On the sharp estimates for convolution operators with oscillatory kernel

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Motivation of the Problem
The Cauchy problem for strictly hyperbolic equations

Consider the Cauchy problem (C.P.) [1]:

$$P(D_t, D_x)U = 0, \quad D_t^l U|_{t=0} = g_l, \quad l = 0, \dots, m-1,$$

where

$$D_t := \frac{\partial}{i\partial t}, \quad D_x := \left(\frac{\partial}{i\partial x_1}, \dots, \frac{\partial}{i\partial x_\nu}\right)$$

and $P(\tau, \xi)$ is a homogeneous polynomial of degree m. We will assume that the characteristic equation $P(\tau, \xi) = 0$ has m distinct smooth real solutions $\varphi_1(\xi), \dots, \varphi_m(\xi)$ in $\mathbb{R}^\nu \setminus \{0\}$, e.g. $P(D_t, D_x)U = 0$ is the strictly hyperbolic equation. In addition, it is assumed that $\varphi_j(\xi)\,(j = 1, \dots, m)$ preserves sign e.g. $\varphi_j(\xi)\, j = 1, \dots, m$ is positive or negative in $\mathbb{R}^\nu \setminus \{0\}$, provided that $\nu \geq 2$.

[1] M. Sugimoto, Estimates for Hyperbolic Equations of Space Dimension 3, Journal of Functional Analysi, 160, 382-407 (1998).

Related convolution operator

We will assume that the initial dates $g_l \in L^p(\mathbb{R}^\nu)$ and consider the solutions to the (C.P.) in a distributional sense.
Then the solutions to the Cauchy problem up to a smooth function can be written as a sum of the convolution operators having the form:

$$M_k(t) = F^{-1} e^{it\varphi(\xi)} a_k(\xi) F,$$

where F is the Fourier transform operator, $\varphi \in C^\infty(\mathbb{R}^\nu \setminus \{0\})$ is a smooth out of the origin homogeneous function of order 1, which coincides one of the $\varphi_j (j = 1, \ldots, m)$, $a_k \in C^\infty(\mathbb{R}^\nu)$ is a smooth function and it is homogeneous of order $-k$ for large $|\xi|$.

Scaling arguments

By using scaling arguments, the boundedness problem for the convolution operator $M_k(t)$, is reduced an analogical problem for the following operator:

$$M_k = F^{-1} e^{i\varphi(\xi)} a_k(\xi) F.$$

In this talk we consider the $L^p(\mathbb{R}^\nu) \mapsto L^{p'}(\mathbb{R}^\nu)$ boundedness Problem for the operator M_k, where $1 \le p \le 2$ and p' is the conjugate exponent, e.g. $1/p + 1/p' = 1$.

The main problem

In this talk we discuss the Problem:
Let $1 \leq p \leq 2$ be a fixed number. For which k_p the convolution operator M_k is bounded from $L^p(\mathbb{R}^\nu)$ to $L^{p'}(\mathbb{R}^\nu)$, whenever $k > k_p$?

Critical exponent

Further, we use notation:

$$k_p := \inf_{k>0}\{k > 0 : M_k \,\text{is}\, L^p(\mathbb{R}^\nu) \to L^{p'}(\mathbb{R}^\nu) \,\text{bounded for any}\, a_k\}. \qquad (2)$$

The number k_p is called to be a critical exponent.

The related level set

Since φ preserves sign for the sake of simplicity we will assume that $\varphi(\xi) > 0$ for any $\xi \neq 0$. We consider the set

$$\Sigma := \{\xi \in \mathbb{R}^\nu \setminus \{0\} : \varphi(\xi) = 1\}. \tag{1}$$

Since φ is a homogeneous function of degree one by Euler's relation we have

$$\sum_{j=1}^{\nu} \xi_j \frac{\partial \varphi(\xi)}{\partial \xi_j} = \varphi(\xi) = 1$$

on Σ and it is a smooth (analytic) hypersurface, provided that φ is a smooth (real analytic) function respectively.

The Fourier transform of the surface-carried measures

Let $\Sigma \subset \mathbb{R}^\nu$ be a smooth hypersurface, and let $\psi \in C_0^\infty(\Sigma)$ be a smooth function with compact support. Consider the signed measure $d\mu(X) := \psi(X)d\Sigma$, where $d\Sigma$ is the induced Lebesgue measure on the hypersurface Σ. In particular, if ψ is a nonnegative function, then we are dealing with a Borel measure. The Fourier transform of the signed measure $d\mu$ is defined by the following integral:

$$\widehat{d\mu}(\xi) := \int_\Sigma e^{i(X,\xi)} d\mu(X),$$

where $(X, \xi) := X_1\xi_1 + \cdots + X_n\xi_n$ is an inner product of the vectors X and ξ. Surely, $\widehat{d\mu}$ corresponds to the Fourier transform of the distribution determined by the charge $d\mu$.

Essential remark

It turns out that behavior of the integral $\widehat{d\mu}(\xi)$ when $|\xi|$ gets large as well as the number $k_p(\Sigma) := k_p$ depend on geometric properties of the hypersurface Σ. As we will see later actually, the number $k_p(\Sigma)$ is closely related to behavior of the integral $\widehat{d\mu}(\xi)$.

Classical results

In the case $\Sigma = S^{\nu-1}$ (where $S^{\nu-1}$ is the unit sphere centered at the origin of $\mathbb{R}^\nu$), which is related to the classical wave equation, the $L^p(\mathbb{R}^\nu) \to L^{p'}(\mathbb{R}^\nu)$ boundedness of M_k is obtained by Strichartz [2]. It should be noted that the sphere has the positive, constant Gaussian curvature, which is essentially used in the estimation of the convolution operator.
These results were extended to the case when Σ has a non-vanishing Gaussian curvature by Brenner [3], and the case when Σ is convex (but, not necessarily strictly convex) by Sugimoto [4]. Note that the obtained estimates, in that paper by M. Sugumoto, are sharp for a partial class of convex hypersurfaces.

[2] R. Strichartz, Convolutions with kernels having singularities on a sphere, Trans. Amer. Math. Soc. 148 (1970), 461-471.

[3] Brenner P., On $L^p \to L^{p'}$ estimates for the wave-equation, Math. Z. 145 (1975), 251-254.

[4] M. Sugimoto, A priori estimates for higher order hyperbolic equations, Math. Z. 215 (1994), 519-531.

M. Sugimoto results on boundedness for the convolution operators

Let $\Sigma \subset \mathbb{R}^\nu$ be a smooth hypersurface.
M. Sugimoto introduces the concept of an index $\gamma_0(\Sigma)$: first, for a fixed point $X \in \Sigma$ and the two-dimensional plane H containing the normal at the point $X \in \Sigma$, we denote by $\gamma(\Sigma; X; H)$ the number determined by the order of tangency of the curve $\Sigma \cap H$ and the straight line $T_X\Sigma \cap H$ at the point X, where $T_X\Sigma$ is the affine tangent hyperplane to Σ at X. Then, we set

$$\gamma(\Sigma; X) := \inf_H \gamma(\Sigma; X; H) \quad \text{and} \quad \gamma_0(\Sigma) := \sup_{X \in \Sigma} \gamma(\Sigma; X).$$

M. Sugimoto Theorem on an upper bound for $k_p(\Sigma)$.

Theorem [M. Sugimoto]. *Suppose* $\nu \geq 2$. *Then* M_k *is* $L^p(\mathbb{R}^\nu) \mapsto L^{p'}(\mathbb{R}^\nu)$*-bounded if* $k > (2\nu - 2/\gamma_0(\Sigma))(1/p - 1/2) \geq k_p(\Sigma)$. *This inequality can be replaced by an equality if* $p \neq 1$. *Furthermore,* M_k *is not necessarily* $L^p \mapsto L^{p'}$*-bounded if* $k < (2\nu - 2/\gamma_0(\Sigma))(1/p - 1/2)$ *and* $p = 1, 2$.

On the sharpness of the M. Sugimoto Theorem

Actually, the number $\gamma_0(\Sigma)$ defines the sharp uniform (with respect to directions of ξ)) bound for the Fourier transform only for the case when $\nu = 2$, which is given by

$$\widehat{d\mu}(\xi) = O(|\xi|^{-\frac{1}{\gamma_0(\Sigma)}}) \quad (\text{as} \quad |\xi| \to \infty).$$

It can be seen from the simple example $\Sigma = S^{\nu-1}$ with $\nu \geq 2$. Note that in this case $\gamma_0(S^{\nu-1}) = 2$. But, we have $\widehat{d\mu}(\xi) = O(|\xi|^{-\frac{\nu-1}{2}})$(as $|\xi| \to \infty$) and

$$k_p(S^{\nu-1}) = (\nu+1)(1/p - 1/2) \leq (2\nu-1)(1/p - 1/2) = \\ (2\nu - 2/\gamma_0(\Sigma))(1/p - 1/2).$$

On the other hand as we will see later in general, even the sharp, uniform with respect to directions, estimate for the Fourier transform of measures does not define the sharp value of $k_p(\Sigma)$ for the case when $\nu \geq 3$.

Localization arguments

The following standard localization arguments are used in the paper by M. Sugimoto: We can assume a_k is supported in a sufficiently "small"conic (cone with vertex at the origin) neighborhood Γ of a particular point $v \in S^{\nu-1}$ say $v = (0, \dots, 0, 1)$ and $\varphi \in C^\infty(\Gamma)$. Then in a sufficiently small neighborhood of the point v on Σ (after possible linear change of variables) the hypersurface can be expressed as

$$\Sigma \cap \Gamma = \{\xi \in \Gamma : \varphi(\xi) = 1\} = \{(y, 1 + \phi(y)) \in \mathbb{R}^\nu : y \in U\},$$

where $\phi \in C^\infty(U)$ is a smooth function with $\phi(0) = 0, \nabla\phi(0) = 0$ and $U \subset \mathbb{R}^{\nu-1}$ is an open neighborhood of the origin.

The local critical exponent

Further, we assume that the amplitude function $a_k(\xi)$ is concentrated in a sufficiently small conic neighborhood Γ of a fixed point $v \in S^{\nu-1}$ and $\varphi(\xi) \in C^\infty(\Gamma)$. Fixing such a point $v \in S^{\nu-1}$, let us define the following local critical exponent $k_p(v)$ attached to this point:

$$k_p(v) := \inf_{k>0} \{k : \exists \Gamma \ni v,\, M_k : L^p(\mathbb{R}^\nu) \mapsto L^{p'}(\mathbb{R}^\nu) \\ \text{is bounded, whenever}\, supp(a_k) \subset \Gamma\}.$$

The three-dimensional case.

By M. Sugimoto [5] it was considered the problem for the case when $\Sigma \subset \mathbb{R}^3$ is an analytic surface with $\gamma_0(\Sigma) = 2$. It was obtained an upper bound for the number $k_p(v)$ (where $v \in \Sigma$ is a fixed point).
More precisely, M.Sugimoto introduced three classes of hypersurfaces in $\mathbb{R}^3$ with $\gamma_0(\Sigma) = 2$. For each class, he got upper bounds for the number $k_p(v)$. Moreover, he gave examples for each classes showing sharpness of the bounds for that examples.

[5] M. Sugimoto, Estimates for Hyperbolic Equations of Space Dimension 3, Journal of Functional Analysi, 160, 382-407 (1998).

The main question

The natural question is: **Whether the upper estimate for the number $k_p(v)$ given by M. Sugimoto is the sharp bound for arbitrary hypersurfaces of the appropriate classes ?**
We obtain the exact value of $k_p(v)$ improving the results proved by M. Sugimoto for arbitrary analytic hypersurfaces having at least one non-vanishing principal curvature.

Relation with Arnol'd's singularities

We will assume that the function ϕ has singularities of type $A_n (1 \leq n \leq \infty)$ at the origin (see [6]).
Let ϕ be a smooth function with $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Singularity of the function ϕ is called to be an A_n type singularity if there exists a local smooth diffeomorphism $x = f(y)$ with $f(0) = 0$ and the relation

$$\phi(f(y)) = \pm y_1^2 \pm y_2^{n+1}$$

holds true, whenever $n < \infty$ and $\phi(f(y)) = \pm y_1^2 + b_0(y_2)$ (with a flat function b_0) for the case $n = \infty$.

[6] V. I. Arnol'd, S. M. Gusein-zade, and A. N. Varchenko, "Singularities of differentiable mappings,"in Classification of Critical Points of Caustics and Wavefronts 1985, Vol. 1, 1985, Birkhäuser, Boston, Basel, Stuttgard.

The main assumption

The main Assumption: Further, we consider surfaces $\Sigma \subset \mathbb{R}^3$ with $\gamma_0(\Sigma) = 2$.
Since we consider the local Problem it is enough to assume $\gamma(\Sigma; v) = 2$.
The following statements holds:
Proposition. *The following conditions are equivalent:*
(i) $\gamma(\Sigma; v) = 2$*;*
(ii) The surface Σ *has at least one non-vanishing principal curvature at the point* v*;*
(iii) The function ϕ *has singularities of type* A *at the point* $(0, 0)$*.*

Simply notation

Further, we use the standard notation assuming F being a sufficiently smooth function:

$$\partial^{\alpha} F(x) := \partial_1^{\alpha_1} \dots \partial_{\nu}^{\alpha_{\nu}} F(x) := \frac{\partial^{|\alpha|} F(x)}{\partial x_1^{\alpha_1} \dots \partial x_{\nu}^{\alpha_{\nu}}},$$

where $\alpha = (\alpha_1, \dots, \alpha_{\nu}) \in \mathbb{Z}_+^{\nu}$ is a multiindex, with $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, and $|\alpha| := \alpha_1 + \dots + \alpha_{\nu}$.

On Normal form with respect to linear transforms

We use the following Proposition [7]:

Proposition

Assume that ϕ *is a smooth function defined in a neighborhood of the origin of* $\mathbb{R}^2$ *satisfying the conditions:* $\partial_2^2\phi(0,0) \neq 0$ *and also* $\partial^\alpha\phi(0,0) = 0$ *for any* $\alpha \in \mathbb{Z}_+^2$ *with* $|\alpha| = \alpha_1 + \alpha_2 \leq 2$ *except* $\alpha \neq (0,2)$*.*
Then, ϕ *can be written in the following form on a sufficiently small neighborhood of the origin:*

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_0(x_1), \qquad (1.2.2)$$

where b, b_0 *and* ψ *are smooth functions with* $b(0,0) \neq 0$*. The function* $\psi(b_0)$ *can be written as* $\psi(x_1) = x_1^m\omega(x_1)$ *with* $\omega(0) \neq 0$, $m \geq 2$ *and* $(b_0(x_1) = x_1^n\beta(x_1)$, *with* $\beta(0) \neq 0$, $n \geq 2)$ *unless* $\psi(b_0)$ *is a flat function.*

[7] I.A. Ikromov and D. Müller. Fourier restriction and Newton polyhedra, Monograph series Princeton University press, 2016 (258 pages)

Discrete characteristics at a point

It is easy to show that the numbers m, n (where $2 \leq m, n \leq \infty$) are well-defined for arbitrary smooth function ϕ having A type singularities (see [8] and also [9]). Moreover, to each point $v \in \Sigma$ of the surface with at least one non-vanishing principal curvature we can attach a pair $(m(v), n(v))$ due to the Proposition.

[8] M. Sugimoto, Estimates for Hyperbolic Equations of Space Dimension 3, Journal of Functional Analysis, 160, 382-407 (1998).

[9] I.A. Ikromov and D. Müller. Fourier restriction and Newton polyhedra, Monograph series Princeton University press, 2016 (258 pages)

Sugimoto classes of surfaces

Following M. Sugimoto we can introduce the following three classes of hypersurfaces at the point $v = (0, 0, 1)$:
We say that Σ is of type I with order n if $b_0(x_1) = x_1^n \beta(x_1)$, where β is a smooth function with $\beta(0) \neq 0$;
Σ is of type II with order m if b_0 is a flat function at the origin and also $\psi(x_1) = x_1^m \omega(x_1)$, where ω is a smooth function with $\omega(0) \neq 0$,
and finally, Σ is of type III if both functions ψ, b_0 are flat at the origin.

R-condition

Further, we will assume that if Σ is a C^{∞} hypersurface of type II then $b_0 \equiv 0$. This condition matches with the so-called "$R-$condition"introduced in the monograph [10]. Surely, any real analytic function satisfies the "$R-$condition".

[10] I.A. Ikromov and D. Müller. Fourier restriction and Newton polyhedra, Monograph series Princeton University press, 2016 (258 pages)

The main results

We prove the following statement, which is the main our result.

Theorem

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface having at least one non-vanishing principal curvature at the point $v := (0,0,1)$ and $1 \le p \le 2$ be a fixed number and also (m,n) be the pair defined by the Proposition. Then the following statements hold:
(i) If $2m \ge n$ then $k_p(v) = (5 - \frac{2}{n})(\frac{1}{p} - \frac{1}{2})$;
(ii) If Σ is a smooth hypersurface satisfying the $R-$condition and $m \ge 3$ and also $2m < n \le \infty$ then

$$k_p(v) = \max\left\{\left(5 - \frac{1}{m}\right)\left(\frac{1}{p} - \frac{1}{2}\right), \left(6 - \frac{2(m+1)}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} + \frac{m}{n}\right\}. \tag{1}$$

Discussion of results for the class I

Note that in the case (i) formally it is possible $m = \infty$ e.g. the ψ can be a flat function. M. Sugimoto [11] suggested the example:

$$\phi_I(y) = 1 - (y_2^2 - y_1^n), \tag{2}$$

which corresponds to the case (i), with $\psi(y_1) \equiv 0$. From our results it follows that the Sugimoto result is sharp in that case.
Moreover, the Sugimoto result is sharp, for surfaces of the class I with order n, if and only if $2m \geq n$.

[11] M. Sugimoto, Estimates for Hyperbolic Equations of Space Dimension 3, Journal of Functional Analysis, 160, 382-407 (1998).

Estimates for the class III

If $n = \infty$ e.g. if b_0 is a flat function at the origin then so is ψ, under the condition $2m \geq n$. Hence, the Sugimoto result is sharp in that case also, in other words, his results are sharp for arbitrary smooth surface of the class III. On the other hand if $2m < n < \infty$ then the result of Sugimoto [[12]] is not sharp for the hypersurfaces Σ of the class I. Our results show that one can not be ignored influence of the number m for the surfaces of the class I.

[12] M. Sugimoto, Estimates for Hyperbolic Equations of Space Dimension 3, Journal of Functional Analysis, 160, 382-407 (1998).

Discussion results for the class II

For the case $n = \infty$ e.g. for hypersurfaces of the class II M. Sugimoto obtained the sharp bound for a subset of analytic surfaces of the class II. It turns out that the analogical result holds true for arbitrary analytic hypersurfaces of the class II and also for arbitrary smooth surfaces of the class II under the $R-$ condition. More precisely, from our result it follows that actually the statement of the Theorem 2 proved by M. Sugimoto in the paper [13] (page no. 396) holds true for arbitrary analytic hypersurface having type II and also for analogical smooth hypersurfaces under the R-condition.

[13] M. Sugimoto, Estimates for Hyperbolic Equations of Space Dimension 3, Journal of Functional Analysis, 160, 382-407 (1998).

Comparing with M. Sugimoto Results

Theorem [M. Sugimoto]. *Suppose that a real analytic function ϕ is of type II (e.g. $n = \infty$) with $m \geq 3$ and satisfies*

$$\frac{\partial^\mu}{\partial x_1^\mu} \left\{ \frac{\partial^\kappa \phi(x_1, \psi(x_1))}{\partial x_2^\kappa} \right\} |_{x_1=0} = 0$$

for $\mu = 1, 2, \ldots, m-1$ and $\kappa = 2, 3, \ldots$. Then M_k is $L^p(\mathbb{R}^3) - L^{p'}(\mathbb{R}^3)$ bounded if

$$k > \max \left\{ \left(5 - \frac{1}{m}\right) \left(\frac{1}{p} - \frac{1}{2}\right), 6 \left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} \right\}.$$

This matches with our result in the case $n = \infty$.

Estimates for the class II

More precisely, our main Theorem yields:

Corollary. *Suppose that a real analytic function* ϕ *is of type II (e.g.* $n = \infty$*) with* $m \geq 3$*. Then* M_k *is* $L^p(\mathbb{R}^3) - L^{p'}(\mathbb{R}^3)$ *bounded if*

$$k > \max\left\{\left(5 - \frac{1}{m}\right)\left(\frac{1}{p} - \frac{1}{2}\right), 6\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}\right\}.$$

Comparing with the restriction estimates

Note that the first case (i) is agree with the so-called linearly adapted condition introduced in the monograph [[14]] (see also [15]). Also note that under the linearly adapted case the sharp uniform estimate for the Fourier transform of measures gives the sharp bound for the exponent p in the $L^p(\mathbb{R}^3) \mapsto L^2(\Sigma)$ Fourier restriction problem. As had been shown in [16] it is only the case. Our results show that in the linearly adapted case the value of $k_p(v)$ can be derived from the sharp uniform estimate for the Fourier transform of the corresponding measure.

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[14] I.A. Ikromov and D. Müller. Fourier restriction and Newton polyhedra, Monograph series Princeton University press, 2016 (258 pages)

[15] Ikromov, I. A., Müller, D., Uniform estimates for the Fourier transform of surface carried measures in $\mathbb{R}^3$ and an application to Fourier restriction. *I. Fourier Anal. Appl.,* 17 (2011), no. 6, 1292-1332

Sketches of proof (Related oscillatory integrals)

Obviously, the Fourier transform of signed measure can be written as the following oscillatory integral:

$$I(\lambda, z) := \int_{\mathbb{R}^{\nu-1}} e^{i\lambda((z,y)+\phi(y))} g(y)dy, \quad \lambda > 0,\ z \in \mathbb{R}^{\nu-1},\ g \in C_0^\infty(U).$$

It is related with the convolution kernel

$$K_k(x) = F^{-1}(e^{i\varphi(\xi)} a_k(\xi))(x).$$

Relation on estimates for the oscillatory integrals and boundedness for the convolution operators

Proposition 1 [M. Sugimoto]. *Let* $q \geq 2$ *and* $\rho \geq 0$*. Suppose, for all* $g \in C_0^\infty(U)$ *and* $\lambda > 0$

$$\|I(\lambda, \cdot)\|_{L^q(\mathbb{R}_z^{\nu-1})} \leq C_g \lambda^{-\rho},$$

where C_g *is independent of* λ*. Then* $K_k \in L^q(\mathbb{R}^\nu)$*; hence* M_k *is* $L^p(\mathbb{R}^\nu) \mapsto L^{p'}(\mathbb{R}^\nu)$ *bounded for* $p = (2q)/(2q-1)$*, if* $k > \nu - \rho - 1/q$*.*
The proposition 1 is proved by using Hausdorff- Young's inequality for convolution.

Estimate by using uniform with respect to directions estimates

Proposition 2 [M. Sugimoto]. *Let* $\rho \geq 0$. *Suppose, for all* $g \in C_0^\infty(U)$ *and* $\lambda > 0$

$$\|I(\lambda, \cdot)\|_{L^\infty(\mathbb{R}_z^{\nu-1})} \leq C_g \lambda^{-\rho}, \quad (3)$$

where C_g *is independent of* λ. *Then* $\{K_{k,j}\}_{j=0}^\infty$ *is bounded in* $L^\infty(\mathbb{R}^\nu)$ *if* $k > \nu - \rho$. *Hence* M_k *is* $L^p(\mathbb{R}^\nu) \mapsto L^{p'}(\mathbb{R}^\nu)$ *bounded if* $k > (2\nu - 2\rho)(1/p - 1/2)$. *This inequality can be replaced by an equation if* $p \neq 1$.

The proposition 2 is proved by using Besov's space technics and interpolation arguments.

An upper bound for $k_p(v)$

Note that we dealt with two-dimensional oscillatory integral $I(\lambda, z)$. If ϕ has singularities at the origin and $|z| > \delta$ then the phase function $\phi(x_1, x_2) + xz$ has no critical points provided U is a sufficiently small neighborhood of the origin. Therefore we can use integration by parts arguments and obtain:

$$|I(\lambda, z)| \lesssim \frac{1}{|z\lambda|^2},$$

which is better than wanted.

Stationary phase method for inner integral

Further, we assume that $|z| << 1$ and U is a sufficiently small neighborhood of the origin. Then we can use stationary phase method with x_2 variables and obtain

$$I(\lambda, z) = \frac{C}{\lambda^{\frac{1}{2}}} \int_{\mathbb{R}} e^{i\lambda(x_1^n \beta(x_1) + \phi_1(x_1, z_2) + z_2 x_1^m \omega(x_1) + z_1 x_1))} g(x_2^c, x_1) dx_1 + R(\lambda, z),$$

where R is a remainder term satisfying the estimate $|R(\lambda, z)| \lesssim \lambda^{-\frac{3}{2}}$. Moreover the phase function $\phi_1(x_1, z_2)$ can be written as:

$$\phi_1(x_1, z_2) = z_2^2 B(z_2) + z_2^2 x_1 q(x_1, z_2),$$

where B, q are smooth functions with $B(0) \neq 0$ (see [16].

[16] S. Buschenhenke, I.A. Ikromov, D. Müller Estimates for maximal functions associated to hypersurfaces in $\mathbb{R}^3$ with height $h < 2$: Part II A geometric conjecture and its proof for generic 2-surfaces, Arkiv, 2022.

Then by using the Van der Corput type lemma [17] we see that the estimate (3) (in the Proposition 2 by M.Sugimoto) holds true with $\rho = \frac{1}{2} + \frac{1}{n}$. In this case we can use the Proposition 2 [M. Sugimoto] and have the following upper bound for $k_p(v)$:

$$k_p(v) \leq \left(5 - \frac{2}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right). \tag{4}$$

This case includes also the class of type III e.g. the case $n = \infty$.
Note that the upper bound does not depend on the number m.
As noted before the estimate for $k_p(v)$ is sharp if and only if $2m \geq n$.

[17] Stein, E. M., Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals. *Princeton Mathematical Series* 43.
Princeton University Press, Princeton, NI, 1993.

The upper bound for $k_p(v)$ in the case when $2m < n$.

Now, we consider the more subtle case $2m < n$. In this case we use the following Lemma (compare with Theorem 2 of [18]):
Main Lemma. *Let* $2m < n \leq \infty$ *and* $m \geq 3$. *Then for a positive number* $\varepsilon > 0$ *the following estimate*

$$\|I(\lambda, \cdot)\|_{L^{m+1}(\mathbb{R}^2)} \leq C\lambda^{-(\frac{1}{2}+\frac{2}{m+1})+\varepsilon}.$$

holds true.

[18] M. Sugimoto, Estimates for Hyperbolic Equations of Space Dimension 3, Journal of Functional Analysis, 160, 382-407 (1998).

Interpolation arguments

From the Main Lemma it follows the required upper bound for the number $k_p(v)$ for the case $2m < n$. Indeed,
1. first, we use the Proposition 1 by Sugimoto and obtain $L^{p_0} \mapsto L^{p_0'}$ boundedness of the convolution operator with $p_0 = \frac{2m+2}{2m+1}$ for $k_0 > \frac{5}{2} - \frac{3}{m+1}$.
2. Due to Proposition 2 by Sugimoto we obtain $L^{p_1} \mapsto L^{p_1'}$ boundedness of the convolution operator for $p_1 = 1$ and $k_1 > \frac{5}{2} - \frac{1}{n}$
and also
3. by classical Plancherel's identity $L^{p_2} \mapsto L^{p_2'}$ boundedness of the convolution operator for $p_2 = p_2' = 2$ and $k_2 = 0$.
Then by analytic interpolation arguments we obtain the upper bound for the number $k_p(v)$.

Lower bounds for $k_p(v)$.

Lemma on lower bounds. *If* $2m \geq n$, *then there exists an amplitude function* a_k *such that the associated operator* M_k *is not* $L^p(\mathbb{R}^3) \mapsto L^{p'}(\mathbb{R}^3)$ *bounded, whenever* $k < (5 - \frac{2}{n})(\frac{1}{p} - \frac{1}{2})$.

An amplitude function

Let us take a smooth function in $\mathbb{R}^3$ such that $a_k(\xi) = |\xi|^{-k}$ for large ξ. For instance, we can take the function $a_k(\xi) = (1 - \chi_0(\xi))|\xi|^{-k}$ where χ_0 is a smooth non-negative cut-off function satisfying the conditions

$$\chi_0(\xi) = \begin{cases} 1 & \text{for} \quad |\xi| \leq \varepsilon, \\ 0 & \text{for} \quad |\xi| \geq 2\varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a fixed positive number.

Cut-off functions

Define a non-negative smooth function $\chi_0(0) = 1$ concentrated in a sufficiently small neighborhood of the origin, and a smooth function with $\chi_1(1) = 1$ and with support in a sufficiently small neighborhood of the point 1.
Also, following M. Sugimoto we introduce the function:
$G(y) = 1 + \phi(y_1, y_2) - y\nabla\phi(y)$.

Test functions

We set

$$u_j(x) = 2^{j\left(\frac{5}{2}-\frac{1}{n}\right)\left(-\frac{1}{p'}\right)} F^{-1}(v_j(2^{-j}\xi))(x),$$

where

$$v_j(\xi) = \frac{\chi_0(2^{\frac{j}{2}}\frac{\xi_1}{\varphi(\xi)})\chi_0(2^{\frac{j}{n}}\frac{\xi_2}{\varphi(\xi)})\chi_1(\varphi(\xi))|\xi|^k}{\varphi(\xi)^2 G(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)})} \in C_0^\infty(\mathbb{R}^3).$$

Note that $v_j(2^{-j}\xi)$ is supported on the set $|\xi| \backsim 2^j$ and the sequence $\{u_j\}_{j=1}^\infty$ is bounded in the space $L^p(\mathbb{R}^3)$.

Action of the operator to the test function

On the other hand we have the relation:

$$M_k u_j(x) = 2^{j(\frac{5}{2}-\frac{1}{n})(-\frac{1}{p'})-kj+2j}$$

$$F^{-1}\left(e^{i\varphi(\xi)}\frac{\chi_0\left(2^{\frac{j}{2}}\frac{\xi_1}{\varphi(\xi)}\right)\chi_0\left(2^{\frac{j}{n}}\frac{\xi_2}{\varphi(\xi)}\right)\chi_1(2^{-j}\varphi(\xi))}{\varphi(\xi)^2 G\left(\frac{\xi_1}{\varphi(\xi)},\frac{\xi_2}{\varphi(\xi)}\right)}\right).$$

We perform the change of variables given by the scaling $2^{-j}\xi \mapsto \xi$ and obtain:

$$M_k u_j(x) = \frac{2^{j((\frac{5}{2}-\frac{1}{n})(-\frac{1}{p'})-k+3)}}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{i2^j(\varphi(\xi)-x\xi)} \frac{f(2^{\frac{j}{2}}\frac{\xi_1}{\varphi(\xi)})\chi_0(2^{\frac{j}{n}}\frac{\xi_2}{\varphi(\xi)})\chi_1(\varphi(\xi))}{\varphi^2(\xi)G(\frac{\xi_1}{\varphi(\xi)},\frac{\xi_2}{\varphi(\xi)})} d\xi.$$

Then following M. Sugimoto we use change of variables $\xi = (\lambda y, \lambda(1+\phi(y)))$ and get:

$$M_k u_j(x) = \frac{2^{j((\frac{5}{2}-\frac{1}{n})(-\frac{1}{p'})-k+3)}}{\sqrt{(2\pi)^3}} \int e^{i2^j\lambda(1-(x_1y_1+x_2y_2+x_3(1+\phi(y))))}$$

$$\chi_0(2^{\frac{j}{2}}y_1)\chi_0(2^{\frac{j}{n}}y_2)\chi_1(\lambda)d\lambda dy.$$

Scaling arguments

Finally, we use change of variables $2^{\frac{j}{2}}y_1 \mapsto y_1$, $2^{j/n}y_2 \mapsto y_2$ and obtain:

$$M_k u_j(x) = 2^{j((\frac{5}{2}-\frac{1}{n})(-\frac{1}{p'})-k-\frac{1}{2}-\frac{1}{n}+3)}$$

$$\int_{\mathbb{R}^3} e^{2^j i\lambda((x_3-1)-2^{-\frac{j}{2}}y_1x_1-2^{-\frac{j}{n}}y_2x_2-x_3\phi(2^{-\frac{j}{2}}y_1,2^{-\frac{j}{n}}y_2))}\chi_0(y_1)\chi_0(y_2)\chi_1(\lambda)d\lambda dy.$$

Lower bounds

If $|x_3 - 1| \ll 2^{-j}$, $|x_1| \ll 2^{-j/2}$, $|x_2| \ll 2^{\frac{-j(n-1)}{n}}$, then the phase is a non-oscillating function, because $\phi(2^{-j/2}y_1, 2^{-j/n}y_2) = o(2^{-j})$ provided the support of χ_0 is small enough. Where we essentially use the condition $n \leq 2m$. Consequently, we have the following lower bound:

$$\|M_k u_j\|_{L^{p'}} \gtrsim 2^{j((\frac{5}{2}-\frac{1}{n})(-\frac{1}{p'})-k+\frac{5}{2}-\frac{1}{n}-(\frac{5}{2}-\frac{1}{n})(\frac{1}{p'}))} = 2^{j((5-\frac{2}{n})(-\frac{1}{p'})+\frac{5}{2}-\frac{1}{n}-k)} = 2^{j((5-\frac{2}{n})(\frac{1}{p}-\frac{1}{2})-k)}.$$

Therefore, if $k < k_p(v) := (5-\frac{2}{n})(\frac{1}{p}-\frac{1}{2})$, then $\|M_k u_j\|_{L^{p'}} \to \infty (\text{as}\, j \to +\infty)$. Thus, the operator $M_k : L^p(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$ is unbounded.

The case $2m < n$

Further, we consider the case $2m < n$. The proof of the **Lemma on lower bounds** shows that if $2m < n$ and $k < (5 - \frac{1}{m})(\frac{1}{p} - \frac{1}{2})$, then the operator $M_k : L^p(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$ is not bounded. Indeed, we can repeat all arguments of the Lemma on lower bounds taking the function

$$u_j(x) = 2^{j\left(\frac{5}{2} - \frac{1}{2m}\right)\left(-\frac{1}{p'}\right)} F^{-1}(v_j(2^{-j}\xi))(x),$$

with

$$v_j(\xi) = \frac{\chi_0(2^{\frac{j}{2}} \frac{\xi_1}{\varphi(\xi)})\chi_0(2^{\frac{j}{2m}} \frac{\xi_2}{\varphi(\xi)})\chi_1(\varphi(\xi))|\xi|^k}{\varphi(\xi)^2 G(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)})} \in C_0^\infty(\mathbb{R}^3)$$

for the case $2m < n$ and obtain the following lower bound:

$$k_p(v) \geq \left(5 - \frac{1}{m}\right)\left(\frac{1}{p} - \frac{1}{2}\right). \tag{3}$$

for the number $k_p(v)$.

Further lower bound

Now, we consider the case $2m < n$.

Lower bound Lemma in the NLA case. *If* $2m < n$, *and* $m \geq 3$ *then*

$$k_p(v) \geq \left(6 - \frac{2(m+1)}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} + \frac{m}{n}. \tag{4}$$

Test functions for the case $2m < n$

We a little modified the M. Sugimoto sequence and consider the sequence

$$u_j = 2^{-\frac{3j}{p'} + \frac{j(m+1)}{n}} F^{-1}(v_j(2^{-j}\cdot))(x),$$

where

$$v_j(\xi) = \chi_0\left(2^{\frac{jm}{n}}\left(\frac{\xi_2}{\varphi(\xi)} - \left(\frac{\xi_1}{\varphi(\xi)}\right)^m \omega\left(\frac{\xi_1}{\varphi(\xi)}\right)\right)\right) \frac{\chi_0\left(2^{\frac{j}{n}}\frac{\xi_1}{\varphi(\xi)}\right)\chi_1(\varphi(\xi))|\xi|^k}{\varphi^2(\xi) G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)},$$

where $\chi_0, \chi_1 \in C_0^\infty(\mathbb{R})$ are non-negative smooth functions satisfying the conditions: $\chi_0(0) = 1$, $\chi_1(1) = 1$, and supports of χ_0 lie in a sufficiently small neighborhood of the origin of $\mathbb{R}$ and support of χ_1 belongs to a sufficiently small neighborhood of one. Obviously $v_j \in C_0^\infty(\mathbb{R}^3)$ and $\|v_j\|_{L^{p'}(\mathbb{R}^3)} \sim 2^{-j\frac{m+1}{p'n}}$, where the symbol " $\sim$ " means that there exit non-zero constants $c_1, c_2 > 0$ such that

$$c_1 2^{-j\frac{m+1}{n}} \le \int_{\mathbb{R}^3} |v_j(\xi)|^{p'} d\xi \le c_2 2^{-j\frac{m+1}{n}}.$$

Indeed, in the integral $\int_{\mathbb{R}^3} |v_j(\xi)|^{p'} d\xi$ we use change of variables $\xi = \lambda(y_1, y_2, 1 + \phi(y_1, y_2))$. Note that on the support of v_j make sense the change of variables, provided j is big enough. Then we get:

$$\int_{\mathbb{R}^3} |v_j(\xi)|^{p'} d\xi = \int_{\mathbb{R}^3} \chi_0^{p'}(2^{j\frac{m}{n}}(y_2 - y_1^m \omega(y_1)))\chi_0^{p'}(2^{\frac{j}{n}} y_1)\chi_1^{p'}(\lambda)$$

$$\lambda^{(k-2)p'+2}(y_1^2 + y_2^2 + (1 + \phi(y_1, y_2))^2)^{\frac{kp'}{2}} G^{2-p'}(y_1, y_2) dy_1 dy_2 d\lambda \sim 2^{-j\frac{m+1}{n}},$$

Thus, for large j, we have

$$\|u_j\|_{L^p(\mathbb{R}^3)} \sim 1.$$

Now, we consider estimate for $\|M_k u_j\|_{L^{p'}(\mathbb{R}^3)}$.
We have:

$$M_k u_j = F^{-1} e^{i\varphi(\xi)} a_k(\xi) F u_j = 2^{-\frac{3j}{p'} + j\frac{m+1}{np'}} F^{-1}(e^{i\varphi(\xi)} a_k(\xi) v_j(2^{-j}\xi))(x).$$

Change of variables

We perform change of variables given by scaling $2^j\xi \to \xi$ and obtain:

$$M_k u_j(x) = \frac{2^{\frac{3j}{p}+\frac{j(m+1)}{np'}-kj}}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{i2^j(\varphi(\xi)-\xi x)}$$

$$\chi_0\left(2^{\frac{jm}{n}}\left(\frac{\xi_2}{\varphi(\xi)} - \left(\frac{\xi_1}{\varphi(\xi)}\right)^m \omega\left(\frac{\xi_1}{\varphi(\xi)}\right)\right)\right) \frac{\chi_0(2^{\frac{j}{n}}\frac{\xi_1}{\varphi(\xi)})\chi_1(\varphi(\xi))}{\varphi^2(\xi)G(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)})} d\xi.$$

Finally, we use change of variables $\xi \to \lambda(y_1, y_2, 1+\phi(y_1, y_2))$. Then we have:

$$M_k u_j(x) = \frac{2^{\frac{3j}{p}+\frac{j(m+1)}{p'}-kj}}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{i2^j\lambda(1-x_3-(y_1x_1+y_2x_2+x_3\phi(y_1,y_2)))} \times$$

$$\times \chi_0(2^{\frac{jm}{n}}(y_2 - y_1^m\omega(y_1)))\chi_0(2^{\frac{j}{n}}y_1)\chi_1(\lambda) d\lambda dy_1 dy_2.$$

Now, we perform the change of variables

$$y_1 = 2^{-\frac{j}{n}} z_1,\, y_2 = y_1^m \omega(y_1) + 2^{-j\frac{m}{n}} z_2.$$

Thus

$$M_k u_j(x) = 2^{\frac{3j}{p} + \frac{m+1}{np'} j - \frac{m+1}{n} j - kj} \int e^{i2^j \lambda \Phi_3(z,x,j)} \chi_0(z_2)\chi_0(z_1)\chi_1(\lambda) d\lambda dz_1 dz_2,$$

where

$$\Phi_3(z,x,j) := 1 - x_3 - (2^{-\frac{j}{n}} x_1 z_1 + x_2 2^{-\frac{jm}{n}} z_1^m \omega(2^{-\frac{j}{n}} z_1) + z_2 2^{-\frac{jm}{n}} x_2 + x_3 2^{-\frac{2jm}{n}} z_2^2 b(2^{-\frac{j}{n}} z_1, 2^{-\frac{jm}{n}}(z_1^m \omega(2^{-\frac{j}{n}} z_1) + z_2) + 2^{-j} z_1^n \beta(2^{-\frac{j}{n}} z_1))).$$

Stationary phase method

We use stationary phase method assuming ,
$|1-x_3| << 2^{-j}$, $|x_1| << 2^{-\frac{n-1}{n}j}$, $|x_2| << 2^{-\frac{j(n-m)}{n}}$ and obtain:

$$M_k u_j(x) = 2^{j(\frac{3}{p}+\frac{m+1}{np'}-\frac{1}{n}-\frac{1}{2}-k)}$$

$$\int_{\mathbb{R}^2} e^{i2^j\lambda\Phi_4)}\chi_0(z_2^c(z_1,x_2))\chi_0(z_1)\chi_1(\lambda)d\lambda dz_1 + O(2^{j(\frac{2m}{n}-1))}),$$

where

$$\Phi_4 := \Phi_4(z_1,x,j) := 1-x_3-x_1z_1 2^{-\frac{j}{n}} + x_2 2^{-\frac{jm}{n}} z_1^m \omega(2^{-\frac{j}{n}}z_1) + 2^{-j} z_1^n \beta(2^{-\frac{j}{n}}z_1) + x_2^2 2^{-\frac{2jm}{n}} B(z_1,x_2).$$

From here we obtain the lower bound:

$$\|M_k u_j\|_{L^{p'}(\mathbb{R}^3)} \geq 2^{j(\frac{3}{p}+\frac{m+1}{np'}-\frac{1}{n}-\frac{1}{2}-\frac{1}{p'}(3-\frac{m+1}{n})-k)}c,$$

where $c > 0$ is a constant which does not depend on j. Thus, if $k < \left(6 - \frac{2(m+1)}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} + \frac{m}{n}$ then the operator M_k is not $L^p(\mathbb{R}^3) \mapsto L^{p'}(\mathbb{R}^3)$ bounded.
Analogical result holds true for the case $n = \infty$.
Thus, if $k < k_p(v)$ then the M_k is no $L^p - L^{p'}$ operator. This completes a proof of the main Theorem.

THANKS FOR YOUR ATTENTIONS !