On various classes of topologically semi-transitive operators on Banach spaces

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The talk is mainly based on the following paper:

S. Ivković, *On various classes of supercyclic operators on Banach spaces*, https://export.arxiv.org/abs/2406.06011

Definition

Let X be a separable Banach space. A sequence $(T_n)_{n\in\mathbb{N}}$ of bounded operators in $B(X)$ is called *supercyclic* if there is an element $x \in X$ (called *supercyclic vector*) such that the set $\{\lambda T_n x : n \in \mathbb{N}, \lambda \in \mathbb{C} \setminus \{0\}\}$ is dense in X. The set of all supercyclic vectors of a sequence $(T_n)_{n\in\mathbb{N}}$ is denoted by $SC((T_n)_{n\in\mathbb{N}})$. If $SC((T_n)_{n\in\mathbb{N}})$ is dense in X, the sequence $(T_n)_{n\in\mathbb{N}}$ is called *densely supercyclic*. An operator $T \in B(X)$ is called *densely supercyclic* if the sequence $(T^n)_{n\in\mathbb{N}}$ is densely supercyclic.

Definition

Let X be a Banach space and $T \in B(X)$. We say that T is *topologically semi-transitive* on X if for each pair of open non-empty subsets O_1 and O_2 of X there exists some $n \in \mathbb{N}$ and some $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda T^n(O_1) \cap O_2 \neq \varnothing$.

Let $\mathcal{A}$ be a non-unital C^*-algebra such that $\mathcal{A}$ is a closed two-sided ideal in a unital C^*-algebra $\mathcal{A}_1$. Let Φ be an isometric $*$-isomorphism of $\mathcal{A}_1$ such that $\Phi(\mathcal{A}) = \mathcal{A}$. Assume that there exists a net $\{p_\alpha\}_\alpha \subseteq \mathcal{A}$ consisting of self-adjoint elements with $\| p_\alpha \| \leq 1$ for all α and such that $\{p_\alpha^2\}_\alpha$ is an approximate unit for $\mathcal{A}$. Suppose in addition that for all α there exists some $N_\alpha \in \mathbb{N}$ such that $\Phi^n(p_\alpha) \cdot p_\alpha = 0$ for all $n \geq N_\alpha$ (which gives that $0 = (\Phi^n(p_\alpha) \cdot p_\alpha)^* = p_\alpha \cdot \Phi^n(p_\alpha)$ since Φ is a $*$-isomorphism). Let $b \in G(\mathcal{A}_1)$ and $T_{\Phi,b}$ be the operator on $\mathcal{A}_1$ defined by

$$T_{\Phi,b}(a) = b \cdot \Phi(a)$$

for all $a \in \mathcal{A}_1$. Then $T_{\Phi,b}$ is a bounded linear operator on $\mathcal{A}_1$ and since $\mathcal{A}$ is an ideal in $\mathcal{A}_1$, it follows that $T_{\Phi,b}(\mathcal{A}) \subseteq \mathcal{A}$ because $\Phi(\mathcal{A}) = \mathcal{A}$.

The inverse of $T_{\Phi,b}$, which we will denote by $S_{\Phi,b}$, is given as $S_{\Phi,b}(a) = \Phi^{-1}(b^{-1}) \cdot \Phi^{-1}(a)$ for all $a \in \mathcal{A}_1$. Again, since $\Phi^{-1}(\mathcal{A}) = \mathcal{A}$ and $\mathcal{A}$ is a two-sided ideal in $\mathcal{A}_1$, we have that $S_{\Phi,b}(\mathcal{A}) \subseteq \mathcal{A}$, hence $T_{\Phi,b}(\mathcal{A}) = \mathcal{A} = S_{\Phi,b}(\mathcal{A})$.
By some calculations one can check that
$T^n_{\Phi,b}(a) = b \cdot \Phi(b) \dots \Phi^{n-1}(b)\Phi^n(a)$ and
$S^n_{\Phi,b}(a) = \Phi^{-1}(b^{-1})\Phi^{-2}(b^{-1}) \dots \Phi^{-n}(b^{-1}) \cdot \Phi^{-n}(a)$ for all $a \in \mathcal{A}$.

Theorem

Under the above assumptions, the following statements are equivalent.
i) $T_{\Phi,b}$ *is topologically semi-transitive on* $\mathcal{A}$ *.*
ii) For every p_α *there exists a strictly increasing sequence* $\{n_k\}_k \subseteq \mathbb{N}$ *and sequences* $\{q_k\}_k, \{d_k\}_k$ *in* $\mathcal{A}$ *such that*

$$\lim_{k\to\infty} \| q_k - p_\alpha^2 \| = \lim_{k\to\infty} \| d_k - p_\alpha^2 \| = 0$$

and

$$\lim_{k\to\infty} (\| \Phi^{n_k-1}(b^{-1}) \dots \Phi(b^{-1}) b^{-1} d_k \| \cdot \| \Phi^{-n_k}(b) \dots \Phi^{-1}(b) q_k \|) = 0.$$

Let X be a locally compact Hausdorff space, $C_0(X)$ be the C^*-algebra of all continuous functions on X vanishing at infinity equipped with the supremum norm, $C_b(X)$ be the the C^*-algebra of all continuous bounded functions on X equipped with the supremum norm, and $C_c(X)$ be the space of continuous functions on X with compact support. In this case, we let $\mathcal{A} = C_0(X)$, $\mathcal{A}_1 = C_b(X)$ and Φ be given by $\Phi(f) = f \circ \alpha$ for all $f \in C_b(X)$ where α is a homeomorphism of X. Put

$$S = \{f \in C_c(X) \mid 0 \leq f \leq 1 \text{ and } f_{|_K} = 1 \text{ for some compact } K \subset X\}.$$

If $\tilde{S} = \{f^2 \mid f \in S\}$, then $\tilde{S}$ is an approximate unit for $C_0(X)$.

Suppose that α is *aperiodic*, that is for each compact subset K of X, there exists a constant $N_K > 0$ such that for each $n \geq N_K$, we have $K \cap \alpha^n(K) = \varnothing$. Then, for every $f \in S$, there exists some $N_f \in \mathbb{N}$ such that $\Phi^n(f) \cdot f = 0$ for all $n \geq N_f$. By some calculations it is not hard to see that in this case the conditions of Theorem 3 are equivalent to the requirement that for every compact subset K of Ω there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty}[\left(\sup_{t\in K}\prod_{j=0}^{n_k-1}\left(b\circ\alpha^{j-n_k}\right)(t)\right)\cdot\left(\sup_{t\in K}\prod_{j=0}^{n_k-1}\left(b\circ\alpha^{j}\right)^{-1}(t)\right)]=0.$$

As a concrete example, let $X=\mathbb{R}$, $\alpha:\mathbb{R}\to\mathbb{R}$ be given by $\alpha(t)=t-1$ for all $t\in\mathbb{R}$ and b be a continuous bounded positive function on $\mathbb{R}$ satisfying that b^{-1} is also bounded. If there exist some $M,\delta,K_1,K_2>0$ such that $1<M-\delta\le b(t)\le M$ for all $t\le -K_1$ and $\frac{1}{M}\le b(t)\le 1$ for all $t\ge K_2$, then the conditions of Theorem 3 are satisfied. Similarly if $\alpha(t)=t+1$ for all $t\in\mathbb{R}$, $\frac{1}{M}\le b(t)\le\frac{1}{M-\delta}$ for all $t\le -K_1$ and $1\le b(t)\le M$ for all $t\ge K_2$, then the conditions of Theorem 3 are also satisfied. Moreover, if $\alpha(t)=t-1$ for all $t\in\mathbb{R}$, $1\le b(t)\le M$ for all $t\le -K_1$ and $\dfrac{1}{M}\le b(t)\le\dfrac{1}{M-\delta}$ for all $t\ge K_2$, then the conditions of Theorem 3 are satisfied. Finally, if $\alpha(t)=t+1$ for all $t\in\mathbb{R}$, $\dfrac{1}{M}\le b(t)\le 1$ for all $t\le -K_1$ and $M-\delta\le b(t)\le M$ for all $t\ge K_2$, then the conditions of Theorem 3 are also satisfied.

Let H be a separable Hilbert space with an orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$ and U be a unitary operator on H. Set Φ to be the $*$-isomorphism on $B(H)$ given by $\Phi(F) = U^* F U$. For each $m \in \mathbb{N}$, let P_m denote the orthogonal projection onto $Span\{e_{-m}, \ldots, e_m\}$. Then, $\{P_m\}_{m\in\mathbb{N}}$ is an approximate unit for $B_0(H)$, where $B_0(H)$ denotes the C^*-algebra of all compact operators on H. Thus, here we consider the case when $\mathcal{A}_1 = B(H)$ and $\mathcal{A} = B_0(H)$.
Suppose that for every $m \in \mathbb{N}$ there exists an $N_m \in \mathbb{N}$ such that $P_m U^n P_m = 0$ for $n \geq N_m$. Then, for all $n \geq N_m$ we have $\Phi^n(P_m)P_m = U^{*n} P_m U^n P_m = 0$. Assume that α is a translation on $\mathbb{Z}$. Set $U_\alpha(e_j) := e_{\alpha(j)}$ for all $j \in \mathbb{Z}$. Then, U_α is a unitary operator on H satisfying this property.

If W is an invertible bounded linear operator on H, we can consider the operator $\tilde{T}_{U,W}$ on $B_0(H)$ given by $\tilde{T}_{U,W}(F) = WFU$ for all $F \in B_0(H)$ and we have $\tilde{T}_{U,W} = T_{\Phi,WU}$. The conditions in Theorem 3 are in this case equivalent to the condition that for every $m \in \mathbb{N}$ there exist sequences of operators $\{D_k\}_k, \{G_k\}_k$ in $B_0(H)$ and a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} \| D_k - P_m \| = \lim_{k\to\infty} \| G_k - P_m \| = 0$$

and

$$\lim_{k\to\infty} \| W^{n_k} D_k \| \cdot \| W^{-n_k} G_k \| = 0.$$

Theorem

If there exist dense subsets H_1 and H_2 of H and a strictly increasing sequence $\{n_k\}_k$ of natural numbers such that for every $f \in H_1$ and $g \in H_2$ we have that

$$\lim_{k\to\infty} (\| W^{n_k} f \| \cdot \| W^{-n_k} g \|) = 0,$$

then $\tilde{T}_{U,W}$ is topologically semi-transitive on $B_0(H)$.

Let $H = L^2(\mathbb{R})$ and W be the operator on H defined by $W(f) = b \cdot (f \circ \alpha)$ for all $f \in H$, where α is a homeomorphism of $\mathbb{R}$ and b is a continuous, bounded positive function on $\mathbb{R}$ satisfying that b^{-1} is also bounded. Then, W is a bounded invertible linear operator on H. If $m \in \mathbb{N}$ and $f \in L^2(\mathbb{R})$ with *supp* $f \subseteq [-m, m]$, then, by some calculations, it is not hard to see that for all $n \in \mathbb{N}$ we have that

$$\int \left|W^{-n}(f)\right|^2 d\mu \leq \left(\sup_{t \in [-m,m]} \prod_{j=0}^{n-1} (b \circ \alpha^j)^{-1}(t) \right)^2 \|f\|_2^2$$

and

$$\int \left|W^{n}(f)\right|^2 d\mu \leq \left(\sup_{t \in [-m,m]} \prod_{j=0}^{n-1} (b \circ \alpha^{j-n})(t) \right)^2 \|f\|_2^2 .$$

Therefore, for any $f, g \in C_c(\mathbb{R})$, we obtain that

$$\lim_{n\to\infty} \| W^n(f) \|_2 \cdot \| W^{-n}(g) \|_2 \leq$$

$$\lim_{n\to\infty} \left[\left(\sup_{t\in[-m,m]} \prod_{j=0}^{n-1} (b \circ \alpha^{j-n})(t) \right) \cdot \left(\sup_{t\in[-m,m]} \prod_{j=0}^{n-1} (b \circ \alpha^{j})^{-1}(t) \right) \right] \| f \|_2 \| g \|_2$$

where $m \in \mathbb{N}$ is chosen in a such way that *supp* f, *supp* $g \subseteq \{-m, \ldots, m\}$. Hence, if we let α and b be as in the previous example, then

$$\lim_{n\to\infty} \| W^n(f) \|_2 \cdot \| W^{-n}(g) \|_2 = 0.$$

We let now $\mathcal{A} = C_0(\mathbb{R})$ and $\tau \in C_b(\mathbb{R})$, that is τ is a bounded continuous function on $\mathbb{R}$. Put

$$\mathcal{A}_\tau := \{f \in \mathcal{A} : \sum_{k=0}^{\infty} \|f\tau^k\|_\infty < \infty\}.$$

For each $f \in \mathcal{A}_\tau$ we define

$$\|f\|_\tau := \sum_{k=0}^{\infty} \|f\tau^k\|_\infty.$$

Then, $\mathcal{A}_\tau$ is a Banach algebra. We will call this algebra *Segal algebra corresponding to* τ. We shall denote by $K_\epsilon^{(\tau)}$ a compact subset of $|\tau|^{-1}([0,\epsilon])$, where $\epsilon \in (0,1)$.

Let w be a positive function on $\mathbb{R}$ with $w, w^{-1} \in C_b(\mathbb{R})$. If $\tau \in C_b(\mathbb{R})$ and α is a homeomorphism of $\mathbb{R}$ such that $\tau \circ \alpha = \tau$, then the operator $\tilde{T}_{\alpha,w}$ defined by $\tilde{T}_{\alpha,w}(f) = w \cdot (f \circ \alpha)$ for all $f \in \mathcal{A}$ is a bounded linear self-mapping on $\mathcal{A}_\tau$. In the sequel, we shall assume that α is aperiodic.

Theorem

The following statements are equivalent.
(i) $\tilde{T}_{\alpha,w}$ is topologically semi-transitive on $\mathcal{A}_\tau$.
(ii) For each positive ϵ and every compact subset $K_\epsilon^{(\tau)} \subseteq |\tau|^{-1}([0,\epsilon])$ there exists a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$\lim_{k\to\infty} [\left(\sup_{t \in K_\epsilon^{(\tau)}} \prod_{j=0}^{n_k-1} \left(w \circ \alpha^{j-n_k}\right)(t) \right) \cdot \left(\sup_{t \in K_\epsilon^{(\tau)}} \prod_{j=0}^{n_k-1} \left(w \circ \alpha^{j}\right)^{-1}(t) \right)] = 0.$$

Now, if we consider $\tilde{T}_{\alpha,w}$ as an operator on $C_0(\Omega)$ where Ω is a locally compact Hausdorff space, then the adjoint $\tilde{T}^*_{\alpha,w}$ is a bounded linear operator on $M(\Omega)$ where $M(\Omega)$ stands for the Banach space of all Radon measures on Ω equipped with the total variation norm. It is straightforward to check that

$$\tilde{T}^*_{\alpha,w}(\mu)(E) = \int_E w \circ \alpha^{-1} \, d\mu \ \circ \alpha^{-1}$$

for every $\mu \in M(\Omega)$ and every measurable subset E od Ω. Here $\mu \ \circ \alpha^{-1}(E) = \mu(\alpha^{-1}(E))$ for every $\mu \in M(\Omega)$ and every measurable subset E od Ω.

In the sequel, for every Radon measure μ on Ω, we let as usual $|\mu|$ denote the total variation of μ. Also, we assume as before that α is an aperiodic homeomorphism of Ω. Under these assumptions and keeping this notation, we provide the following theorem.

Theorem

The following statements are equivalent.
(i) $\tilde{T}^*_{\alpha,w}$ *is topologically semi-transitive on* $M(\Omega)$.
(ii) For every compact subset K *of* Ω *and any two measures* μ, v *in* $M(\Omega)$ *with* $|\mu|(K^c) = |v|(K^c) = 0$ *there exist a strictly increasing sequence* $\{n_k\}_k \subseteq \mathbb{N}$ *and sequences* $\{A_k\}_k, \{B_k\}_k$ *of Borel subsets of* K *such that* $\alpha^{n_k}(K) \cap K = \varnothing$ *for all* $k \in \mathbb{N}$ *and*

$$\lim_{k\to\infty} |\mu|(A_k) = \lim_{k\to\infty} |v|(B_k) = 0,$$

$$\lim_{k\to\infty} [\left(\sup_{t \in A_k^c \cap K} \prod_{j=0}^{n_k-1} \left(w \circ \alpha^j\right)(t) \right) \cdot \left(\sup_{t \in B_k^c \cap K} \prod_{j=1}^{n_k} \left(w \circ \alpha^{-j}\right)^{-1}(t) \right)] = 0.$$

In the last two theorems, the assumption that α is aperiodic is only needed for the proof of the implication $(i) \Rightarrow (ii)$. Therefore, we obtain the following corollary, which holds for general (not necessarily aperiodic) homeomorphism α of Ω.

Corollary

We have that ii) $\Rightarrow$ i).
i) $\tilde{T}^*_{\alpha,w}$ is topologically semi-transitive on $M(\Omega)$.
ii) For every compact subset K of Ω, we have that

$$\lim_{n\to\infty}\left[\left(\sup_{t\in K}\prod_{j=0}^{n-1}\left(w\circ\alpha^{j}\right)(t)\right)\cdot\left(\sup_{t\in K}\prod_{j=1}^{n}(w\circ\alpha^{-j})^{-1}(t)\right)\right]=0.$$

Let $\Omega = \mathbb{R}$, $\alpha : \mathbb{R} \to \mathbb{R}$ be given by $\alpha(t) = t + 1$ for all $t \in \mathbb{R}$ and w be a continuous bounded positive function on $\mathbb{R}$ such that w^{-1} is also bounded. If there exist some $M, \delta, K_1, K_2 > 0$ such that $1 < M - \delta \leq w(t) \leq M$ for all $t \leq -K_1$ and $\frac{1}{M} \leq w(t) \leq 1$ for all $t \geq K_2$, then the conditions of Corollary 7 are satisfied. Similarly if $\alpha(t) = t - 1$ for all $t \in \mathbb{R}$, $\frac{1}{M} \leq w(t) \leq \frac{1}{M-\delta}$ for all $t \leq -K_1$ and $1 \leq w(t) \leq M$ for all $t \geq K_2$, then the conditions of Corollary 7 are also satisfied. Moreover, if $\alpha(t) = t + 1$ all $t \in \mathbb{R}$, $1 \leq w(t) \leq M$ for all $t \leq -K_1$ and $\dfrac{1}{M} \leq w(t) \leq \dfrac{1}{M-\delta}$ for all $t \geq K_2$, then the conditions of Corollary 7 are satisfied. Finally, if $\alpha(t) = t - 1$ for all $t \in \mathbb{R}$, $\dfrac{1}{M} \leq w(t) \leq 1$ for all $t \leq -K_1$ and $M - \delta \leq w(t) \leq M$ for all $t \geq K_2$, then the conditions of Corollary 7 are also satisfied.

In sequel, the set of all Borel measurable complex-valued functions on a topological space X is denoted by $\mathcal{M}_0(X)$. Also, χ_A denotes the characteristic function of a set A. We recall the following definitions.

Definition

Let X be a topological space and $\mathcal{F}$ be a linear subspace of $\mathcal{M}_0(X)$. If $\mathcal{F}$ equipped with a given norm $\|\cdot\|_{\mathcal{F}}$ is a Banach space, we say that $\mathcal{F}$ is a *Banach function space on* X.

Definition

Let $\mathcal{F}$ be a Banach function space on a topological space X, and $\alpha : X \longrightarrow X$ be a homeomorphism. We say that $\mathcal{F}$ is α*-invariant* if for each $f \in \mathcal{F}$ we have $f \circ \alpha^{\pm 1} \in \mathcal{F}$ and $\|f \circ \alpha^{\pm 1}\|_{\mathcal{F}} = \|f\|_{\mathcal{F}}$.

Definition

A Banach function space $\mathcal{F}$ on X is called *solid* if for each $f \in \mathcal{F}$ and $g \in \mathcal{M}_0(X)$, satisfying $|g| \leq |f|$, we have $g \in \mathcal{F}$ and $\|g\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}}$.

Definition

Let X be a topological space, $\mathcal{F}$ be a Banach function space on X, and α be an aperiodic homeomorphism of X. We say that $\mathcal{F}$ satisfies condition Ω_α if the following conditions hold:

1. $\mathcal{F}$ is solid and α-invariant;
2. for each compact set $E \subseteq X$ we have $\chi_E \in \mathcal{F}$;
3. $\mathcal{F}_{bc}$ is dense in $\mathcal{F}$, where $\mathcal{F}_{bc}$ is the set of all bounded compactly supported functions in $\mathcal{F}$.

Theorem

The following statements are equivalent.

i) $\tilde{T}_{\alpha,w}$ *is topologically semi-transitive on* $\mathcal{F}$*.*

ii) For each compact subset K *of* X*, there exist a sequence of Borel subsets* $\{E_k\}_{k=1}^{\infty}$ *of* K *and a strictly increasing sequence* $\{n_k\}_k \subseteq \mathbb{N}$ *such that*

$$\lim_{x\to\infty} \left\|\chi_{K\setminus E_k}\right\|_{\mathcal{F}} = 0$$

and

$$\lim_{k\to\infty} [\left(\sup_{x\in E_k} \prod_{j=0}^{n_k-1} \left(w\circ\alpha^{j}\right)^{-1}(x)\right)\cdot\left(\sup_{x\in E_k} \prod_{j=1}^{n_k} (w\circ\alpha^{-j})(x)\right)] = 0.$$

We notice once again that the assumption that α is aperiodic is only needed for the proof of the implication $i) \Rightarrow ii)$ in Theorem 12. Therefore, we obtain the following corollary, which holds for a general homeomorphism α of X.

Corollary

We have that $ii) \Rightarrow i)$
i) $\tilde{T}_{\alpha,w}$ is topologically semi-transitive on $\mathcal{F}$.
ii) For every compact subset K of X, we have

$$\lim_{n\to\infty}\left[\left(\sup_{x\in K}\prod_{j=0}^{n-1}\left(w\circ\alpha^j\right)^{-1}(x)\right)\cdot\left(\sup_{x\in K}\prod_{j=1}^{n}(w\circ\alpha^{-j})(x)\right)\right]=0.$$

If $\mathcal{F} = L^2(\mathbb{R})$ and $\alpha, w = b$ are as in previous example, then the conditions of the previous corollary are satisfied.

Definition

Let X be a Banach space and $B(X)$ denote the space of all linear bounded operators on X. We say the operators $T_1, T_1, \ldots, T_N \in B(X)$ are disjoint topologically semi-transitive provided for every non-empty open subsets $V_0, V_1, \ldots V_N$ of X, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that

$$V_0 \cap (\lambda T_1^{-n})(V_1) \cap \cdots \cap (\lambda T_N^{-n})(V_N) \neq \varnothing.$$

Theorem

Let $r_1, \ldots, r_N \in \mathbb{N}$ with $r_1 < \ldots < r_N$. Then, under the assumptions of Proposition 12 and keeping the same notation, the following statements are equivalent.

(i) The operators $\tilde{T}^{r_1}_{\alpha,w_1}, \ldots, \tilde{T}^{r_N}_{\alpha,w_N}$ are disjoint topologically semi-transitive on $\mathcal{F}$.

(ii) For each compact subset K of X there exist a sequence $\{E_k\}_k$ of Borel subsets of K and a strictly increasing sequence $\{n_k\}_k$ of natural numbers such that $\lim_{k\to\infty} \chi_{E_k} = \chi_K$ in $\mathcal{F}$, for each $s, l \in \{1, 2, \ldots, N\}$ we have

$$\lim_{k\to\infty} \left[\left(\sup_{x\in E_k} \prod_{j=0}^{r_l n_k - 1} \left(w_l \circ \alpha^j \right)^{-1}(x) \right) \left(\sup_{x\in E_k} \prod_{j=1}^{r_s n_k} (w_s \circ \alpha^{-j})(x) \right) \right] = 0,$$

and for each distinct $j, l \in \{1, \ldots, N\}$ we have

$$\lim_{k\to\infty} \sup_{x\in E_k} \frac{\prod_{i=1}^{r_l n_k} \left(w_l \circ \alpha^{r_j n_k - i} \right)(x)}{\prod_{i=0}^{r_j n_k - 1} (w_j \circ \alpha^i)(x)} = 0.$$

Corollary

We have that (ii) implies (i).
(i) The operations $\tilde{T}^{r_1}_{\alpha,w_1}, \ldots, \tilde{T}^{r_N}_{\alpha,w_N}$ *are disjoint topologically semi-transitive.*
(ii) For each compact subset K *of* X *and reach* $s, l \in \{1, \ldots N\}$ *we have*

$$\lim_{n\to\infty} \left[\left(\sup_{x\in K} \prod_{i=0}^{r_l n-1} \left(w_l \circ \alpha^i\right)^{-1}(x) \right) \left(\sup_{x\in K} \prod_{j=1}^{r_s n} (w_s \circ \alpha^{-j})(x) \right) \right] = 0,$$

and for each distinct $j, l \in \{1, \ldots, N\}$ *we have*

$$\lim_{n\to\infty} \sup_{x\in K} \frac{\prod_{i=1}^{r_l n} \left(w_l \circ \alpha^{r_j n - i}\right)(x)}{\prod_{i=0}^{r_j n-1} (w_j \circ \alpha^i)(x)} = 0.$$

Let $r_1 \in \mathbb{N}, r_2 = 6r_1, X = \mathbb{R}$ and $\alpha(t) = t - 1$ for all $t \in \mathbb{R}$. Put $w_1 = \chi_{\mathbb{R}^-} + \frac{1}{2}\chi_{\mathbb{R}^+}$ and $w_2 = 3\chi_{\mathbb{R}^-} + \frac{1}{3}\chi_{\mathbb{R}^+}$. By some calculations one can check that the operators $\tilde{T}^{r_1}_{\alpha,w_1}$ and $\tilde{T}^{r_{21}}_{\alpha,w_2}$ satisfy the conditions of the part (ii) in Corollary 16.

Let Ω be a compact Hausdorff space, $C_{\mathbb{R}}(\Omega)$ be the space of all real-valued continuous functions on Ω equipped with the supremum-norm, and $M_r(\Omega)$ denote the space of all signed Radon measures on Ω with the norm $\| v \| = |v|(\Omega)$.
For $v \in M_r(\Omega)$, let $\phi_v : C_{\mathbb{R}}(\Omega) \to \mathbb{R}$ be defined by

$$\phi_v(f) = \int_\Omega f dv.$$

Then $v \to \phi_v$ is an isometric isomorphism of $(M_r, (\Omega), \| \cdot \|)$ onto $((C_{\mathbb{R}}(\Omega))^*, \| \cdot \|)$.
Let $k : \Omega \times \Omega \to \mathbb{R}$ be a continuous non - negative function and let μ be a positive Radon measure on Ω such that

$$\int_\Omega k(x, y) d\mu(y) > 0 \text{ for all } x \in \Omega.$$

Put then $\tilde{k} : \Omega * \Omega \to \mathbb{R}$ to be defined as

$$\tilde{k}(x,y) = \frac{k(x,y)}{\int_\Omega k(x,y)d\mu(y)}.$$

Then $\tilde{k}$ is continuous, non-negative and $\int_\Omega \tilde{k}(x,y)d\mu(y) = 1$.

We consider now the integral operator T_k on $C_{\mathbb{R}}(\Omega)$ given by

$$T_k(f)(x) = \int_\Omega \tilde{k}(x,y)f(y)d\mu(y) \text{ for all } x \in \Omega.$$

Theorem

Under the above assumptions, if $||\tilde{k}||_\infty < \frac{2}{\mu(\Omega)}$, *then there exists a unique invariant probability Radon measure* $\tilde{v}$ *on* Ω *such that*

$$|(T_k^*)^n(v) - \tilde{v}|\ (\Omega) \to 0 \text{ as } n \to \infty$$

for all probability Radon measures v *on* Ω.

As a concrete example, let now $\Omega = [0, 2\pi]$, μ be the Lebesgue measure on $[0, 2\pi]$ and $k : \Omega \times \Omega \to \mathbb{R}$ be given as $k(x, y) = \frac{1}{4}sin(\frac{1}{4}(x + y))$.

Thank you for attention !

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[1] F. Bayart and É. Matheron, Dynamics of Linear Operators, Cambridge Tracts in Math. **179**, Cambridge University Press, Cambridge, 2009.

[2] C-C. Chen, S.M. Tabatabaie, *Chaotic and Hypercyclic Operators on Solid Banach Function Spaces*, Probl. Anal. Issues Anal. Vol. 9 (27), No 3, 2020, pp. 83–98, DOI: 10.15393/j3.art.2020.8750

[3] K-G. Grosse-Erdmann and A. Peris, Linear Chaos, Universitext, Springer, 2011.

[4] J. Inoue and S.-E. Takahasi, *Segal algebras in commutative Banach algebras*, Rocky Mountains of Math., **44**(2) (2014), 539-589.

[5] S. Ivković, *Hypercyclic operators on Hilbert C*-modules*, Filomat **38** (2024), 1901–1913.

[6] S. Ivković, S. M. Tabatabaie, *Disjoint Linear Dynamical Properties of Elementary Operators* , Bull. Iran. Math. Soc., **49**, 63 (2023). https://doi.org/10.1007/s41980-023-00808-1

[7] S. Ivković, S. M. Tabatabaie, *Hypercyclic Generalized Shift Operators*, Complex Anal. Oper. Theory, **17**, 60 (2023). https://doi.org/10.1007/s11785-023-01376-2

[8] S. Ivković *Dynamics of operators on the space of Radon measures*, https://doi.org/10.48550/arXiv.2310.10868

[9] Y. Liang, . Z. Zhou, *Disjoint supercyclic weighted composition operators*, Bull. Korean Math. Soc. 55(4), (2018), 1137-1147

[10] O. Martin and R. Sanders, *Disjoint supercyclic weighted shifts*, Integr. Equ. Oper. Theory, 85, 191-220, 2016

[11] H. Salas, *Supercyclicity and weighted shifts.* Studia Math. 135(1), 55- 74 (1999).

[12] Y. Sawano, S.M. Tabatabaie and F. Shahhoseini, *Disjoint dynamics of weighted translations on solid spaces*, Topology Appl. **298**, 107709, 14 pp. (2021) DOI:10.1016/J.TOPOL.2021.107709

[13] Ya Wang, Cui Chen, Ze-Hua Zhou, *Disjoint hypercyclic weighted pseudoshift operators generated by different shifts.* Banach J. Math. Anal. 13 (4) 815 - 836, October 2019. https://doi.org/10.1215/17358787-2018-0039

[14] L. Zhang and Z-H. Zhou, *Disjointness in supercyclicity on the algebra of Hilbert-Schmidt operators*, Indian J. Pure Appl. Math. **46** 219–228 (2015). https://doi.org/10.1007/s13226-015-0116-9