h, q, p\right) : \\ \left(q^h, p^h\right) \in H_{1,2,\theta,fd}, \ \ (q, p) \in H_{1,2,\theta}, \\ \left(q^h, p^h\right)(x_i, y_j, \theta) = (q, p)(x_i, y_j, \theta), \\ i, j = 0, ..., n+1, \\ \left\|\left(q^h, p^h, q, p\right)\right\|^2_{H_{1,2,2,\theta,fd}} = \\ = \left\|\left(q^h, p^h\right)\right\|^2_{H_{1,2,\theta,fd}} + \|(q, p)\|^2_{H_{1,2,\theta}}, \\ \forall \theta \in [0, 2\pi], \end{array} \right\}$$

$$H^0_{1,2,\theta,fd} = \left(q^h, p^h\right) : \left\{ \begin{array}{c} q^h\left(\mathbf{x}^h_{i,j}, \theta\right) |_{\partial\Omega^h} = p^h\left(\mathbf{x}^h_{i,j}, \theta\right) |_{\partial\Omega^h} = 0, \\ i,j = 0, ..., n+1, \\ \partial_x q^h_{n+1,k} = \partial_x p^h_{n+1,k} = 0, k = 0, ..., n+1, \\ \text{for } \partial_x q^h_{n+1,k}, \partial_x p^h_{n+1,k}, \\ \|\cdot\|_{H^0_{1,2,\theta,fd}} = \|\cdot\|_{H_{1,2,\theta,fd}} \end{array} \right\},$$

$$H^0_{1,2,2,\theta,fd} = \left\{ \begin{array}{c} \left(q^h, p^h, q, p\right) \in H_{1,2,2,\theta,fd} : \\ \left(q^h, p^h\right) \in H^0_{1,2,\theta,fd}, (q, p) \in H^0_{1,2,\theta}, \\ \|\cdot\|_{H^0_{1,2,2,\theta,fd}} = \|\cdot\|_{H_{1,2,2,\theta,fd}}, \ \forall\theta \in [0, 2\pi] \end{array} \right\},$$

$$H^2_{1,2,\theta} = \left\{ \begin{array}{c} (q,p)(\mathbf{x},\theta) \in H^2(\Omega) \times H^2(\Omega) : \\ \|(q,p)(\mathbf{x},\theta)\|^2_{H^2_{1,2,\theta}} = \|q(\mathbf{x},\theta)\|^2_{H^2(\Omega)} + \|p(\mathbf{x},\theta)\|^2_{H^2(\Omega)}, \\ \forall\theta \in [0,2\pi]. \end{array} \right\}$$

$$H^2_{1,2,2,\theta,fd} = \left\{ \begin{array}{c} \left(q^h,p^h,q,p\right) : \\ \left(q^h,p^h\right) \in H_{1,2,\theta,fd}, \ \ (q,p) \in H^2_{1,2,\theta}, \\ \left\|\left(q^h,p^h,q,p\right)\right\|_{H^2_{1,2,2,\theta,fd}} = \\ = \left\|\left(q^h,p^h\right)\right\|^2_{H_{1,2,\theta,fd}} + \|(q,p)\|^2_{H^2_{1,2,\theta}} \ \ \forall\theta \in [0,2\pi]. \end{array} \right\},$$

**Definition.** *$G^h_\theta(A)$ is the set of all vector functions $\left(q^h, p^h, q, p\right) \in H_{1,2,2,\theta,fd}$ such that $(q,p) \in G_\theta(A)$. Boundary conditions for*
$\left(q^h, p^h\right)$ at $(x_{n+1}, y_j) = (a+c, y_j)$.
Hence,

$$G^h_\theta(A) \subset H_{1,2,2,\theta,fd},$$

$$\left\| \left(q^h, p^h\right) \right\|^2_{H_{1,2,\theta,fd}}(\theta) = \|(q,p)\|^2_{H^2_{1,2,\theta}}(\theta) + O(h),$$
$$\forall \left(q^h, p^h, q, p\right) \in G^h_\theta(A),\ \forall \theta \in [0, 2\pi].$$

We obtain

$$
\begin{gathered}
q^h\left(x_i, y_j, \theta\right)|_{\left(x_i, y_j\right)\in\partial\Omega^h} = \partial_\theta s_0\left(x_i, y_j, \theta\right), \\
\partial_x^h q^h\left(x_{n+1}, y_j, \theta\right) = \partial_\theta s_1\left(x_{n+1}, y_j, \theta\right), \;\; \left(x_{n+1}, y_j\right) \in \Gamma_0, \\
p^h\left(x_i, y_j, \theta\right)|_{\left(x_i, y_j\right)\in\partial\Omega^h} = \left(\partial_\theta s_0 - \varepsilon s_0\right)\left(x_i, y_j, \theta\right), \\
\partial_x^h p^h\left(x_{n+1}, y_j, \theta\right) = \left(\partial_\theta s_1 - \varepsilon s_1\right)\left(x_{n+1}, y_j, \theta\right), \left(x_{n+1}, y_j\right) \in \Gamma_0, \\
i, j = 0, ..., n+1, \\
\forall\left(q^h, p^h, q, p\right) \in G_\theta^h\left(A\right), \; \forall\theta \in [0, 2\pi],
\end{gathered}
$$

# Semi-discretization of the functional $J_{\kappa,\alpha}$

We consider the discrete analog $J_{1,h,\kappa,\alpha}$ of the functional $J_{1,\kappa,\alpha}$

$$J_{1,h,\kappa,\alpha}\left(q^h, p^h\right)(\theta) =$$

$$= \sqrt{\varepsilon} \int\limits_{\Omega, rt} \left[ \left( F_1^h \left(q^h, p^h\right) \left(\mathbf{x}_{i,j}^h, \theta\right) \right)^2 + \left( F_2^h \left(q^h, p^h\right) \left(\mathbf{x}_{i,j}^h, \theta\right) \right)^2 \right] W_\kappa^h \left(\mathbf{x}_{i,j}^h\right.$$

$$\forall \left(q^h, p^h\right) \left(\mathbf{x}_{i,j}^h, \theta\right) \in H_{1,2,\theta,fd}, \ \forall \theta \in [0, 2\pi].$$

$J_{2,\kappa,\alpha}(q,p)(\theta)$ remains the same as before.

$$J_{h,\kappa,\alpha}\left(q^h, p^h, q, p\right)(\theta) = J_{1,h,\kappa,\alpha}\left(q^h, p^h\right)(\theta) + J_{2,\kappa,\alpha}(q,p)(\theta),$$
$$\forall \left(q^h, p^h, q, p\right) \in G_\theta^h(A), \ \forall \theta \in [0, 2\pi].$$

# Theorems About the Semi-Discrete Functional $J_{h,\kappa,\alpha}$

**Theorem 4** (the $h-$strong convexity).

*1. For each $\kappa > 0$, for each $\theta \in [0, 2\pi]$ and for each vector function $\left(q^h, p^h, q, p\right) \in \overline{G_\theta^h(A)}$ the functional $J_{h,\kappa,\alpha}\left(q^h, p^h, q, p\right)(\theta)$ has the Fréchet derivative*

$$J'_{h,\kappa,\alpha}\left(q^h, p^h, q, p\right)(\theta) \in H^0_{1,2,2,\theta,fd},$$

*where $H^0_{1,2,2,\theta,fd}$ is the function space defined above. Furthermore,*

$$J'_{h,\kappa,\alpha}\left(q_1^h, p_1^h, q_1, p_1\right)(\theta)\left(q_2^h - q_1^h, p_2^h - p_1^h, q_2 - q_1, p_2 - p_1\right)\left(x_{i,j}^h, \theta\right) =$$

$$= J'_{\kappa,\alpha}(q_1, p_1)(\theta)(q_2 - q_1, p_2 - p_1) + O(h)\exp\left(2\kappa(a+c)^2\right),$$

$$\forall\left(q_1^h, p_1^h, q_1, p_1\right), \left(q_2^h, p_2^h, q_2, p_2\right) \in G_\theta^h(A), \ \forall\theta \in [0, 2\pi],$$

*where $J'_{\kappa,\alpha}(q, p)(\theta)$ is the Fréchet derivative of the functional $J_{\kappa,\alpha}(q, p)(\theta)$,* *2. Let $\kappa_0 = \kappa_0(\Omega) \geq 1$ and $\kappa_1 = \kappa_1(A, \Omega, \varepsilon) \geq \kappa_0$ be the numbers of Theorems 1 and 2 respectively. Then there exists a sufficiently large number $\kappa_2 = \kappa_2(A, \Omega, \varepsilon) \geq \kappa_1$ such that for each $\kappa \geq \kappa_2$ and for each $\theta \in [0, 2\pi]$ the functional $J_{h,\kappa,\alpha}(q^h, p^h, q, p)(\theta)$ is $h-$strongly convex on the set $\overline{G_\theta^h(A)}$, i.e.*

$$J_{h,\kappa,\alpha}\left(q_2^h, p_2^h, q_2, p_2\right)(\theta) - J_{h,\kappa,\alpha}\left(q_1^h, p_1^h, q_1, p_1\right)(\theta) -$$

$$-J'_{h,\kappa,\alpha}\left(q_1^h, p_1^h, q_1, p_1\right)(\theta)\left(q_2^h - q_1^h, p_2^h - p_1^h, q_2 - q_1, p_2 - p_1\right) \geq$$

$$\geq C_1 \exp\left(2\kappa(a-c)^2\right)\left\|\left(q_2^h - q_1^h, p_2^h - p_1^h\right)\right\|_{H_{1,2,\theta,fd}}^2 +$$

$$+\alpha\left\|(q_2 - q_1, p_2 - p_1)\right\|_{H_{1,2,\theta}}^2 - C_1 h \exp\left(2\kappa(a+c)^2\right),$$

$$\forall\left(q_1^h, p_1^h, q_1, p_1\right), \left(q_2^h, p_2^h, q_2, p_2\right) \in \overline{G_\theta^h(A)},$$

$$\forall\theta \in [0, 2\pi],\ \forall\kappa \geq \kappa_2.$$

*3. Assume that there exists a pair of numbers* $(\kappa_3, \theta_0)$ *satisfying* $\kappa_3 \geq \kappa_2$, $\theta_0 \in [0, 2\pi]$ *and such that there exists a minimizer*

$$X_{\min} = \left( \widetilde{q}^h_{\min,\kappa_3,\alpha}, \widetilde{p}^h_{\min,\kappa_3,\alpha}, \widetilde{q}_{\min,\kappa_3,\alpha}, \widetilde{p}_{\min,\kappa_3,\alpha} \right) \in \overline{G^h_{\theta_0}(A)}$$

*of the functional* $J_{h,\kappa_3,\alpha}\left(q^h, p^h, q, p\right)(\theta_0)$ *on the set* $\overline{G^h_{\theta_0}(A)}$. *In other words, assume that*

$$J_{h,\kappa_3,\alpha}(X_{\min})(\theta_0) \leq J_{h,\kappa,\alpha}\left(q^h, p^h, q, p\right)(\theta_0),$$
$$\forall \left(q^h, p^h, q, p\right)(\theta_0) \in \overline{G^h_{\theta_0}(A)}.$$

Then

$$J'_{h,\kappa_3,\alpha}(X_{\min})\left(X_{\min} - \left(q^h, p^h, q, p\right)\right)(\theta_0) \leq 0,$$
$$\forall \left(q^h, p^h, q, p\right) \in \overline{G^h_{\theta_0}(A)}.$$

**Theorem 5.** *Assume that conditions of Theorems 3 and 4 hold. In particular, let* $\kappa_3$ *and* $\theta_0$ *be the pair of numbers of item 3 of Theorem 4 and let* $\left(q_{\min,\kappa_3,\alpha}, p_{\min,\kappa_3,\alpha}\right)(\mathbf{x},\theta_0) \in \overline{G_{\theta_0}(A)}$ *be the corresponding minimizer of the functional* $J_{\kappa_3,\alpha}(q,p)(\theta)$ *on the set* $\overline{G_{\theta_0}(A)}$*. Let*

$$Z_{\min} = \left(q^h_{\min,\kappa_3,\alpha}, p^h_{\min,\kappa_3,\alpha}, q_{\min,\kappa_3,\alpha}, p_{\min,\kappa_3,\alpha}\right) \in \overline{G^h_{\theta_0}(A)}$$

*be the corresponding point of the space* $H_{1,2,2,\theta_0,fd}$*. Also, let* $X_{\min} \in \overline{G^h_{\theta_0}(A)}$ *be the minimizer of the functional* $J_{h,\kappa_3,\alpha}\left(q^h, p^h, q, p\right)(\theta_0)$ *on the set* $\overline{G^h_{\theta_0}(A)}$, *which was mentioned in item 3 of Theorem 4. Denote*

$$Z^* = \left(q^{*h}, p^{*h}, q^*, p^*\right) \in G^h_{\theta_0}(A),$$

*see item 2 of Theorem 3 about the pair* $(q^*, p^*)(\mathbf{x}, \theta_0) \in G_{\theta_0}(A)$.
*Then the following accuracy estimates hold*

$$\|Z_{\min} - X_{\min}\|_{H^2_{1,2,2,\theta_0,fd}} \leq C_1\left(\sqrt{\alpha} + \sqrt{h}\right)\exp\left(\kappa_3 (a+c)^2\right),$$

$$\|Z^* - X_{\min}\|_{H^2_{1,2,2,\theta_0,fd}} \leq C_1\left(\sqrt{\alpha} + \sqrt{h}\right)\exp\left(\kappa_3 (a+c)^2\right).$$

# Minimizing sequence

Let $\kappa_2 = \kappa_2(A, \Omega, \varepsilon) \geq 1$ be the number of Theorem 4. In particular, unlike Theorem 5, where the numbers $\kappa_3 \geq \kappa_2$ and $\theta_0 \in [0, 2\pi]$ are fixed, we assume here that $\kappa \geq \kappa_2$ and $\theta \in [0, 2\pi]$ are two arbitrary numbers. Denote

$$m(h, \kappa, \alpha, \varepsilon, A, \theta) = \inf_{\overline{G_\theta^h(A)}} J_{h,\kappa,\alpha}\left(q^h, p^h, q, p\right), \ \forall \kappa \geq \kappa_2, \ \forall \theta \in [0, 2\pi].$$

There exists a minimizing sequence $\left\{\left(q_n^h, p_n^h, q_n, p_n\right)\right\}_{n=1}^{\infty} \subset \overline{G_\theta^h(A)}$ such that

$$\begin{gathered}\lim_{n\to\infty} J_{h,\kappa,\alpha}\left(q_n^h, p_n^h, q_n, p_n\right) = m(h, \kappa, \alpha, \varepsilon, A, \theta), \\ J_{h,\kappa,\alpha}\left(q_n^h, p_n^h, q_n, p_n\right) \geq m(h, \kappa, \alpha, \varepsilon, A, \theta), \ \forall n \geq 1.\end{gathered}$$

Denote

$$Z_n = \left(q_n^h, p_n^h, q_n, p_n\right) \in \overline{G_\theta^h(A)}.$$

**Theorem 6.** *Assume that is replaced with*

$$Z^* = \left(q^{*h}, p^{*h}, q^*, p^*\right) \in G_\theta^h(A), \ \forall \theta \in [0, 2\pi].$$

*Let* $\kappa \geq \kappa_2$ *be an arbitrary number. Let* $(q_{\min,\kappa,\alpha}, p_{\min,\kappa,\alpha})(\mathbf{x}, \theta) \in \overline{G_\theta(A)}$ *be the minimizer of the functional* $J_{\kappa,\alpha}(q,p)(\theta)$ *on the set* $\overline{G_\theta(A)}$*, which was found in Theorem 2. Let* $\left\{\left(q_n^h, p_n^h, q_n, p_n\right)\right\}_{n=1}^{\infty} \subset \overline{G_\theta^h(A)}$ *be the above minimizing sequence of the functional* $J_{h,\kappa,\alpha}\left(q^h, p^h, q, p\right)$*. Then for each* $\theta \in [0, 2\pi]$ *there exists such an integer* $N = N(h, \kappa, \alpha, \varepsilon, A, \theta) > 1$ *depending only on listed parameters that the following analogs of accuracy estimates hold*

$$\|q_{\min,\kappa,\alpha} - q_n, p_{\min,\kappa,\alpha} - p_n\|_{H^2_{1,2,\theta}} \leq$$
$$\leq C_1 \left(\sqrt{\alpha} + \sqrt{h}\right) \exp\left(\kappa (a + c)^2\right), \ \forall n \geq N,$$

$$\|q^* - q_n, p^* - p_n\|_{H^2_{1,2,\theta}} \leq$$
$$\leq C_1 \left(\sqrt{\alpha} + \sqrt{h}\right) \exp\left(\kappa (a + c)^2\right), \ \forall n \geq N,$$

$$\|Z^* - Z_n\|_{H^2_{1,2,2,\theta,fd}} \leq C_1 \left(\sqrt{\alpha} + \sqrt{h}\right) \exp\left(\kappa (a + c)^2\right), \ \forall n \geq N.$$

# Deep Learning With A Priori Accuracy Estimates on the Training Stage

Let $\kappa_3$ and $\theta_0$ be two numbers of Theorem 5. Since we now have only one value of the number $\theta = \theta_0 \in [0, 2\pi]$ instead of the whole interval $\theta \in [0, 2\pi]$, then

$$r_{\kappa_3,\alpha}\left(\mathbf{x}_{i,j}^h\right) = -\left(\Delta^h \psi_{\min,\kappa_3,\alpha} + \left(\nabla \psi_{\min,\kappa_3,\alpha}\right)^2\right)\left(\mathbf{x}_{i,j}^h, \theta_0\right).$$

Here the function $r_{\kappa_3,\alpha}\left(\mathbf{x}_{i,j}^h\right)$ is our computational result. Next, we interpolate formula on a fine grid in $\Omega$. We obtain this way an approximation $\sigma_{\kappa_3,\alpha}(\mathbf{x})$ of the true coefficient $\sigma^*(\mathbf{x})$. Accuracy of the latter approximation is the same as

$$\left\|\sigma_{\kappa_3,\alpha} - \sigma^*\right\|_{L_2(\Omega)} \leq C_1\left(\sqrt{\alpha} + \sqrt{h}\right)\exp\left(\kappa_3 (a+c)^2\right).$$

# Hybrid Deep Learning Model

We propose an unsupervised hybrid image-to-image model that combines a Denoising Convolutional Neural Network (DnCNN) with a Residual U-Net enhanced by Squeeze-and-Excitation (SE) attention blocks.

The network is trained to minimize a loss function measuring the dissimilarity between its output

$$\sigma_{\mathrm{pred}}^{(i)} = \mathrm{U\text{-}Net}\Big(\sigma_{\mathrm{conv}}^{(i)}\Big)$$

and the true solution $\sigma^{(i)*}$ for each $i = 1, \ldots, N$.
For each training case $i = 1, \ldots, N$, we use the reconstruction obtained by the convexification method on the coarse grid, since it guarantees the accuracy estimate

$$O\Big(\sqrt{\alpha} + \sqrt{h}\Big) \qquad \text{as } \sqrt{\alpha} + \sqrt{h} \to 0.$$

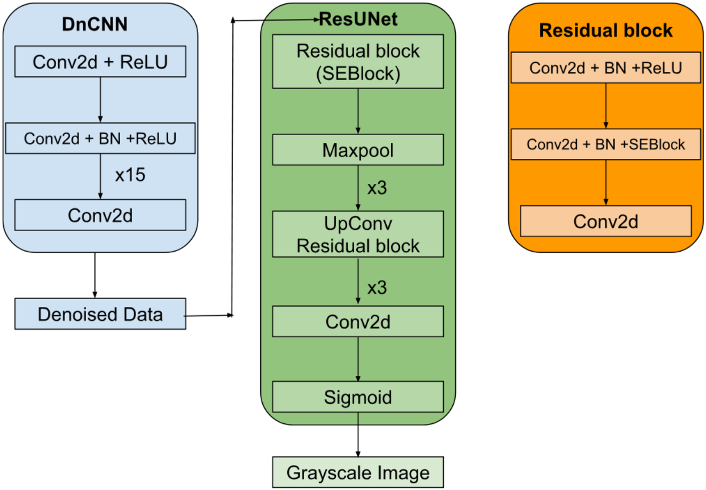


Figure 2: Architecture of the proposed DnCNN + ResUNet model with SEBlock.

To train the network, we require a large dataset of paired images

$$\big(\sigma^{(i)}_{\mathrm{conv}}, \sigma^{(i)*}\big), \qquad i = 1, \ldots, N,$$

where $\sigma^{(i)*}$ is the true solution for the case $i$, and $\sigma^{(i)}_{\mathrm{conv}}$ is the corresponding noisy reconstruction obtained from the semi-discrete convexification method on the coarse grid and interpolated then on a fine grid in $\Omega$.

$$\mathcal{L}_{\mathrm{total}} = \gamma\big(1 - \mathrm{MS\text{-}SSIM}\big) + (1-\gamma)\,\mathrm{L1L\textsc{oss}}, \qquad \gamma = 0.84.$$

# Combined DnCNN + ResUNet Architecture

Table 1: Architecture summary of the combined DnCNN + ResUNet model.

| **Component** | **Details** |
|---|---|
| DnCNN | 17-layer residual denoiser (Conv-BN-ReLU) |
| ResUNet | 3-level encoder-decoder with skip connections |
| Residual Block | 2 convolutional layers + SE attention + identity skip |
| SE Block | Channel-wise self-attention via squeeze-and-excitation |
| Loss | $\gamma = 0.84$: MS-SSIM + L1 |
| Output | Sigmoid activation to scale prediction to $[0, 1]$ |

# Numerical Setup

For each $\theta \in [0, 2\pi)$ and for the corresponding position $\mathbf{x}_0 = \mathbf{x}_0(\theta) \in E_B(a, b)$, the forward problem is solved by the finite element method (FEM) in the disk $P_D$.

The radius of this disk is $D = 3$, and its center is at the point

$$(a, b) = (1.5, 1.5),$$

see Figure 1. The circle $E_B(a, b)$ is therefore $E_B(1.5, 1.5)$ with radius

$$B = 2.$$

The spatial mesh size of the FEM is

$$\frac{1}{160}.$$

The square $\Omega$ is

$$\Omega = \{\mathbf{x} = (x, y) : 1 < x, y < 2\},$$

i.e., $c = 0.5$.

The source function $g(\mathbf{x} - \mathbf{x}_0)$ is defined as before with

$$\xi = 0.1.$$

We use 199 discrete source positions corresponding to

$$\theta_n = n\rho_\theta, \qquad n = 1, 2, \ldots, 199, \qquad \rho_\theta = \frac{\pi}{100}.$$

For the convexification method, we used a finite difference scheme to discretize the partial differential operators in the computational domain $\Omega$.

We used two coarse grids:

$$10 \times 10 \quad \text{and} \quad 20 \times 20$$

pixels, which correspond to grid step sizes

$$h_1 = 0.1, \qquad h_2 = 0.05,$$

respectively.

On the other hand, the grid step size for the convexification method on the fine grid was

$$h_3 = \frac{1}{40} = 0.025,$$

which is two times smaller than $h_2$. This explains the

$$14.25 = \frac{57}{4}$$

fold speedup in computations on the grid with $h = h_2 = 0.05$.

We used the parameters

$$\alpha = 0.01, \qquad \varepsilon = 0.0002, \qquad \kappa = 3.$$

Since the minimization in the convexification method is carried out on the set $\overline{G_\theta(A)}$ with constraints, we used the MATLAB built-in function **fmincon**.
The true solutions and the output of the deep learning network are defined on a finer grid of

$$128 \times 128$$

pixels.

# Numerical Results

We compare the reconstruction results obtained from two different spatial discretizations of the coarse grids:

1. A too coarse grid with step size

$$h_1 = 0.1.$$

2. A coarse grid with a finer grid with step size

$$h_2 = 0.05.$$

Figure 3: Left column: image obtained by the semi-discrete version of the convexification method on a too coarse grid with the grid step size $h_1 = 0.1$. Right column: the true image. Middle column: the reconstructed image after deep learning. These results are visibly worse than those obtained with the a finer coarse grid step size $h_2 = 0.05$

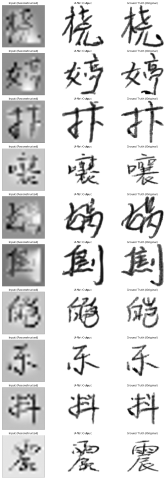


Figure 4: Left column: the image obtained by the semi-discrete version of the convexification method on a coarse mesh with a finer coarse grid with step size $h_2 = 0.05$. Middle column: the reconstructed image after deep learning.

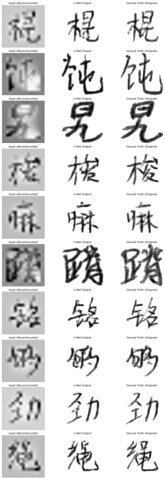

Figure 5: This is the second series of images with the grid step size of a finer coarse grid $h_2 = 0.05$. Left column: image obtained by the semi-discrete version of the convexification method with $h_2 = 0.05$. Right column: the true image. Middle column: the reconstructed image after deep learning.

Thank you for your attention!

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