On the theory of functions of omega-bounded type

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Early results

Biberbach, 1914, approximation by rational functions in the Diichlet space

$$\iint_{|z|<1} |f'(z)|^2 d\sigma(z) < +\infty$$

of holomorphic in the unit disc functions.

W. Wirtinger, 1932, orthogonal projection and representation by the square of the Cauchy kernel in the Hilbert space

$$\iint_{|z|<1} |f(z)|^2 \, d\sigma(z) < +\infty.$$

of holomorphic in the unit disc functions.

Pages 151-152 of the Walsh’s book

R. Nevanlinna's book, for $0 < \alpha < +\infty$, 1936

$$+\infty > \int_0^1 (1-r)^{\alpha-1} T(r,f)dr = \sup_{0<x<1} \int_0^x (x-r)^{\alpha-1} T(x,f)dr$$

$$= \sup_{0<x<1} \int_0^x (x-r)^{\alpha-1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(xe^{i\vartheta})| d\vartheta \right\} dr$$

$$+ \sup_{0<x<1} \int_0^x (x-r)^{\alpha-1} N(r,f)dr$$

$$\Longrightarrow \quad \sum_k (1-|a_k|)^{1+\alpha} < +\infty, \qquad \sum_n (1-|b_n|)^{1+\alpha} < +\infty,$$

Theorem III. *If a function $F(z)$ is meromorphic in $|z|<1$ and*

$$(\alpha+1)\int_0^1 (1-\rho)^\alpha T(\rho)d\rho < +\infty \quad (\alpha > -1),$$

then this function is representable in the following canonical form:

$$F(z) = Kz^\lambda \frac{\pi_\alpha(z,\{a_\mu\})}{\pi_\alpha(z,\{b_\nu\})} \exp\left\{\frac{\alpha+1}{\pi}\int_0^1\int_0^{2\pi}(1-\rho^2)^\alpha \frac{\lg|F(\rho e^{i\vartheta})|}{(1-z\rho e^{-i\vartheta})^{\alpha+2}}\rho d\rho d\vartheta\right\},$$

where $\pi_\alpha(z,\{a_\mu\})$ and $\pi_\alpha(z,\{b_\nu\})$ are defined by (8) and

$$K = \overline{C}_\lambda^{-1} \exp\left\{4\lambda(\alpha+1)\int_0^1 (1-\rho^2)^\alpha \lg\frac{1}{\rho}\rho d\rho d\vartheta\right\}.$$

1. We define $B_\delta(\alpha)$ $(\delta > 0,\ \alpha > -1)$ as the set of all those functions $f(z)$ meromorphic in $|z| < 1$, for which the integral

$$\frac{\alpha+1}{\pi}\int_0^1\int_0^{2\pi}(1-\rho^2)^\alpha|f(\rho e^{i\vartheta})|^\delta\rho d\rho d\vartheta \tag{1}$$

exists.

Theorem I. *If $f(z) \in B_\delta(\alpha)$ $(\delta \geq 1)$, then the following integral representations are true in $|z| < 1$:*

$$f(z) = \frac{\alpha+1}{\pi}\int_0^1\int_0^{2\pi}(1-\rho^2)^\alpha\frac{f(\rho e^{i\vartheta})}{(1-z\rho e^{-i\vartheta})^{\alpha+2}}\rho d\rho d\vartheta, \tag{2}$$

$$f(z) = -\overline{f(0)} + \frac{2(\alpha+1)}{\pi}\int_0^1\int_0^{2\pi}(1-\rho^2)^\alpha\frac{\operatorname{Re} f(\rho e^{i\vartheta})}{(1-z\rho e^{-i\vartheta})^{\alpha+2}}\rho d\rho d\vartheta. \tag{2^1}$$

Applying the operator immediately to the function

Fourier transform $\Longrightarrow$ Riemann-Liouville integrodifferentation

$$D^{-\alpha}\log|f(z)| \equiv \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\log|f(tz|dt, \quad 0 < alpha < +\infty,$$

$$D^{\alpha}\log|f(z)| \equiv D^{\alpha-1}|z|\frac{d}{d|z|}\log|f(z)|, \quad 0 < \alpha < 1,$$

$$D^0\log|f(z)| \equiv \log|f(z)|.$$

This leads to the M.M.Djrbashian factorization theory of the classes N_α of functions meromorphic in the unit disc.

While the change of $(1-t)^{alpha-1}$ to a wide class of ω functional parametars in $[0,1]$, which can have any rate of decrease near 1 and the operators which can be simply written in the form

$$L_\omega\log|f(z)| \equiv -\int_0^1 \log|f(tz)|d\omega(t),$$

leads to his theory of $N\{\omega\}$ classes, the union of which covers the whole set of functions meromorphic in $|z| < 1$.

Delta-subharmonic extension of the M.M.Djrbashian factorization theory and a similar theory in the half-plane

A passage to the more general delta subharmonic functions $u(z)$ instead of $\log|f(z)|$ leads me to a simple connection of M.M.Djrbashian's characteristic with Nevanlinna's one:

$$T_\omega(u,r) = T(L_\omega u, r), \quad 0 < r < +\infty,$$

and construct an a theory of Riesz type representations of some classes N_ω, the union of which covers the whole set of functions delta-subharmonic in $|z| < 1$.

A similar theory was constructed, where the Fourier-Taylor series is replaced by the Laplace transform and the operator

$$L_\omega u(z) \equiv \int_0^{+\infty} u(z+it)\,d\omega(t)$$

is used .

A^p_ω and N°_ω in the disc, in the half-plane and in the whole finite complex plane

Applying the operator to the integral means of the p-th degree of the integral means of a holomorphic function leads to some theories in the disc and in the half-plane, where:

- The representations of functions are given by the M.M.Djrbashian omega special omega-kernels.
- These representations generate the orthogonal projection for p=2 and an isomorphism for any other p.
- For p=2 an explicit isometry formula is found with the Hardy space.
- Riesz type representations are found for delta-subharmonic functions, the Nevanlinna and Tsuji characteristics of which can have any growth rate.

Banach spaces of functions delta-subharmonic in the disc and in the half-plane

I shall bring here only one result in the disc. The results in the half-plain are similar.

Let $\omega \in \Omega_B(\mathbb{D})$ and $0 < d < 1$. Then $\tilde{\mathcal{D}}_{\omega,d}$ is the set of those delta-subharmonic in $\mathbb{D}$ functions u with associated charges ν, the supports of which are located in the ring $\{\zeta : d \le |\zeta| < 1\}$ and

$$||u||_\omega = \sup_{0<r<1} \frac{1}{2\pi}\left\{\int_0^{2\pi} \left[L_\omega u(re^{i\vartheta})\right]^2 d\vartheta\right\}^{1/2} + \iint_{\mathbb{D}}\left(\int_0^{1-|\zeta|} \omega\left(|\zeta|(1-t)\right)\frac{dt}{\sqrt{t}}\right)|d\nu(\zeta)| < +\infty.$$

Let $\omega \in \Omega_B(\mathbb{D})$ and $d \in (0,1)$. Then $\tilde{\mathcal{D}}_{\omega,d}$ is a Banach space

THEOREM 19. *The class H_2' which is the closed extension of the set* $1, z, z^2, \cdots$ *on* C': $|z| < 1$ *is essentially the set of functions of class* L^2 *on* C' *analytic in* C', *or in other words is essentially the class of functions represented by* (57).

The analogue of the latter part of Theorem 15 is

THEOREM 20. *Let* $F(z)$ *be of class* L^2 *on* C'. *The essentially unique function* $f(z)$ *of class* H_2' *such that*

$$\iint_{C'} |F(z) - f(z)|^2 \, dS$$

=0, if F=f

is least is given by

$$f(z) \equiv \frac{1}{\pi} \iint_{C'} F(\zeta) \frac{dS}{(1 - \bar{\zeta} z)^2}, \qquad |z| < 1. \tag{58}$$

The formal development of $F(z)$ on C' in terms of the functions z^k is

$$\sum_{k=0}^{\infty} a_k z^k, \qquad a_k = \frac{k+1}{\pi} \iint_{C'} F(\zeta) \bar{\zeta}^k \, dS; \tag{59}$$

this series converges to $f(z)$ of class H_2' in the mean on C', hence (§5.8, Theorem 17) converges to $f(z)$ uniformly on any closed set interior to C'. Interior to C', the function represented by (59) is

$$f(z) \equiv \frac{1}{\pi} \iint_{C'} F(\zeta) [1 + 2\bar{\zeta} z + 3\bar{\zeta}^2 z^2 + \cdots] \, dS, \qquad |z| < 1,$$

for the series in square brackets converges uniformly for $|\zeta| \leqq 1$ when z is fixed. This equation for $f(z)$ can be rewritten in form (58). Of course if $F(z)$ is an arbitrary function of class H_2', then (58) is valid with $f(z) \equiv F(z)$.

Theorem 20 is due to Wirtinger [1932], by a quite different method.

By the method of proof of Theorem 17 the reader may prove

THEOREM 21. *Let* $n(z)$ *be the square of the modulus of a function* $N(z)$ *analytic, bounded, and bounded from zero on* C': $|z| < 1$. *Let the function* $F(z)$ *be of class* L^2 *on* C'. *The essentially unique function* $f(z)$ *of class* H_2' *such that*

$$\iint_{C'} n(z) |F(z) - f(z)|^2 \, dS \tag{60}$$

is least is given by

$$f(z) \equiv \frac{1}{\pi N(z)} \iint_{C'} \frac{F(\zeta) N(\zeta) \, dS}{(1 - \bar{\zeta} z)^2}, \qquad |z| < 1.$$

If auxiliary conditions (53) are prescribed in the situation of Theorem 21, the determination of $f(z)$ is more difficult. However, if all of the points β_j coincide at the origin, we shall indicate that a solution of the problem lies at hand. In the case $n(z) \equiv 1$ let $p(z)$ denote the polynomial of degree $\nu - 1$ which satisfies the auxiliary conditions. The problem of determining $f(z)$ so as to minimize (60) is equivalent to that of determining the function $f_1(z)$ of class H_2' such that

$$\iint_{C'} | F(z) - p(z) - z^\nu f_1(z) |^2 \, dS$$

is least. The formal development of $F(z) - p(z)$ in terms of the functions $z^\nu, z^{\nu+1}, z^{\nu+2}, \cdots$ orthogonal on C' is

$$\sum_{k=\nu}^{\infty} a_k z^k, \qquad a_k = \frac{k+1}{\pi} \iint_{C'} [F(\zeta) - p(\zeta)] \, \bar{\zeta}^k \, dS,$$

whence as with (59)

$$z^\nu f_1(z) \equiv \frac{1}{\pi} \iint_{C'} [F(\zeta) - p(\zeta)] \frac{(\nu + 1) \, \bar{\zeta}^\nu z^\nu - \nu \bar{\zeta}^{\nu+1} z^{\nu+1}}{(1 - \bar{\zeta} z)^2} \, dS;$$

the minimizing function $f(z)$ is $p(z) + z^\nu f_1(z)$. The introduction of a norm function as in Theorem 21 presents no difficulty.

Theorems 20 and 21 and the remark just made extend to more general regions by the use of conformal mapping; compare §11.4.

The study of extremal problems and their solution by methods of approximation is to be resumed in §§11.3 and A 3.

Of course one may study approximation in a *multiply connected* region (compare §§1.6 and 1.7) in the sense of least squares, by orthogonalizing a suitable set of rational functions; see Ghika [1936] and Bergman [1950].