Separation of solutions and the attractivity of fractional-order positive linear delay systems with variable coefficients

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OTHA Seminar, May 16, 2024

Introduction (I)

Fractional differential equations is a topic of research that has received a lot of attention. Nowadays, ones find more and more applications of them in different fields of science and technology. Their latest applications can be found in the review paper [SZBCC18] and recently published monographs, see, e.g., [BL19(V1), BL19(V2), P19, T19(V1), T19(V2)] and references therein.

Introduction (II)

As suggested from ordinary differential equations, when studying the stability of fractional order differential equations, one often uses the following two approaches: the linearization stability method and the Lyapunov candidate function method, see, e.g., [CTT20] and references therein. Unfortunately, however, both of the approaches mentioned above do not seem to be applicable to systems with many fractional-order derivatives. This is the reason why it is necessary to build new theoretical foundations for the qualitative theory of such systems.

Introduction (III)

Up to now, there are two strategies commonly employed to study the asymptotic behavior of solutions to fractional systems with multi-fractional derivatives (incommensurate-order systems). First, take the Laplace transform to obtain a characteristic pseudo-polynomial. Then, rely on different techniques (Nyquist's theorem, see, e.g., [TBMP09, SFT13], Mikhailov stability criterion, see, e.g., [S22], Rouché's theorem, see, e.g., [BK21], Cauchy's argument, see, e.g., [DTT22], and so on) to investigate the position of the roots of this polynomial. Depending on the situation where the zeros are in stable (or unstable) regions, it is concluded that the solutions of the system converge to the origin or to infinity. The second way is to use the principles of comparison. This method is especially useful for classes of systems where the order relation in phase space is preserved. Fortunately, fractional-order positive systems (with or without delays) have this property. Below we review some of the important contributions where comparative arguments play an essential role in analyzing the variation of solutions to such systems according to the time variable.

Introduction (IV)

In [SL16], J. Shen and J. Lam have explored a stability criterion for multi-order fractional positive linear systems with bounded delays. After that, in [TTL21], H.T. Tuan, H. Trinh, and J. Lam have provided an extended version for the stability of such systems when the delays are unbounded. By combining the candidate Lyapunov type functions and comparison arguments, K.L. Bichitra and N.B. Swaroop [BS22], J.A. Gallegos et al. [GAD20] have shown the global attractivity of systems under the influence of multiple time-varying delays while J. Jia et al. [JWZ21] have studied the global stabilization of incommensurate-order memristor-based neural networks with time-varying delays. Based on developing new comparison techniques that are compatible with fractional order derivatives and delays, H.T. Tuan and L.V. Thinh [TT23] have newly obtained the attractivity of coupled positive linear delay systems. However, to the best of our knowledge, there have been almost no studies on the stability of fractional-order positive linear systems with variable coefficient matrices (both with and without delays) announced.

Introduction (V)

Let $\hat{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0,1] \times \cdots \times (0,1] := (0,1]^d$ and $h > 0$, in current work, we focus on the fractional-order system

$$^{C}D^{\hat{\alpha}}_{0^+} y(x) = A(x)y(x) + B(x)y(x - q(x)), \forall x \in (0, T], \tag{1}$$

$$y(x) = \omega(x), \ \forall x \in [-h, 0], \tag{2}$$

here the notation $^{C}D^{\hat{\alpha}}_{0^+} y(x)$ is defined below, $A(\cdot)$, $B(\cdot) : [0, T] \to \mathbb{R}^{d \times d}$ are continuous matrix-valued functions, $q(\cdot) : [0, T] \to \mathbb{R}$ is a continuous delay satisfying $0 \leq q(x) \leq h$ on $[0, T]$, and $\omega(\cdot) : [-h, 0] \to \mathbb{R}^d$ is a given continuous initial condition.

Introduction (VI)

The question of whether solutions of a fractional differential equation can intersect was discussed by K. Diethelm [D08, D10], K. Diethelm and N.J. Ford [DF12]. The main difficulty of the problem for those classes is the history-dependence of their solutions. In [CT17], N.D. Cong and H.T. Tuan have rigorously proved that solutions starting at different points of a scalar fractional differential equation do not intersect [Theorem 3.5, CT17]. However, in the multi-dimensional case, it is very surprising that they have shown that solutions derived from different initial conditions of a fractional-order system can intersect in general [Theorem 6.1, CT17]. This is another special property of solutions to fractional-order systems compared with ordinary differential equations.

Introduction (VII)

Motivated by N.D. Cong and H.T. Tuan [CT17] and inspired by recent knowledge published in [TT20, TTL21, TT23], our first aim is to prove the separation of solutions and lower estimates of solutions to the system (1)–(2). Then, by comparative arguments, we search for different criteria to characterize the global attractivity of multi-order fractional positive linear systems with variable coefficients and bounded delays. In addition, in the case when the fractional orders are equal, we describe precisely the rate of convergence of the solutions to the origin. These are the first results to appear in the literature on the subject.

Notation and Preliminary (I)

Let $\mathbb{N}$, $\mathbb{R}$ be the set of natural numbers and real numbers, respectively. $\mathbb{R}^d$ stands for the d-dimensional real Euclidean space. Let $\mathbb{R}^d_{\geq 0}$, $\mathbb{R}^d_+$ be the set of all vectors in $\mathbb{R}^d$ with nonnegative and positive entries, that is, $\mathbb{R}^d_{\geq 0} = \{y = (y_1, ..., y_d)^{\mathrm{T}} \in \mathbb{R}^d : y_i \geq 0,\ 1 \leq i \leq d\}$ and $\mathbb{R}^d_+ = \{y = (y_1, ..., y_d)^{\mathrm{T}} \in \mathbb{R}^d : y_i > 0,\ 1 \leq i \leq d\}$, respectively. For two vectors $u, v \in \mathbb{R}^d$, we write

- $u \preceq v$ if $u_i \leq v_i$ for all $1 \leq i \leq d$,
- $u \precneqq v$ if $u_i \leq v_i$ for all $1 \leq i \leq d$ and there is $i_0 \in \{1, \cdots, d\}$ so that $u_{i_0} < v_{i_0}$,
- $u \prec v$ if $u_i < v_i$ for all $1 \leq i \leq d$.

Notation and Preliminary (II)

Since all the norms in $\mathbb{R}^d$ are equivalent, here and in what follows, we only choose the L1-norm to work. More precisely, for any $y \in \mathbb{R}^d$, we set $\|y\| := \sum_{i=1}^{d} |y_i|$. Given a vector $v \in \mathbb{R}^d_+$, the weighted norm $\|\cdot\|_v$ on $\mathbb{R}^d$ is given by

$$\|y\|_v := \max_{1 \le i \le d} \frac{|y_i|}{v_i}, \ \forall y \in \mathbb{R}^d.$$

A matrix $A = (a_{ij}) \in \mathbb{R}^{d \times d}$ is Metzler if and only if its off-diagonal entries a_{ij}, $i \neq j$, are nonnegative. A is Hurwitz if its spectrum $\sigma(A)$ satisfies the stable condition

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : \Re(\lambda) < 0\}.$$

Notation and Preliminary (III)

For an interval $[a, b] \subset \mathbb{R}$, and X is a set determined later, $C([a, b]; X)$ denotes the set of all real-valued continuous functions from $[a, b]$ to X. For $\alpha \in (0, 1]$ and $T > 0$, the Riemann-Liouville fractional integral of a function $y : [0, T] \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} y(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} y(u) du, \text{ for almost every } x \in (0, T],$$

and its Caputo fractional derivative of the order $\alpha \in (0, 1]$ as

$${}^C D_{0^+}^{\alpha} y(x) := \frac{d}{dx} I_{0^+}^{1-\alpha}(y(x) - y(0)), \text{ for almost every } x \in (0, T],$$

in which $\Gamma(\cdot)$ is the Gamma function and $\frac{d}{dx}$ is the first derivative. We suggest the interested readers to [Chapter III, D10] and [V16] for more detail about the Caputo type fractional derivative. Let $\hat{\alpha} \in (0, 1]^d$ and $y = (y_1, \ldots, y_d)^{\mathrm{T}}$, where $y_i \in C([0, T]; \mathbb{R})$, $i = 1, \ldots, d$, are vector valued functions, then ${}^C D_{0^+}^{\hat{\alpha}} y(x) := \left({}^C D_{0^+}^{\alpha_1} y_1(x), \cdots, {}^C D_{0^+}^{\alpha_d} y_d(x)\right)^{\mathrm{T}}$.

Notation and Preliminary (IV)

Consider the fractional-order system (1)–(2):

$$
\begin{aligned}
{}^{C}D^{\hat{\alpha}}_{0^+}y(x) &= A(x)y(x) + B(x)y(x - q(x)),\ \forall x \in (0, T],\\
y(x) &= \omega(x),\ \forall x \in [-h, 0].
\end{aligned}
$$

From [Lemma 2.1,TT20], a function $\Phi(\cdot,\omega) \in C\left([-h, T];\mathbb{R}^d\right)$ is a solution of the initial value problem (1)–(2) on $[-h, T]$ if and only if it is a solution of the following integral system on $[0, T]$:

$$
\begin{aligned}
y_i(x) =& \omega_i(0) + \frac{1}{\Gamma(\alpha_i)} \int_0^x (x-u)^{\alpha_i - 1} \Big(\sum_{j=1}^d a_{ij}(u) y_j(u)\\
&+ \sum_{j=1}^d b_{ij}(u) y_j(u - q(u))\Big) du,\ 1 \le i \le d,
\end{aligned} \tag{3}
$$

and the initial condition $y(x) = \omega(x)$ on $[-h, 0]$ holds. Due to [Theorem 2.2, TT20], it can be shown that for any $\omega(\cdot) \in C([-h, 0];\mathbb{R}^d)$, $A(\cdot), B(\cdot) \in C([0, T];\mathbb{R}^{d\times d})$ and $q(\cdot) \in C([0, T], [0, h])$, the system (1)–(2) has a unique global solution $\Phi(\cdot,\omega)$ on $[-h, T]$.

Notation and Preliminary (V)

Definition 1

The system (1) is said to be positive if for any $\omega(\cdot) \succeq 0$ on $[-h, 0]$, the solution $\Phi(\cdot, \omega)$ of the initial value problem (1)–(2) satisfies

$$\Phi(x, \omega) \succeq 0 \text{ for all } x \in [0, T].$$

The following lemma is a sufficient condition for the positivity of the system (1). It is a natural extension of the known result in the case of autonomous systems to non-autonomous systems.

Lemma 2

The system (1) *is positive if for each* $x \in [0, T]$, $A(x)$ *is a Metzler matrix and* $B(x)$ *is a nonnegative matrix.*

Notation and Preliminary (VI)

To conclude this section, we introduce some useful auxiliary results.

Lemma 3 (Proposition 2, R18)

Let $A \in \mathbb{R}^{d\times d}$ and suppose that it is a Metzler matrix. Then, A is Hurwitz if and only if there exists a vector $\lambda \succ 0$ such that $A\lambda \prec 0$.

Lemma 4 (Lemma 25, CTT20)

Let $y : [0, T] \to \mathbb{R}$ be a continuous function, $\alpha \in (0, 1]$ and the Caputo derivative ${}^{C}D^{\alpha}_{0^+}y(\cdot)$ exists on the interval $(0, T]$. If there is some $x_0 > 0$ with $y(x_0) = 0$ and $y(x) \le 0$ for all $x \in [0, x_0)$, then ${}^{C}D^{\alpha}_{0^+}y(x_0) \ge 0$.

Lemma 5 (Lemma 3.2, HTN23)

If $\eta > 0$, $\alpha \in (0, 1]$, then for all $t \ge 0$, $s \ge 0$, we have

$$E_\alpha(-\eta t^\alpha)E_\alpha(-\eta s^\alpha) \le E_\alpha(-\eta(t + s)^\alpha).$$

Separation of trajectories of solutions of one-dimensional FDE

In this section, we consider the one-dimensional FDE

$$ {}^{C}D^{\alpha}_{0+}x(t) = f(t, x(t)), \tag{4} $$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Assume that $f(\cdot, \cdot)$ satisfies the following Lipschitz condition on the second variable: there exists a nonnegative continuous function $L : J \to \mathbb{R}_+$ such that

$$ |f(t, x) - f(t, y)| \leq L(t)|x - y| \quad \text{for all} \quad t \in J \quad \text{and all} \quad x, y \in \mathbb{R}. \tag{5} $$

Theorem 6 (Different trajectories do not meet)

Assume that $f(\cdot,\cdot)$ *satisfies the Lipschitz condition* (5). *Then for any two different initial values* $x_{10} \neq x_{20}$ *in* $\mathbb{R}$ *the trajectories of solutions of the FDE* (4) *do not meet on* J, *i.e., the solutions* $x_1(\cdot), x_2(\cdot)$ *of* (4) *starting from* $x_{10} = x_1(0)$ *and* $x_{20} = x_2(0)$ *verify* $x_1(t) \neq x_2(t)$ *for all* $t \in J$.

Bounds for solutions of scalar FDEs

Theorem 7 (Convergence rate for solutions of 1-dim FDEs)

Assume that f satisfies the Lipschitz condition (5). *For any two solutions $x_1(\cdot), x_2(\cdot)$ of the FDE* (4) *and any $t \in J$ the following estimate holds*

$$|x_2(t) - x_1(t)| \geq |x_2(0) - x_1(0)| E_\alpha\big(- (\max_{0 \leq \tau \leq t} L(\tau)) t^\alpha \big).$$

Corollary 8 (Lower bound for solutions of 1-dim FDEs)

Assume that f satisfies the Lipschitz condition (5). Assume additionally that $f(t,0)=0$ for all $t\in J$, $x\in\mathbb{R}$. Then for any solution $x(\cdot)$ of the FDE (4) and any $t\in J$ the following estimate holds

$$|x(t)| \geq |x(0)| E_\alpha\Big(-\big(\max_{0\leq\tau\leq t} L(\tau)\big)t^\alpha\Big).$$

Theorem 9 (Divergence rate for solutions of 1-dim FDEs)

Assume that f satisfies the Lipschitz condition (5)*. For any two solutions $x_1(\cdot), x_2(\cdot)$ of the FDE* (4) *and for any $t \in J$ the following estimate holds*

$$|x_2(t) - x_1(t)| \leq |x_2(0) - x_1(0)| E_\alpha\Big(\big(\max_{0 \leq \tau \leq t} L(\tau)\big) t^\alpha\Big).$$

Corollary 10 (Upper bound for solutions of 1-dim FDEs)

Assume that f satisfies the Lipschitz condition (5)*. Assume additionally that $f(t,0)=0$ for all $t \in J$, $x \in \mathbb{R}$. Then for any solution $x(\cdot)$ of the FDE* (4) *and any $t \in J$ the following estimate holds*

$$|x(t)| \leq |x(0)| E_\alpha\Big(\big(\max_{0 \leq \tau \leq t} L(\tau)\big) t^\alpha\Big).$$

One-dimensional FDEs generate two-parameter flows

Now we are in a position to show that one-dimensional FDEs generate two-parameter flows. First we define the evolution mappings of (4).

Definition 11

The mapping

$$\Phi_{0,T_1} : \mathbb{R} \to \mathbb{R}, \quad x_0 \mapsto x(T_1), \tag{6}$$

where $x_0 \in \mathbb{R}$ is an arbitrary initial value of (4), $x(\cdot)$ is the solution of (4) starting from $x(0) = x_0$ and $x(T_1)$ is the evaluation of $x(\cdot)$ at T_1, is called *the evolution mapping of* (4).

Definition 12

A two-parameter family of mappings

$$\varphi_{s,t}(\cdot) : \mathbb{R} \to \mathbb{R}, \qquad s, t \in J,$$

is called *a two-parameter flow in* $\mathbb{R}$ if $\varphi_{s,t}(x)$ is continuous as a function of three variables $s, t \in J, x \in \mathbb{R}$, for any fixed $s, t \in J$ the mapping $\varphi_{s,t}$ is a homeomorphism of $\mathbb{R}$, and this family satisfies the following flow property

$$\varphi_{s,t} \circ \varphi_{u,s} = \varphi_{u,t} \qquad \text{for all} \quad u, s, t \in J.$$

Theorem 13 (One-dimensional FDEs generate two-parameter flows in $\mathbb{R}$)

The following statements hold for the one-dimensional FDE (4).

1. *The evolution mapping* $\Phi_{0,t}$ *generated by* (4) *is a bijection for any* $t \in J$.
2. *The FDE* (4) *generates a two-parameter family of bijections on* J *by its evolution mappings as follows*

$$\Phi_{s,t} := \Phi_{0,t} \circ \Phi_{0,s}^{-1} \qquad \text{for all} \quad s, t \in J, \tag{7}$$

   where $\Phi_{0,\cdot}$ *is the evolution mapping of* (4) *defined in Definition 11.*
3. *The family* $\Phi_{s,t}$, $s, t \in J$, *generated by the FDE* (4) *is a two-parameter flow in* $\mathbb{R}$.
4. *If* f *is linear in* x *then the two-parameter flow generated by the FDE* (4) *is a flow of linear operators.*

Definition 14

The two-parameter flow $\Phi_{s,t}$, specified in Theorem 13, generated by the FDE (4) is called the *nonlocal dynamical system generated by* (4).

Remark

Some distinguished features of the two-parameter flow generated by the FDE (4):
(i) The flow has history memory. Though the past has impact on the behavior of the solutions, the solutions form a two-parameter flow of homeomorphisms.
(ii) The flow is in general α-Hölder, but not C^1.

A general high dimensional system of FDEs does not generate a dynamical system

Theorem 15 (Different trajectories of a high dimensional system of FDEs can meet)

For any $d \geq 2$ there exists a system of type (4) with a property that it has two different solutions $x_1(\cdot), x_2(\cdot)$ with $x_1(0) \neq x_2(0)$ which intersect each other at some finite time moment $0 < T < \infty$, i.e., $x_1(T) = x_2(T)$.

Separation of solutions and lower estimates of solutions to multi-order fractional positive linear delay systems with variable coefficients

In this section, we discuss on the separation of solutions to multi-order fractional positive linear systems with variable coefficients and bounded delays. Our first contribution is the result below.

Theorem 16 (Separation of solutions)

Consider the system (1). *Suppose that for each* $x \in [0, T]$, $A(x)$ *is a Metzler matrix and* $B(x)$ *is nonnegative. Let* $\omega(\cdot), \psi(\cdot) \in C\left([-h, 0], \mathbb{R}^d\right)$ *with* $\omega(x) \preceq \psi(x),\ \forall x \in [-h, 0]$ *and* $\omega(0) \precnsim \psi(0)$, *then two solutions* $\Phi(\cdot, \omega), \Phi(\cdot, \psi)$ *of the system* (1) *do not meet on* $[0, T]$.

Remark

When the ordered initial conditions $\omega, \psi \in C([-h, 0], \mathbb{R}^d)$ satisfy $\psi(0) \succ \omega(0)$, we can easily deduce that the solutions $\Phi(\cdot, \psi)$, $\Phi(\cdot, \omega)$ of (1) are strictly ordered in $\mathbb{R}^d$, that is,

$$\Phi(x, \psi) \succ \Phi(x, \omega) \text{ for all } x \in [0, T].$$

Let the initial conditions be as in Theorem 16 and more constraints on the positivity are imposed on the coefficient matrix $A(\cdot)$, we obtain the strict separation of the solutions to the system (1)–(2) as follows.

Theorem 17 (Strict separation of solutions)

Consider the system (1)*. Let* $A(\cdot)$, $B(\cdot)$ *be as in Theorem 16. In addition, assume that* $a_{ij}(x) > 0$ *for all* $x \in (0, T]$ *and* $i \neq j$*. For any* $\omega(\cdot) \preceq \psi(\cdot)$ *on* $[-h, 0]$ *and* $\omega(0) \precneqq \psi(0)$*, we observe that*

$$\Phi(x, \omega) \prec \Phi(x, \psi), \ \forall x > 0.$$

From the point of view of pure mathematics, lower estimates for the separation of solutions allow to inspire interesting and useful conclusions about their behavior. In addition, applications of such estimates are found in the numerical analysis of fractional-order terminal value problems, see [DT22]. One of the remarkable results in this direction is obtained by N.D. Cong and H.T. Tuan in [CT17]. This was the key tool by which they proved the separability of the scalar equation. This result is then improved by K. Diethelm and H.T. Tuan in [Theorem 4, DT22].

Theorem 18 (Theorem 4, DT22)

Let $\alpha \in (0,1]$ and consider the equation

$$^{C}D^{\alpha}_{0^+}y(x) = m(x)y(x),\ \forall x \in (0, T], \tag{8}$$

where $m : [0, T] \to \mathbb{R}$ is continuous. Then, the following estimate holds

$$|y_1(x) - y_2(x)| \geq |y_1(0) - y_2(0)| E_\alpha(x^\alpha m_*(x)\cdot),\ x \in [0, T],$$

where $m_*(x) := \min_{0\leq s\leq x} m(s)$ and $y_1(\cdot)$, $y_2(\cdot)$ are two solutions of (8).

Based on the positivity of the system and Theorem 18, we introduce a generalized version of [Theorem 4, DT22] for mixed-order systems.

Theorem 19

Consider the system (1). Suppose that $A(x)$ is a Metzler matrix and $B(x)$ is nonnegative for each $x \in [0, T]$. For any $\omega(\cdot) \preceq \psi(\cdot)$ on $[-h, 0]$ and $\omega(0) \precneqq \psi(0)$. Put $I = \{i = \overline{1, d} : \omega_i(0) < \psi_i(0)\}$, then the following assertions hold

- *If $k \in I$, then*

$$|\Phi_k(x, \psi) - \Phi_k(x, \omega)| \geq |\omega_k(0) - \psi_k(0)| \cdot E_{\alpha_k}(x^{\alpha_k} a_k^*(x)) \text{ for all } x \geq 0.$$

- *If $k \notin I$, then $|\Phi_k(x, \psi) - \Phi_k(x, \omega)| \geq \sum_{j \in I} |\psi_j(0) - \omega_j(0)| w_j(x)$,*

where

$$w_j(x) := \int_0^x (x-u)^{\alpha_k - 1} E_{\alpha_k, \alpha_k}((x-u)^{\alpha_k} a_k^*(x-u)) a_{kj}(u) E_{\alpha_j}(u^{\alpha_j} a_j^*(u)) du$$

for all $x \geq 0$, $j \in I$, and $a_k^*(x) := \inf_{0 \leq s \leq x} a_{kk}(s)$.

Global attractivity of fractional-order positive linear delay systems

Let $\hat{\alpha} \in (0,1]^d$, $h > 0$, consider the system

$$^{C}D^{\hat{\alpha}}_{0^+} y(x) = A(x)y(x) + B(x)y(x - q(x)),\ x \in (0,\infty), \tag{9}$$
$$y(x) = \omega(x),\ x \in [-h, 0], \tag{10}$$

where $A(\cdot)$, $B(\cdot) : [0,\infty) \to \mathbb{R}^{d\times d}$ are continuous matrix-valued functions, $q(\cdot) : [0,\infty) \to [0,h]$ is continuous and $\omega(\cdot) : [-h,0] \to \mathbb{R}^d$ is a given continuous initial condition. Assume that the system (9) is positive.

Definition 20

The positive system (9) is said to be globally attractive if for any $\omega \in C([-h,0]; \mathbb{R}^d_{\geq 0})$, we have

$$\lim_{x\to\infty} \|\Phi(x,\omega)\| = 0.$$

The result below plays a key role in the rest of the talk.

Proposition 21 (Theorem 5.2, TT23)

Let $\hat{\alpha} \in (0,1]^d$, $h > 0$ and $f : [0,\infty) \to \mathbb{R}^d$ is continuous. Consider the following system

$$\begin{cases} {}^C D^{\hat{\alpha}}_{0+} y(x) & = Ay(x) + By(x - q(x)) + f(x),\ x \in (0,\infty), \\ y(x) & = \omega(x),\ x \in [-h, 0], \end{cases} \tag{11}$$

where $A, B \in \mathbb{R}^{d\times d}$, $q : [0,\infty) \to [0,h]$ is continuous and $\omega : [-h,0] \to \mathbb{R}^d_{\geq 0}$ is continuous. Suppose that A is Metzler, B is nonnegativie and $f \in C([0,\infty);\mathbb{R}^d_{\geq 0})$ with $\lim_{x\to\infty} f(x) = 0$. Then, if $A + B$ is Hurwitz, the solution $\Phi(\cdot;\omega)$ of (11) converges to 0 as $x \to \infty$.

Fractional-order system with the orders are different

We first address the case when the system (9) dominated by a positive linear delay system with constant coefficients that is also globally attractive.

Theorem 22

Consider the system (9)–(10). Assume that the following conditions are true.

(H1) *For each $x \geq 0$, $A(x)$ is a Metzler matrix and $B(x)$ is a nonnegative matrix such that $A(x) \preceq \hat{A},\ B(x) \preceq \hat{B},\ \forall x \geq 0$.*

(H2) *$\hat{A} + \hat{B}$ is a Hurwitz matrix.*

Then, every nontrivial solution of (9)–(10) converge to the origin. Moreover, for each $\omega \in C([-h, 0]; \mathbb{R}^d_{\geq 0})$, the solution $\Phi(\cdot, \omega)$ is dominated on $[0, \infty)$ by a continuous nonnegative function $\Phi^*(\cdot, \omega)$ satisfying

$$x^\gamma \Phi^*(x, \omega) \to \infty \text{ as } x \to \infty$$

for any $\gamma > \bar{\alpha} := \max\{\alpha_1, \dots, \alpha_d\}$.

In the case when the system (9) is not uniformly bounded by a system with constant coefficients that is globally attractive but is only essentially dominated by a system with this property, based on a novel comparison technique proposed in [TT23], we also obtain a criterion for attractiveness of this system. However, due to the complexity of the research subject (the delay systems contains different fractional-order derivatives), the rate of convergence of the solutions to the equilibrium point of the system is still an open problem.

Theorem 23

Consider the system (9). Assume that the following conditions are true.

(K1) *For each $x \geq 0$, $A(x)$ is a Metzler matrix and $B(x)$ is a nonnegative matrix. Moreover,*

$$a_{ij}(x) \leq \tilde{a}_{ij}, \quad b_{ij}(x) \leq \tilde{b}_{ij} \text{ for all } x \geq 0 \text{ and } i \neq j.$$

(K2) $\limsup_{x\to\infty} a_{ii}(x) < \tilde{a}_{ii}$ *and* $\limsup_{x\to\infty} b_{ii}(x) < \tilde{b}_{ii}$ *for* $i = 1, \dots, d$.

(K3) $\tilde{A} + \tilde{B}$ *is Hurwitz matrix, where* $\tilde{A} = (\tilde{a}_{ij})_{d\times d}$, $\tilde{B} = (\tilde{b}_{ij})_{d\times d}$.

Then, for any $\omega \in C([-h, 0]; \mathbb{R}^d_{\geq 0})$, *we conclude that* $\lim_{x\to\infty} \Phi(x, \omega) = 0$.

Fractional-order system with the orders are equal

This part is devoted to study fractional-order systems where the orders are equal. We will propose criteria to ensure global attractiveness of these systems. Furthermore, we also accurately describe the rate of convergence of their solutions to equilibrium.

Let $\alpha \in (0,1]$, $h > 0$, and consider the system

$$ {}^{C}D^{\alpha}_{0+}y(x) = A(x)y(x) + B(x)y(x - q(x)),\ x \in (0,\infty), \tag{12} $$

$$ y(x) = \omega(x),\ x \in [-h, 0], \tag{13} $$

where ${}^{C}D^{\alpha}_{0+}y(x) = ({}^{C}D^{\alpha}_{0+}y_1(x), \dots, {}^{C}D^{\alpha}_{0+}y_d(x))^{\mathrm{T}}$, $A(\cdot)$, $B(\cdot) : [0,\infty) \to \mathbb{R}^{d\times d}$ are continuous, $q(\cdot) : [0,\infty) \to [0,h]$ is continuous and $\omega(\cdot) \in C([-h,0];\mathbb{R}^d)$ is a given function.

Theorem 24

Suppose that that the conditions (H1) *and* (H2) *in Theorem 22 are verified. Then, for each* $\omega \in C([-h,0];\mathbb{R}^d_{\geq 0})$, *there exist positive constants* C *and* η *such that the solution* $\Phi(\cdot,\omega)$ *of the system* (12)–(13) *satisfies*

$$ 0 \leq \Phi_i(x,\omega) \leq CE_\alpha(-\eta x^\alpha),\ \forall x \geq 0,\ \forall i = 1,\dots,d. $$

Finally, inspired by the idea that emerged when proving the stability of delay differential equations using the Hanalay inequality, we obtain the following result on global attractivity of the system (12) and the rate of convergence of its solutions to the origin when requirements for dominance (uniformly) or (asymptotically) of the coefficients as in Theorem 22 and Theorem 23 are not justified.

Theorem 25

Consider the system (12)–(13). Assume that for each $x \geq 0$, $A(x)$ is Metzler and $B(x)$ is nonnegative. Moreover, suppose that there exist $\gamma > 0$ and $w = (w_1, \cdots, w_d)^{\mathrm{T}} \in \mathbb{R}^d_+$ satisfying

$$\sum_{j=1}^{d} a_{ij}(x) w_j + \sum_{j=1}^{d} \frac{b_{ij}(x)}{E_\alpha(-\gamma h^\alpha)} w_j \leq -\gamma w_i, \ x \geq 0, \ i = 1, \ldots, d. \qquad (14)$$

Then, for each $\omega \in C([-h, 0]; \mathbb{R}^d_{\geq 0})$, we can find $M > 0$ such that the following statement holds

$$0 \leq \Phi_i(x, \omega) \leq M E_\alpha(-\gamma x^\alpha), \ \forall i = 1, \ldots, d, \ \forall x \geq 0. \qquad (15)$$

Remark: another proof of Theorem 25 can be obtained using [Theorem 1.4, HTN23].

Examples

This section presents specific examples. These are used to illustrate the proposed criteria. In addition, we also provide numerical simulations to support those proposed results.

Example 1

Consider the system

$$ {}^{C}D^{\hat{\alpha}}_{0+}y(x) = A(x)y(x) + B(x)y(x - q(x)),\ x \in (0, \infty), \tag{16}$$

where $\hat{\alpha} = (0.6, 0.25)$,

$$A(x) = \begin{pmatrix} -3.2 + \sin x & 0.2 + \frac{1}{1+x} \\ 1 - \frac{1}{1+x} & -1.4 \end{pmatrix},\ B(x) = \begin{pmatrix} 0.1 + 0.1\cos x & 0.1 \\ 0.2 & 0.2 + 0.1\sin x \end{pmatrix},$$

and the delay

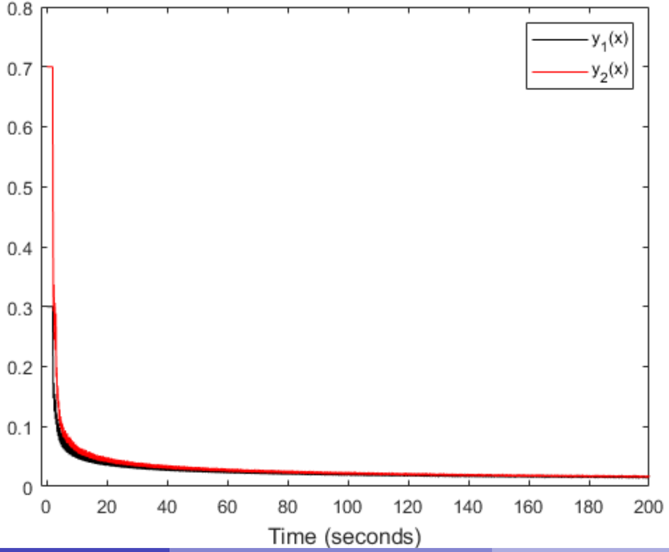
$$q(x) = 1 + \frac{1}{2}\sin x,\ x \geq 0.$$

For each $x \geq 0$, $A(x)$ is Metzler, $B(x)$ is nonnegative. Moreover,

$$A(x) \preceq \hat{A} := \begin{pmatrix} -2.2 & 1.2 \\ 1 & -1.4 \end{pmatrix},\ B(x) \preceq \hat{B} := \begin{pmatrix} 0.2 & 0.1 \\ 0.2 & 0.3 \end{pmatrix},\ x \geq 0,$$

here $\hat{A}$ is Metzler, $\hat{B}$ is nonnegative and $\hat{A} + \hat{B}$ is a Hurwitz matrix. Then, the conditions in Theorem 22 are verified. Thus, the system (16) is globally attractive.

Orbits of the solution to the system (16) with the initial condition $\omega = (0.3, 0.7)^{\mathrm{T}}$.



Example 2

Consider the system

$$ {}^{C}D^{\hat{\alpha}}_{0+}y(x) = A(x)y(x) + B(x)y(x - q(x)),\ x \in (0,\infty), \tag{17} $$

where $\hat{\alpha} = (0.4, 0.65)$,

$$ A(x) = \begin{pmatrix} -3 + \frac{60}{1+x} & 0.2 + e^{-x} \\ 1 - \frac{1}{1+x} & -2 + 3e^{-x} \end{pmatrix}, B(x) = \begin{pmatrix} 0.1 + 0.1\sin x & 0.1 \\ 0.2 + 0.1\cos x & 0.2 + 0.1\sin x \end{pmatrix} $$

and the delay

$$ q(x) = 1 + \frac{1}{2}e^{-x},\ x \geq 0. $$

Notice that for each $x \geq 0$, $A(x)$ is Metzler, $B(x)$ is nonnegative and

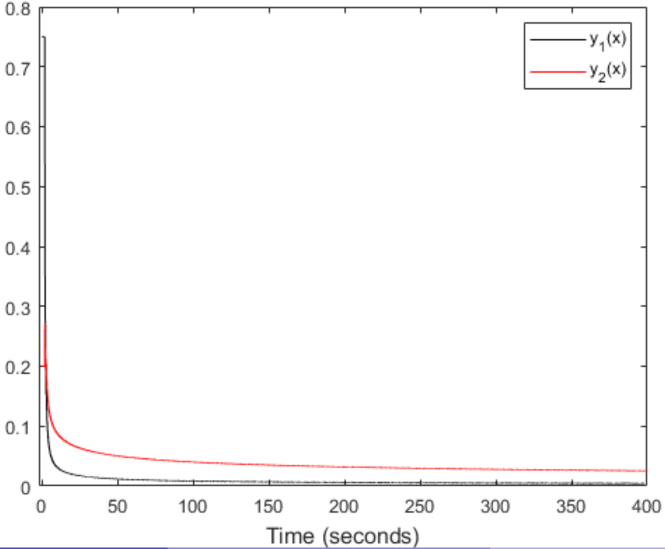
$$A(x) \preceq \hat{A} := \begin{pmatrix} 57 & 1.2 \\ 1 & 1 \end{pmatrix}, \; B(x) \preceq \hat{B} := \begin{pmatrix} 0.2 & 0.1 \\ 0.3 & 0.3 \end{pmatrix} := \hat{B}, \; \forall x \geq 0.$$

However, in this case $\hat{A} + \hat{B}$ is not a Hurwitz matrix. So, Example 2 is not within the scope of Theorem 22. Because $a_{12}(x) \leq 1.2$, $a_{21}(x) \leq 1$ for all $x \geq 0$, $\limsup_{x\to\infty} a_{11}(x) = -3$, $\limsup_{x\to\infty} a_{22}(x) = -2$, we choose

$$\tilde{A} = \begin{pmatrix} -2.9 & 0.3 \\ 1 & -1.9 \end{pmatrix}, \; \tilde{B} = \begin{pmatrix} 0.21 & 0.1 \\ 0.3 & 0.31 \end{pmatrix}.$$

Since $\tilde{A}$ is Metzler, $\tilde{A} + \tilde{B}$ is Hurwitz, by Theorem 23, we conclude the system (17) is globally attractive.

Orbits of the solution to the system (17) subjected the initial condition $\omega = (0.75, 0.2)^T$



Example 3

Consider the system

$$ {}^{C}D^{\alpha}_{0+}y(x) = A(x)y(x) + B(x)y(x - q(x)), \ x \in (0, \infty), \tag{18} $$

where $\alpha = (\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4})$,

$$ A(x) = \begin{pmatrix} -3 + \sin x & 1 - \sin x \\ 0.2 + \dfrac{1}{1+x} & -2 - \dfrac{1}{1+x} \end{pmatrix}, \ B(x) = \begin{pmatrix} \dfrac{0.2x}{1+x} & 0.5 \\ 1 & 0.1 \end{pmatrix}, $$

and the delay

$$ q(x) = \frac{1}{2} + \frac{1}{2}e^{-x}, \ x \geq 0. $$

It is easy to check that in this case $h = 1$,

$$A(x) \preceq \hat{A} := \begin{pmatrix} -2 & 2 \\ 1.2 & -2 \end{pmatrix}, \; B(x) \preceq \hat{B} := \begin{pmatrix} 0.2 & 0.5 \\ 1 & 0.1 \end{pmatrix}, \; x \geq 0,$$

and $\hat{A} + \hat{B} = \begin{pmatrix} -1.8 & 2.5 \\ 2.2 & -1.9 \end{pmatrix}$ is not a Hurwitz matrix. Furthermore, as in Theorem 23, we can choose

$$\tilde{A} := \begin{pmatrix} -1.9 & 2 \\ 1.2 & -1.9 \end{pmatrix}, \; \tilde{B} = \begin{pmatrix} 0.21 & 0.5 \\ 1 & 0.1 \end{pmatrix}.$$

Notice that

$$\tilde{A} + \tilde{B} = \begin{pmatrix} -1.69 & 2.5 \\ 2.2 & -1.8 \end{pmatrix}$$

is also not a Hurwitz matrix. From the above, it deduces that we cannot use Theorem 22 to describe the asymptotic behavior of solutions to the system (18). Moreover, Theorem 23 also seems inappropriate to apply to this example.

Choosing $w = (1, 1)^T$, the inequalities (14) in Theorem 25 become

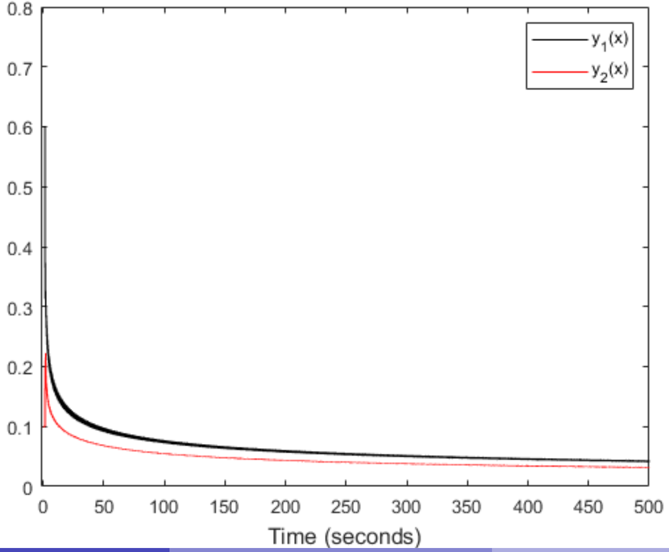
$$\begin{cases} -2 + \dfrac{0.2x}{1+x} \dfrac{1}{E_{\sqrt{6}/4}(-\gamma)} + \dfrac{0.5}{E_{\sqrt{6}/4}(-\gamma)} & \leq -\gamma, \\ -1.8 + \dfrac{1.1}{E_{\sqrt{6}/4}(-\gamma)} & \leq -\gamma. \end{cases} \tag{19}$$

Let

$$g(x, \gamma) = -2 + \frac{0.2x}{1+x} \frac{1}{E_{\sqrt{6}/4}(-\gamma)} + \frac{0.5}{E_{\sqrt{6}/4}(-\gamma)} + \gamma,$$

$g_1(\gamma) = -2 + \frac{0.7}{E_{\sqrt{6}/4}(-\gamma)} + \gamma,\ g_2(\gamma) = -1.8 + \dfrac{1.1}{E_{\sqrt{6}/4}(-\gamma)} + \gamma$, where $x \geq 0$ and $\gamma > 0$. By the fact that g_1, g_2 are strictly increasing on $(0, \infty)$, and $g_1(0) = -1.3 < 0,\ g_2(0) = -0.7 < 0,\ g_1(2) > 0,\ g_2(2) > 0$, we can find $\gamma_1, \gamma_2 \in (0, 2)$ so that $g_1(\gamma_1) = 0,\ g_2(\gamma_2) = 0$. On the other hand, for each $x > 0$, then $g(x, \gamma) < g_1(\gamma),\ \forall \gamma > 0$. Thus, (19) is verified for some $\gamma_* = \min\{\gamma_1/2, \gamma_2/2\}$. With the help of Theorem 25, it implies that the system (18) is globally attractive.

Orbits of the solution of the system (18) with the initial condition $\omega = (0.6, 0.1)^{\mathrm{T}}$



Thank you for your attention!

Our talk is based on following works:

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2. La Van Thinh, Hoang The Tuan, Separation of solutions and the attractivity of fractional-order positive linear delay systems with variable coefficients. *Communications in Nonlinear Science and Numerical Simulation*, **132** (2024), 10789.

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