Remarks on some recent results in metric fixed point theory

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Mathematics Genealogy Project

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The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

G.H. Hardy (1887-1977)

"A century later the multitude is vaster and the utility little greater."

Brailey Sims

Structure

$$f(p) = p$$

1. J. Liouville, Second mémoire sur le développement des fonctions ou parties de fonctions en séries dont divers termes sont assujettis á satisfaire a une m eme équation différentielle du second ordre contenant un paramétre variable, J. Math. Pure et Appi., **2** (1837), 16–35.
2. E. Picard, Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives, J. Math. Pures et Appl., **6** (1890), 145–210.
3. H.Poincaré, Surless courbes define barles equations differentiate less, J.de Math., 2 (1886), 54–65.

One of the accepted classification of Fixed Point Theory is the following:

1. Topological Fixed Point Theory (Brouwer's Fixed Point Theorem)
2. Metric Fixed Point Theory (Banach's Fixed Point Theorem)
3. Discrete Fixed Point Theory (Tarski's Fixed Point Theorem)

Applications are ubiquitous

$$f(p) = p$$

$$T(p) = f(p) - p = 0$$

and, accordingly, finding the zeros of the mapping T and finding the fixed point of f becomes equivalent statement.
For example, finding the fixed point of

$$f(x) = x^2 - 12$$

is equivalent to solving the equation (finding zeros)

$$x = f(x) = x^2 - 12 \Leftrightarrow x^2 - x - 12 = 0.$$

Applications are ubiquitous

Consider one of the classical open problems of number theory, finding perfect numbers: Let $p : \mathbb{N} \to \mathbb{N}$ such that $p(n)$ is the sum of all divisors of n for $n > 1$.
Thus, any fixed points of the function p gives a perfect number, that is,

$$p(n) = n.$$

In particular, 6 is the smallest perfect numbers, (see also, 28, 496, 8128)

$$2^{74207280} \times (2^{74207281} - 1), \text{ with } 44,677,235 \text{ digits},$$

is the biggest known perfect number.

Structure

Statement.

Banach Fixed Point Theorem

Every contraction on a complete normed space has a unique fixed point.

"Stefan Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, Fund. Math., 3, **(1922)**, 133–181. "

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Banach Fixed Point Theorem

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Banach-Caccioppoli Fixed Point Theorem

Every contraction on a complete metric space has a unique fixed point.

R. Caccioppoli, Una teorema generale sull'esistenza di elementi uniti in una transformazione funzionale, Ren. Accad. Naz Lincei **11**(1930), 794-799.

Historical Remarks.

ED. GOURSAT

Sur la théorie des fonctions implicites

Bulletin de la S. M. F., tome 31 (1903), p. 184-192

<http://www.numdam.org/item?id=BSMF_1903__31__184_1>

1. Soit $f(x,y)$ une fonction des deux variables indépendantes réelles x et y, continue dans le voisinage d'un système de valeurs x_0, y_0, tel que $f(x_0, y_0) = 0$. Pour préciser, nous supposerons que cette fonction est continue dans un domaine D défini par les inégalités

$$(1) \qquad x_0 - a \leqq x \leqq x_0 + a, \qquad y_0 - b \leqq y \leqq y_0 + b,$$

a et b étant deux nombres positifs. Nous admettrons de plus que l'on peut choisir les nombres a et b assez petits pour que l'on ait

$$(2) \qquad |f(x,y') - f(x,y'')| < \mathrm{K}\,|y' - y''|,$$

x étant une valeur quelconque comprise entre $x_0 - a$ et $x_0 + a$, y', y'' étant de même deux valeurs quelconques de y comprises entre $y_0 - b$ et $y_0 + b$, et K un nombre positif constant *plus petit que l'unité*. Cette dernière condition sera certainement vérifiée si la fonction $f(x,y)$ admet une dérivée partielle $\frac{\partial f}{\partial y}$ s'annulant pour $x = x_0, y = y_0$, et continue dans le voisinage.

Sketch of the proof.

Formal Form of Banach-Caccioppoli Fixed Point Theorem

Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that for any pair of points $x, y \in X$,

$$d(Tx, Ty) \leq k\, d(x, y),$$

where k is a constant in $[0, 1)$. Then T has a unique fixed point.

Proof of Banach-Caccioppoli's Theorem is awesome:

-Take any $x \in X$ and set $x_0 := x$ and $x_{i+1} := Tx_i$ (Picard operator)!

-Observe T is asymptotically regular:

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} d(Tx_n, Tx_{n-1}) = 0.$$

-Recall that the sequence $\{x_n\}$ is Cauchy and (X, d) is complete!

-Since, T is necessarily continuous, limit of the constructed sequence becomes a fixed point.

-Uniqueness is derived via *Reductio ad Absurdum*.

Applications are ubiquitous

Consider the Volterra integral equation

$$u(x) = f(x) + \int_0^x F(x, y)u(y)dy,$$

where f and the kernel F are defined and continuous on, respectively, $[0, a]$ and $[0, a] \times [0, a]$. Let operator $T : C[0, a] \to [0, a]$ be defined by

$$T(u(x)) = f(x) + \int_0^x F(x, y)u(y)dy,$$

then for $u, v \in C[0, a]$ we derive easily

$$\|Tu - Tv\| \leq aK\|u - v\| \text{ where } K := \sup_{0\leq x,y\leq a} |F(x, y)|,$$

and $\|\cdot\|$ is the usual supremum norm on $C[0, a]$.
Banach's contraction principle thus immediately yields a unique solution (with convergence of successive approximations) on any interval for which $aK < 1$.

Banach-Caccioppoli Theorem: Revisited

Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that for any pair of points $x, y \in X$,

$$d(Tx, Ty) \leq k\, d(x, y),$$

where k is a constant in $[0, 1)$. Then T has a unique fixed point x_0. Further,

$$d(T^n x, x_0) \leq \frac{k}{1-k}\, d(x, Tx), \text{ for all } x \in X, \text{ and for all } n \geq 1.$$

If $k = 1$ then T is called non-expansive mapping.

Whether BCT is valid for non-expansive mapping T ?

Examples of Fixed-Point Free Non-Expansive Mappings

Let $C := \{\{x_n\} \in c_0 : 0 \leq x_n \leq 1 \text{ for all } n \in \mathbb{N}.\}$ Define a self-mapping $T, S : C \to C$ as follows:

$$T(\{x_n\}) := \{1, x_1, x_2, ..., x_n, ...\},$$

and,

$$S(\{x_n\}) := \{1 - x_1, x_2, ..., x_n, ...\}.$$

It is clear that $\|Tx - Ty\| = \|x - y\|$ and, $\|Sx - Sy\| = \|x - y\|$ for any $x, y \in C$. Thus, both self-mappings T and S are non-expansive (in fact, metric isometries).
On the other hand, both T and S are fixed-point free. Indeed, only possible fixed point for T is $p^* = \{1, 1, 1, ..., 1, ...\} \notin c_0$ and for S is $q^* = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, ...\} \notin c_0$. Note that $p^*, q^* \notin C$.

Contractive Mappings

Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that for any pair of points $x, y \in X$ with $x \neq y$,

$$d(Tx, Ty) < d(x, y).$$

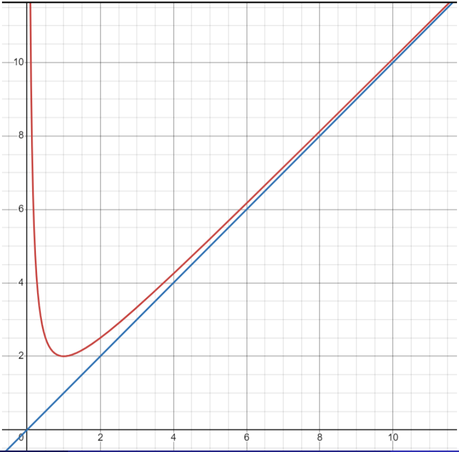
Here, the mapping T is called contractive .

Whether Banach-Caccioppoli Theorem is valid for contractive mappings?

Examples of Fixed-Point Free Contractive Mappings

Example 1

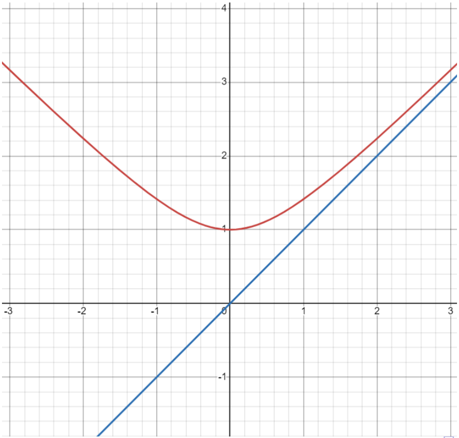
Let $(X := [1, \infty), d)$ be a complete metric space and let $T : X \to X$ be a mapping such that $Tx = x + \frac{1}{x}$.



Examples of Fixed-Point Free Contractive Mappings

Example 2

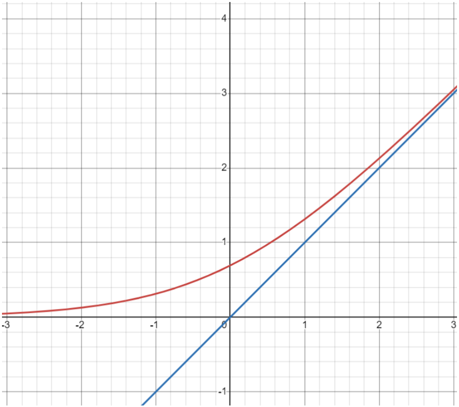
Let $(X := \mathbb{R}, d)$ be a complete metric space and let $T : X \to X$ be a mapping such that $Tx = \sqrt{1 + x^2}$.



Examples of Fixed-Point Free Contractive Mappings

Example 3

Let $(X := \mathbb{R}, d)$ be a complete metric space and let $T : X \to X$ be a mapping such that $Tx = \ln(1 + e^x)$.



Motivation...

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- Easy to state the problem, theorem and solution.
- Not only guarantee the existence and uniqueness, but also show how you get the precise solution!
- Huge application potential!...

A retrospective approach: Linear case

One of the initial significant results, as a generalization of Banach contraction principle, was given in 1969 by Kannan.

Kannan Theorem, 1969

Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that for any pair of points $x, y \in X$,

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)],$$

where k is a constant in $[0, \frac{1}{2})$. Then T has a unique fixed point.

He removed the necessity of the continuity condition. In fact, "continuity" is not the assumption of the Banach contraction mapping principle. It is rather a consequence of the statement. Each contraction mapping is necessarily continuous:

$$d(Tx, Ty) \leq kd(x, y).$$

A retrospective approach: Linear case

Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point in X if it satisfies any of the following conditions:

[Rus, Ćirić, Reich, 1971] There are α, β, γ with $0 \leq \alpha + \beta + \gamma < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty), \quad \forall x, y \in X.$$

[Bianchini, 1972] There exists a number $k, 0 < k < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leq k \max\{d(x, Tx), d(y, Ty)\}$$

A retrospective approach: Linear case

Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point in X if it satisfies any of the following conditions:

[Seghal , 1972] There exists a number $k, 0 < k < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

[Chatterjea, 1972] There is a $k \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq k \left(d(x, Ty) + d(y, Tx)\right), \quad \forall x, y \in X.$$

[Anonymous] There is α, β, γ with $0 \leq \alpha + \beta + \gamma < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Ty) + \gamma d(y, Tx), \quad \forall x, y \in X.$$

A retrospective approach: Linear case

Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point in X if it satisfies any of the following conditions:

[Hardy-Rogers, 1973] There are non-negative constants $\alpha, \beta, \gamma, \delta, \eta$ with $0 \leq \alpha + \beta + \gamma + \delta + \eta < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + \eta d(y, Tx).$$

[Ćirić, 1974] There exists a number $k, 0 < k < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

$\vdots$

A retrospective approach

Example 4

Let $T : [0,2] \to [0,2]$ be defined by $Tx := \begin{cases} 0 & \text{if } x \in [0,1], \\ 1 & \text{if } x \in (1,2]. \end{cases}$

Note that Banach-Caccioppoli's Theorem fails since T is not continuous at 1. On the other hand, $T^2x = T(Tx) = 0$ for all $x \in [0,2]$ and it forms a contraction. Furthermore, 0 is the only fixed point of T.

A retrospective approach

Example 5

Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx := \cos x$.

For this mapping, the Banach-Caccioppoli's Theorem fails. To indicate it, suppose, on the contrary, that there exists $k \in (0, 1)$ such that

$$|\cos x - \cos y| \leq k|x - y| \Leftrightarrow \frac{|\cos x - \cos x|}{|x - y|} \text{ for all } x \neq y.$$

Letting $y \to x$ we derive that $|\sin x| \leq k$ for all $x \in X$, which is a contradiction.

What is the bright side? Consider, $T^2x = T(Tx) = \cos(\cos x)$ forms a contraction for all $x \in \mathbb{R}$. Indeed,

$$\left| \frac{|\cos(\cos x) - \cos(\cos y)}{|x - y|} \right| = |\sin x \sin(\cos c)| < sin1 < 1,$$

due to Mean-Value Theorem. Furthermore, 0.739 is the only fixed point of T.

A retrospective approach

Example 6

Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx := 1 - x$.

Note that Banach-Caccioppoli's Theorem fails since T is not a contraction (it forms non-expansive mapping.) Further, $T^2x = T(Tx) = x$ for all $x \in \mathbb{R}$ and it, still, does form a contraction (it forms non-expansive mapping.) Notice that each $x \in \mathbb{R}$ forms a fixed point for T^2. but $\frac{1}{2}$ is the unique fixed point of T

Caution: "Non-expansive mapping" admit some additional assumption to possess fixed point!!!

A retrospective approach

Caccioppoli Theorem-1930

Let (X, d) be a complete metric space X and T be a self-mapping on X. For each n, if there is a constant k_n such that

$$d(T^n x, T^n y) \leq k_n d(x, y),$$

for all $x, y \in X$, where $\sum_{n=1}^{\infty} k_n < \infty$, then the equation $Tx = x$ has one and only one solution.

A retrospective approach

Kolmogorov-Fomin Theorem

If T is a continuous mapping of a complete metric space X into itself, such that the mapping T is a contraction for some positive integer n, (that is, $d(T^n x, T^n y) \leq kd(x, y)$.) then the equation $Tx = x$ has one and only one solution.

- A. N. Kolmogorov, and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis, Volume I, Metric and Normed Spaces, Graylock Press, Rochester, New York, 1957.
- S. C. Chu and J. B. Diaz, Remarks on a Generalization of Banach's Principle of Contraction Mappings, Journal of Mathematical Analysis and Applications 11(1965) 440–446.
- V. W. Bryant, A remark on a fixed point theorem for iterated mappings, Amer. Math. Monthly 75 (1968), 399-400.

A retrospective approach

Sehgal Theorem

On a complete metric space (X, d), a continuous mapping $T : X \to X$ possesses a unique fixed point provided that there exits $k \in [0, 1)$ such that for each $x \in X$ there exists a positive integer $n(x)$ such that for each $y \in X$

$$d(T^{n(x)}x, T^{n(x)}y) \leq kd(x, y).$$

- V.M. Sehgal, A fixed point theorem for mappings with a contractive iterate, Proc. Amer. Soc., 23(1969), 631–634.
- Guseman, Jr.L.F. Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc. 26 (1970), 615-618.

A retrospective approach: Nonlinear case

Rakotch Theorem, 1962

Let (X, d) be a complete metric space and $T : X \to X$ a map. If there exists a monotone decreasing function $\alpha : (0, \infty) \to [0, 1)$ such that, for each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) \leq \alpha(d(x, y)).$$

Then T has a fixed point in X

A retrospective approach: Nonlinear case

Boyd and Wong, 1969

Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point in X if there exists an upper semi-continuous function $\phi : [0, \infty) \to [0, \infty)$ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \text{ for all } x, y \in X$$

A retrospective approach: Nonlinear case

Browder, 1968

Let (X, d) be a complete metric space and $T : M \to M$ a self-mapping where M is a bounded subset of X. Then T has a fixed point in X if there exists a monotone non-decreasing and continuous from the right function $\phi : [0, \infty) \to [0, \infty)$ with $\psi(t) < t$ for all $t > 0$ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \text{ for all } x, y \in M.$$

Matkowski, 1975

Let (X, d) be a complete metric space and $T : X \to X$ a self-mapping where M is a bounded subset of X. Then T has a fixed point in X if there exists a monotone non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ and satisfying $\lim_{n\to\infty} \psi^n(t) = 0$ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \text{ for all } x, y \in X$$

A retrospective approach: Nonlinear case

Meir and Keeler,1969

Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point in X if it satisfies any of the following conditions:

$$\forall \varepsilon > 0,\ \exists \delta > 0 :\ \varepsilon \leq d(x, y) < \varepsilon + \delta \text{ whenever } d(Tx, Ty) < \varepsilon,\ \forall x, y \in X.$$

A retrospective approach: Nonlinear case

Geraghty, 1973

Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point in X if there exists a mapping $\beta : (0, \infty) \to [0, 1)$ satisfying the condition, $\beta(t_n) \to 1$ implies $t_n \to 0$, such that

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \text{ for any } x, y \in X.$$

A retrospective approach: Nonlinear case

Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point in X if it satisfies any of the following conditions:

(15) [Ya. I.Alber & S. Guerre-Delabriere 1997, Rhoades, 2000] There exists a lower semi-continuous function $\phi : [0, \infty) \to [0, \infty)$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \text{ for all } x, y \in X.$$

(weak-ϕ-contractive).

(16) [Dutta and Choudhury, 2008] There exist alternating distance functions $\phi, \psi : [0, \infty) \to [0, \infty)$ such that

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \text{ for all } x, y \in X.$$

(weak $\psi - \phi$-contractive.)

Note that $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if $\psi(t)$ is continuous and nondecreasing and $\psi(t) = 0$ if and only if $t = 0$.

⋮

(∞) ...

A retrospective approach: Nonlinear case

Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point in X if it satisfies any of the following conditions:

[Das & Gupta, 1975] There exist non-negative constants k_1, k_2 with $0 < k_1 + k_2 < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leq k_1 \frac{[1 + d(x, Tx)]d(y, Ty)}{1 + d(x, y)} + k_2 d(x, y)$$

[Jaggi, 1977] There exist non-negative constants k_1, k_2 with $0 < k_1 + k_2 < 1$, such that, for each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) \leq k_1 \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + k_2 d(x, y)$$

1. J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces, Nonlinear Anal. 74 (3) (2011) 768-774.
2. J. Jachymski, I. Jozwik, Nonlinear contractive conditions: a comparison and related problems, fixed point theory and its applications Polish Acad. Sci., 77 (2007), 123146.
3. J. Jachymski, Equivalence of some contractivity properties over metrical structures, Proceedings of the American Mathematical Society, Volume 125, Number 8, August 1997, Pages 2327-2335

Theorem

Let T be a self-map of an ordered partial metric space $(X, d, \preceq)$. The following statements are equivalent:

(i) *there exist functions $\psi, \eta \in \Im$ such that*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \eta(d(x, y)) \text{ for any } x, y \in X \text{ with } x \preceq y,$$

(ii) *there exists a function $\beta : [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$d(Tx, Ty) \leq \beta(d(x, y))\psi(d(x, y)) \text{ for any } x, y \in X \text{ with } x \preceq y,$$

(iii) *there exists a continuous function $\eta : [0, \infty) \to [0, \infty)$ such that $\eta^{-1}(\{0\}) = 0$ and*

$$d(Tx, Ty) \leq d(x, y) - \eta(d(x, y)) \text{ for any } x, y \in X \text{ with } x \preceq y,$$

(iv) *there exists function $\psi \in \Im$ and a non-decreasing, right continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and for all $t > 0$ with*

$$\psi(d(Tx, Ty)) \leq \varphi(\psi(d(x, y))) \text{ for any } x, y \in X \text{ with } x \preceq y,$$

(v) *there exists a continuous and non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ and for all $t > 0$ with*

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)) \text{ for any } x, y \in X \text{ with } x \preceq y.$$

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A COMPARISON OF VARIOUS DEFINITIONS OF CONTRACTIVE MAPPINGS

BY

B. E. RHOADES

ABSTRACT. A number of authors have defined contractive type mappings on a complete metric space X which are generalizations of the well-known Banach contraction, and which have the property that each such mapping has a unique fixed point. The fixed point can always be found by using Picard iteration, beginning with some initial choice $x_0 \in X$. In this paper we compare this multitude of definitions.

X denotes a complete metric space with distance function d, and f a function mapping X into itself.

1. Definitions of contractive type mappings.

(1) (Banach) There exists a number a, $0 \leqslant a < 1$, such that, for each $x, y \in X$,

$$d(f(x),f(y)) \leqslant a d(x,y).$$

(2) (Rakotch [21]) There exists a monotone decreasing function α: $(0,\infty) \to [0,1)$ such that, for each $x, y \in X$, $x \neq y$,

$$d(f(x),f(y)) \leqslant \alpha d(x,y).$$

(3) (Edelstein [10]) For each $x, y \in X$, $x \neq y$,

$$d(f(x),f(y)) < d(x,y).$$

(4) (Kannan [18]) There exists a number a, $0 < a < \frac{1}{2}$, such that, for each $x, y \in X$,

$$d(f(x),f(y)) \leqslant a[d(x,f(x)) + d(y,f(y))].$$

(5) (Bianchini [3]) There exists a number h, $0 \leqslant h < 1$, such that, for each $x, y \in X$,

$$d(f(x),f(y)) \leqslant h \max\{d(x,f(x)), d(y,f(y))\}.$$

(6) For each $x, y \in X$, $x \neq y$,

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Theorems and Counterexamples on Contractive Mappings

J. Kincses, V. Totik

Presented by Bl. Sendov

Theorems and counterexamples are proved about fixed points of contractive mappings in complete metric spaces. These cover all known results of this kind known to us and solve many open problems.

1. Introduction

In the last 20 years a relatively wide study of contractive type mappings was carried out by different authors (a good summary and detailed reference can be found in [1]). Thus, it seems to be worthwhile examining the subject systematically, which we do in the present paper.

One of the most comprehensible articles about contractive mappings is [1] which contains several results as well as open problems for further study (although we have to remark that [1] contains some mistakes: Theorems 9 and 15 are not true — see Exaple 2 below). We shall keep the definitions of [1] and prove 7 theorems and present 7 counterexamples which cover most of the possible cases (see the "truth table" below). Some spacial cases of our theorems coincide with earlier results, but we do not give exact references, only refer to the summary in [1].

The paper is organized as follows.

Section 2 contains the definitions, section 3 the "truth table" and some comments on it, finally in sections 4 and 5 we prove our theorems and give our counterexamples.

2. Definitions

In what follows (X, d) will always be a complete metric space and $T: X \to X$ a mapping of X into itself. Recall that $x \in X$ is said to be a fixed point of T if $Tx = x$.

Following B. E. Rhoades [1] we define the following contractive type mappings.

(1) (Banach) There exists a number a, $0 \leqq a < 1$ such that, for each $x, y \in X$

Structure

Let $X \neq \emptyset$ and $T : X \to X$ be a self-mapping. A point $x_0 \in X$ is said to be a fixed point of T if $Tx_0 = x_0$. In the metric fixed point theory researchers try to find the answers of the following questions:

Let $X \neq \emptyset$ and $T : X \to X$ be a self-mapping. A point $x_0 \in X$ is said to be a fixed point of T if $Tx_0 = x_0$. In the metric fixed point theory researchers try to find the answers of the following questions:

(1) (Existence) What is(are) the condition(s) on $T : X \to X$ and / or conditions on the structure on X such that there exists at least one point $x_0 \in X$ such that $Tx_0 = x_0$.

(2) (Uniqueness) Is $x_0 \in X$ unique?

(3) (Non-self Mapping) What is(are) the condition(s) for $T : A \to B$ such that $A, B \subset X$ and / or conditions on the structure on X such that there exists at least one point $x_0 \in X$ such that $Tx_0 = x_0$.

(4) What is(are) the condition(s) for $S, T : X \to X$ and / or conditions on the structure on X such that there exists at least one point $x_0 \in X$ such that $Tx_0 = Sx_0$ or $Tx_0 = x_0 = Sx_0$.

$\vdots$

(∞)

Directions!

- By changing the frame (structure of the abstract space)

- By changing the property of the operators (contractions)

- Coincidence Point and Common Fixed Points (for 2 or more operators)

Directions!

- By changing the frame (structure of the abstract space)
  - Cone Metric Spaces
  - Partial Metric Spaces
  - G-Metric Spaces
  - b-Metric Spaces
  - and so on...
- By changing the property of the operators (contractions)
  - Cyclic Contractions
  - $T : X^n \to X$ and finding n-tuple fixed point
    "n=2, Coupled, n=3, Tripled, n=4, Quadruple" fixed points
  - Multivalued Operators, and so on...
- Coincidence Point and Common Fixed Points (for 2 or more operators)
  - Commuting Operators
  - Weakly / Occasionally Weakly - Compatible operators
  - and so on...

Structure of some basic abstract spaces

Let $X \neq \emptyset$ and let $d : X \times X \to [0, \infty)$. For every $x, y, z \in X$, we have

(d_1) $d(x,x) = 0$;

(d_2) $d(x,y) = d(y,x) = 0 \Longrightarrow x = y$;

(d_3) $d(x,y) = d(y,x)$;

(d_4) $d(x,z) \leq d(x,y) + d(y,z)$;

(d_{4a}) $d(x,z) \leq d(x,y) + d(y,z) - d(z,z)$;

(d_{4b}) $d(x,z) \leq d(x,y) + d(y,w) + d(w,z)$ distinct y, w;

(d_{4c}) $d(x,z) \leq s[d(x,y) + d(y,z)]$ for some $s \geq 1$;

(d_{4d}) $d(x,z) \leq d(x,y) + d(y,z)$;

(d_5) $d(x,x) \leq d(x,y)$.

Metric	$(d_1) - (d_4)$	*Bianciari − distance*	$(d_1) - (d_3), (d_{4b})$
Dislocated − metric	$(d_2) - (d_4)$	*Quasi − metric*	$(d_1), (d_2), (d_4)$
Semi − metric	$(d_1), (d_2), (d_3)$	*partialmetric*	$(d_2), (d_3), (d_{4a})$ and (d_5)
b − metric	$(d_1) - (d_3), (d_{4c})$	*ultra − metric*	$(d_1) - (d_3), (d_{4d})$

Structure

2-metric

During the sixties the notation of 2-metric spaces was introduced by Gahler:

Definition: Let X be a nonempty set. A function $d : X \times X \times X \to \mathbf{R}^+$, satisfying the following properties:

(d1) For distinct $x, y \in X$, there exists $z \in X$ such that $d(x, y, z) \neq 0$.

(d2) $d(x, y, z) = 0$ if two of the triple $x, y, z \in X$ are equal.

(d3) $d(x, y, z) = d(x, z, y) = \cdots$ (symmetry in all three variables),

(d4) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$,
for all $x, y, z, a \in X$ (rectangle inequality),

is called a 2- *metric* on X. The set X together with such a 2-metric, d, is called a 2-*metric space* and denoted by (X, d).

S. Gahler, *2-metriche raume und ihre topologische strukture,* Math. Nachr. **26**, (1963), 115-148.

S. Gahler,*Zur geometric 2-metriche raume,* Reevue Roumaine de Math.Pures et Appl., **XI**, 1966, 664-669.

2-metric

Gahler claimed that a 2-metric is a generalization of the usual notion of a metric, but it is not.

It was shown that 2-metric is not continuous function whereas an ordinary metric is.

Gahler mentioned that geometrically $d(x, y, z)$ represents the area of a triangle formed by the points x, y, and z in X, but not necessarily.

K. S. Ha, Y. J. Cho, S. S. Kim, M. S. Khan *Strictly convex and 2-convex 2-normed spaces*, Math. Japonica, **33(3)**, (1988), 375–384.

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D-metric

In 1992 Dhage attempted to develop 2-metric by introducing a new concept of generalized metric:

Definition

Let X be a nonempty set, a function $D : X \times X \times X \to \mathbf{R}^+$ satisfying the following axioms:

(D1) $D(x, y, z) \geq 0$ for all $x, y, z \in X$,

(D2) $D(x, y, z) = 0$ if and only if $x = y = z$,

(D3) $D(x, y, z) = D(x, z, y) = \cdots$ (symmetry in all three variables),

(D4) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$, for all $x, y, z, a \in X$. (rectangle inequality),

is called a *generalized metric*, or a *D-metric* on X. The set X together with such a generalized metric, D, is called a *generalized metric space*, or *D- metric space*, and denoted by (X, D).
It is called *symmetric* if $D(x, x, y) = D(x, y, y)$ for all $x, y \in X$.

D-metric

1. B.C. Dhage, *Generalized Metric Space and Mapping With Fixed Point*, Bull. Cal. Math. Soc. **84**, (1992), 329–336.
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Intuitively $D(x, y, z)$ may be thought of as providing some measure of the perimeter of the triangle with vertices at x, y and z. If $\rho : X \times X \to [0, \infty)$ is any semi-metric on X (that is, ρ is a positive, symmetric function with $\rho(x, y) = 0$ if and only if $x = y$) then it is easily verified that D defined by either,

1. $D(x, y, z) := \rho(x, y) + \rho(x, z) + \rho(y, z)$, or
2. $D(x, y, z) := \max\{\rho(x, y), \rho(x, z), \rho(y, z)\}$,

D-metric

In particular it should be noted that the rectangle inequality satisfied by D–metrics arising in this way from a metric does not depend on the presence of a triangle inequality for the underlying metric, and that the inequality is only sharp in degenerate cases.

In a D–metric space (X, D), three possible notions for the convergence of a sequence (x_n) to a point x present themselves:

(C1) $x_n \to x$ if $D(x_n, x, x) \to 0$ as $n \to \infty$,

(C2) $x_n \to x$ if $D(x_n, x_n, x) \to 0$ as $n \to \infty$,

(C3) $x_n \to x$ if $D(x_m, x_n, x) \to 0$ as $m, n \to \infty$,

Clearly, $(C3) \Rightarrow (C2)$ and if D is symmetric then $(C1) \Leftrightarrow (C2)$.

D-metric

In generalized metric spaces, the proofs of most fixed point results claimed by Dhage and others relied, either directly or indirectly,

-on the continuity of D with respect to convergence in the sense of (C3), or
- with respect to the convergence in the sense of (C2).

However, there are counter examples which were given by Mustafa and Sims.

G-metric

In 2003, Mustafa and Sims attempted to remove the weakness of the D-metric, by introducing G-metrics.

Definition

A G-metric space is a pair (X, G), where X is a nonempty set, and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$ we have:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (symmetry in all three variables); and

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called a *G-metric* on X.

G-metric

Let (X, G) be a G-metric space, let $\{x_n\} \subseteq X$ be a sequence and let $x \in X$. Then the following conditions are equivalent.

(a) $\{x_n\}$ G-converges to x.

(b) $\lim_{n\to\infty} G(x_n, x_n, x) = 0 \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : x_n \in B_G(x, \varepsilon)\, \forall n \geq n_0$.

(c) $\lim_{n\to\infty} G(x_n, x, x) = 0$.

(d) $\lim_{n,m\to\infty,\ m\geq n} G(x_n, x_m, x) = 0$.

(e) $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.

(f) $\lim_{n\to\infty} G(x_n, x, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.

(g) $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ and $\lim_{n\to\infty} G(x_n, x_{n+1}, x) = 0$.

(h) $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$ & $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0$.

(i) $\lim_{n,m\to\infty,\ m>n} G(x_n, x_m, x) = 0$.

G-metric

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X. \tag{1}$$

G-metric

Every G-metric on X defines a metric d_G on X by

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \text{ for all } x,y \in X. \tag{1}$$

Example

Let (X,d) be a metric space. The functions $G_m(d), G_s(d) : X \times X \times X \to [0,+\infty)$, defined by

$$G_m(d)(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}, \tag{2}$$

G-metric

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Example

Let (X, d) be a metric space. The functions $G_m(d), G_s(d) : X \times X \times X \to [0, +\infty)$, defined by

$$G_m(d)(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \tag{2}$$

$$G_s(d)(x, y, z) = \frac{1}{3}[d(x, y) + d(y, z) + d(z, x)], \tag{3}$$

for all $x, y, z \in X$, are standard G-metrics on X.

G-metric

Definition

Let (X, G) be a G-metric space, and let $x_0 \in X$, given $\varepsilon > 0$, define the sets

$$B_G(x_0, \varepsilon) := \{y \in X; G(x_0, y, y) < \varepsilon\}$$

and

$$\overline{B}_G(x_0, \varepsilon) := \{y \in X, G(x_0, y, y) \leq \varepsilon\}$$

Then, $B_G(x_0, \varepsilon)$ and $\overline{B}_G(x_0, \varepsilon)$ are called the *open* and *closed balls*, with centers x_0 and radius ϵ, respectively.

Each $G-$metric G on X generates a topology τ_G on X whose base is a family of open $G-$balls $\{B_G(x, \varepsilon) : x \in X, \varepsilon > 0\}$. A nonempty set A in the $G-$metric space (X, G) is $G-$closed if $\overline{A} = A$. Moreover,

$$x \in \overline{A} \Leftrightarrow B_G(x, \varepsilon) \cap A \neq \emptyset, \quad \text{for all } \varepsilon > 0.$$

Definition

A sequence (x_n) in a G-metric space X is said to converge if there exists $x \in X$ such that $\lim_{n,m \to \infty} G(x, x_n, x_m) = 0$, and one say that the sequence (x_n) is *G-convergent* to x. We call x the limit of the sequence $\{x_n\}$ and write $x_n \to x$ or $\lim_{n \to \infty} x_n = x$.

Theorem

Every G-convergent sequence in a G-metric space (X, G) has a unique limit.

Definition

In a G-metric space X, a sequence (x_n) is said to be G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$, that is $G(x_n, x_m, x_l) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$.

Definition

A G-metric space X is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Analog of Banach Contraction (Mapping) Principle

Theorem

Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \tag{4}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Connection of G-metric and quasi-metric.

Theorem

Let (X, G) be a G-metric space. Let $d : X \times X \to [0, \infty)$ be the function defined by $d(x, y) = G(x, y, y)$. Then

1. (X, d) *is a quasi-metric space;*
2. $\{x_n\} \subset X$ *is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d);*
3. $\{x_n\} \subset X$ *is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, d);*
4. (X, G) *is G-complete if and only if (X, d) is complete.*

Connection of G-metric and metric.

Fact:

Every quasi-metric induces a metric, that is, if (X, d) is a quasi-metric space, then the function $\delta : X \times X \to [0, \infty)$ defined by $\delta(x, y) = max\{d(x, y), d(y, x)\}$ is a metric on X.

Connection of G-metric and metric.

Fact:

Every quasi-metric induces a metric, that is, if (X, d) is a quasi-metric space, then the function $\delta : X \times X \to [0, \infty)$ defined by $\delta(x, y) = max\{d(x, y), d(y, x)\}$ is a metric on X.

As an immediate consequence of this definition with the Theorem above:

Connection of G-metric and metric.

Fact:

Every quasi-metric induces a metric, that is, if (X, d) is a quasi-metric space, then the function $\delta : X \times X \to [0, \infty)$ defined by $\delta(x, y) = max\{d(x, y), d(y, x)\}$ is a metric on X.

As an immediate consequence of this definition with the Theorem above:

Theorem

Let (X, G) be a G-metric space. Let $d : X \times X \to [0, \infty)$ be the function defined by $d(x, y) = G(x, y, y)$. Then

1. *(X, d) is a quasi-metric space;*
2. *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, δ);*
3. *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, δ);*
4. *(X, G) is G-complete if and only if (X, δ) is complete.*

S-metric

Another such generalization was given by S.Sedghi, N. Shobe and A. Aliouche in 2012 as follows:

Definition

Let X be a nonempty set, a function $S : X \times X \times X \to \mathbf{R}^+$ satisfying the following axioms:

(S1) $S(x, y, z) = 0$ if and only if $x = y = z$,

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$, for all $x, y, z, a \in X$. (rectangle inequality),

is called a *S-metric* on X. The pair (X, S) is called as *S- metric space*.

S-metric

Theorem

Let (X, S) be a S-metric space. Let $d : X \times X \to [0, \infty)$ be the function defined by $d(x, y) = S(x, x, y)$. Then

1. (X, d) *is a b-metric space;*
2. $\{x_n\} \subset X$ *is S-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d);*
3. $\{x_n\} \subset X$ *is S-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, d);*

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17. . . .

Cone metric space

Let E be a real Banach space. A subset P of E is called a cone, if and only if, the following hold:

(a_1) P is closed, nonempty, and $P \neq \{0\}$,

(a_2) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply that $ax + by \in P$,

(a_3) $x \in P$ and $-x \in P$ imply that $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$, if and only if, $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal, if there exist a number $K > 1$ such that, $0 \leq x \leq y$ implies $||x|| \leq K||y||$, for all $x, y \in E$.

The least positive number satisfying this, called the normal constant.

Cone metric space

Throughout this talk, E denotes a real Banach space, P denotes a cone in E with $intP \neq \emptyset$, and $\leq$ denotes partial ordering with respect to P.

Definition

Let X be a nonempty set. A function $d : X \times X \to E$ is called a cone metric on X, if it satisfies the following conditions:

(b_1) $d(x,y) \geq 0$, $\forall x,y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$,

(b_2) $d(x,y) = d(y,x)$, for all $x,y \in X$,

(b_3) $d(x,y) \leq d(x,z) + d(y,z)$, for all $x,y,z \in X$.

Then, (X,d) is called a cone metric space.

Definition

Let (X,d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. If for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_0) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent and $\{x_n\}_{n\in\mathbb{N}}$ converges to x and x is the limit of $\{x_n\}_{n\in\mathbb{N}}$.

Cone metric space

Definition

Let (X, d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence in X.

Definition

Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Definition

Let (X, d) be a cone metric space. A self-map T on X is said to be continuous, if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} T(x_n) = T(x)$ for all sequence $\{x_n\}_{n\in\mathbb{N}}$ in X.

Cone metric space

Lemma

Let (X, d) be a cone metric space and P be a cone. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then, $\{x_n\}_{n\in\mathbb{N}}$ converges to x, if and only if,

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{5}$$

Lemma

Let (X, d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then it is a Cauchy sequence.

Lemma

Let (X, d) be a cone metric space and P be a cone in E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if and only if, $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

1. W.-S. Du, A note on cone metric fixed point theory and its equivalence Nonlinear Analysis 72 (2010), no : 5, 2259-2261.
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Complex valued metric spaces

The concept of complex valued metric space which is given by Azam et al.[1]
Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ \textit{if and only if} } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2)$$

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

$$\begin{array}{lll} (h_1) & Re(z_1) = Re(z_2); & Im(z_1) < Im(z_2) \\ (h_2) & Re(z_1) < Re(z_2); & Im(z_1) = Im(z_2) \\ (h_3) & Re(z_1) < Re(z_2); & Im(z_1) < Im(z_2) \\ (h_4) & Re(z_1) = Re(z_2); & Im(z_1) = Im(z_2) \end{array}$$

[1] A. Azam, B. Fisher, M. Khan, Common Fixed point theorems in complex valued metric spaces.Numerical Functional Analysis and Optimization, **32**, 243–253 (2011

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(h_1), (h_2)$ and (h_3) is satisfied and we will write $z_1 \prec z_2$ if only (h_3) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|$$

where $|\cdot|$ represent modulus or magnitude of z, and

$$z_1 \precsim z_2, \; z_2 \prec z_3 \implies z_1 \prec z_3$$

Definition

Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X, if it satisfies the following conditions:

- (b_1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$, if and only if, $x = y$,
- (b_2) $d(x, y) = d(y, x)$, for all $x, y \in X$,
- (b_3) $d(x, y) \precsim d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Here, the pair (X, d) is called a complex valued metric space.

Complex valued metric spaces

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.

Lemma

Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma

Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

Let $(X, d_{\mathbb{C}})$ be a complex-valued metric space where $\mathbb{C}$ is the skew field of complex number z, i.e.,

$$\mathbb{C} = \{x + yi : (x, y) \in \mathbb{R}^2\}.$$

Define

$$\mathcal{P}_{\mathbb{C}} = \{x + yi : x \geq 0, y \geq 0\}.$$

It is apparent that $\mathcal{P}_{\mathbb{C}} \subset \mathbb{C}$. Assume $\mathbf{0}_{\mathbb{C}}$ be the zero of $\mathbb{C}$ from now on. Note that $(\mathbb{C}, |.|)$ is a real Banach space.

Lemma

$\mathcal{P}_{\mathbb{C}}$ is a normal cone in real Banach space $(\mathbb{C}, |.|)$.

Lemma

Any complex-valued metric space $(X, d_{\mathbb{C}})$ is a cone metric space.

Quaternion-valued metric space: Basics

The skew field of quaternion denoted by $\mathbb{H}$ means to write each element $q \in \mathbb{H}$ in the form

$$q = x_0 + x_1 i + x_2 j + x_3 k,$$

$x_n \in \mathbb{R}$; where $1, i, j, k$ are the basis elements of $\mathbb{H}$ and $n = 1, 2, 3$. For these elements we have the multiplication rules

$$\begin{aligned} & i^2 = j^2 = k^2 = -1, \\ & ij = -ji = k, \\ & kj = -jk = -i, \\ & ki = -ik = j. \end{aligned}$$

The conjugate element $\bar{q}$ is given by

$$\bar{q} = x_0 - x_1 i - x_2 j - x_3 k.$$

The quaternion modulus has the form of

$$|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

A quaternion q may be viewed as a four dimensional vector (x_0, x_1, x_2, x_3).

Quaternion-valued metric space: Partial order

Define a partial order $\precsim$ on $\mathbb{H}$ as follows:

$$q_1 \precsim q_2 \iff \begin{cases} Re(q1) \leq Re(q2) \\ Im_s(q_1) \leq Im_s(q_2), \;\; q_1, q_2 \in \mathbb{H}, s = i, j, k. \end{cases}$$

where $Im_i = x_1, Im_j = x_2$ and $Im_k = x_3$.

Quaternion-valued metric space

It follows that $q_1 \lesssim q_2$ if one of the following conditions hold:

$$
\begin{array}{llll}
(I) & Re(q_1) = Re(q_2); & Im_{s_1}(q_1) = Im_{s_1}(q_2) & \text{where } s_1 = j, k;\ Im_i(q_1) < Im_i(q_2) \\
(II) & Re(q_1) = Re(q_2); & Im_{s_2}(q_1) = Im_{s_2}(q_2) & \text{where } s_2 = i, k;\ Im_j(q_1) < Im_j(q_2) \\
(III) & Re(q_1) = Re(q_2); & Im_{s_3}(q_1) = Im_{s_3}(q_2) & \text{where } s_3 = i, j;\ Im_k(q_1) < Im_k(q_2) \\
(IV) & Re(q_1) = Re(q_2); & Im_{s_1}(q_1) < Im_{s_1}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(V) & Re(q_1) = Re(q_2); & Im_{s_2}(q_1) < Im_{s_2}(q_2); & Im_j(q_1) = Im_j(q_2) \\
(VI) & Re(q_1) = Re(q_2); & Im_{s_3}(q_1) < Im_{s_3}(q_2); & Im_k(q_1) = Im_k(q_2) \\
(VII) & Re(q_1) = Re(q_2); & Im_s(q_1) < Im_s(q_2) & \\
(VIII) & Re(q_1) < Re(q_2); & Im_s(q_1) = Im_s(q_2) & \\
(IX) & Re(q_1) < Re(q_2); & Im_{s_1}(q_1) = Im_{s_1}(q_2); & Im_i(q_1) < Im_i(q_2) \\
(X) & Re(q_1) < Re(q_2); & Im_{s_2}(q_1) = Im_{s_2}(q_2); & Im_j(q_1) < Im_j(q_2) \\
(XI) & Re(q_1) < Re(q_2); & Im_{s_3}(q_1) = Im_{s_3}(q_2); & Im_k(q_1) < Im_k(q_2) \\
(XII) & Re(q_1) < Re(q_2); & Im_{s_1}(q_1) < Im_{s_1}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XIII) & Re(q_1) < Re(q_2); & Im_{s_2}(q_1) < Im_{s_2}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XIV) & Re(q_1) < Re(q_2); & Im_{s_3}(q_1) < Im_{s_3}(q_2); & Im_i(q_1) = Im_i(q_2) \\
(XV) & Re(q_1) < Re(q_2); & Im_s(q_1) < Im_s(q_2) & \\
(XVI) & Re(q_1) = Re(q_2); & Im_s(q_1) = Im_s(q_2) &
\end{array}
$$

remark

In particular, we write $q_1 \lnsim q_2$ if $q_1 \neq q_2$ and one from (I) to (XVI) is satisfied. Also, we will write $q_1 < q_2$ if only (XV) is satisfied. It should be remarked that

$$q_1 \lesssim q_2 \;\Rightarrow\; |q_1| \leq |q_2|.$$

Ahmed *et al.* [2], introduced the definition of the quaternion valued metric space as follows:

Definition

Let X be a nonempty set. A function $d_{\mathbb{H}} : X \times X \to \mathbb{H}$ is called a quaternion valued metric on X, if it satisfies the following conditions:

- (d_1) $0 \precsim d_{\mathbb{H}}(x, y)$ for all $x, y \in X$ and $d_{\mathbb{H}}(x, y) = 0$, if and only if, $x = y$,
- (d_2) $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(y, x)$, for all $x, y \in X$,
- (d_3) $d_{\mathbb{H}}(x, y) \precsim d_{\mathbb{H}}(x, z) + d_{\mathbb{H}}(y, z)$, for all $x, y, z \in X$.

Then, $(X, d_{\mathbb{H}})$ is called a quaternion valued metric space.

[2] A. E-S. Ahmed, A. Asad, S. Omran, Fixed point theorems in quaternion-valued metric spaces, Abstract and Applied Analysis,(2014) Article Id. 258985

Quaternion-valued metric space

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \lesssim c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_{\mathbb{H}}(x_n, x) \lesssim c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \lesssim c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_{\mathbb{H}}(x_n, x_{n+m}) \lesssim c$, then $\{x_n\}$ is called a Cauchy sequence in $(X, d_{\mathbb{H}})$. If every Cauchy sequence is convergent in $(X, d_{\mathbb{H}})$, then $(X, d_{\mathbb{H}})$ is called a complete quaternion valued metric space.

Lemma

Let (X, d) be a quaternion valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma

Let $(X, d_{\mathbb{H}})$ be a quaternion valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d_{\mathbb{H}}(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

Quaternion-valued metric space

Let $(X, d_{\mathbb{H}})$ be a quaternion-valued metric space where $\mathbb{H}$ is the skew field of quaternion number q, i.e.,

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k : (x_0, x_1, x_2, x_3) \in \mathbb{R}^4\}.$$

Define

$$\mathcal{P}_{\mathbb{H}} = \{x_0 + x_1 i + x_2 j + x_3 k : x_0 \geq 0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}.$$

It is apparent that $\mathcal{P}_{\mathbb{H}} \subset \mathbb{H}$. Assume $\mathbf{0}_{\mathbb{H}}$ be the zero of $\mathbb{H}$ from now on. Note that $(\mathbb{H}, |.|)$ is a real Banach space.

Lemma

$\mathcal{P}_{\mathbb{H}}$ is a normal cone in real Banach space $(\mathbb{H}, |.|)$.

Quaternion-valued metric space

Lemma

Any quaternion valued metric space $(X, d_{\mathbb{H}})$ *is a cone metric space.*

Lemma

A sequence $\{x_n\}$ *in* $(X, d_{\mathbb{H}})$ *be convergent in the context of quaternion valued metric space if and only if* $\{x_n\}$ *be convergent in the setting of of cone metric space.*

1. E. Karapinar and W.-S. Du, A note on b-cone metric and its related results: Generalizations or equivalence? , Fixed Point Theory and Applications, (2013), 2013:210
2. R. P. Agarwal, H. Alsulami, E.Karapinar and F.Khojasteh, Remarks on some recent fixed point results in quaternion-valued metric spaces, Abstract and Applied Analysis, (2014) Article Id: 171624
3. H. Alsulami, S. Almezel, E. Karapinar, F. Khojasteh, A note on fixed point results in complex valued metric spaces, Journal of Inequalities and Applications, 2015, 2015:33
4. E. Karapinar , Recent Advances on Metric Fixed Point Theory: A Review, Applied and Computational Mathematics an International Journal, Vol 22. issue 1, 2023, pages: 3-30,

Multiplicative Metric[3]

Let X be a nonempty set. An operator $d*: X \times X \to [1, \infty)$ is a multiplicative metric (MM for short) on X, if it satisfies:

(m_1*) $d*(x,y) = 1 \geq 1 \forall x, y \in X$, and $d*(x,y) = 1$ if and only if $x = y$,

(m_2*) $d*(x,y) = d*(y,x)$ for all $x, y \in X$,

(m_3*) $d*(x,z) \leq d*(x,y)d*(y,z)$ for all $x, y, z \in X$,(multiplicative triangle inequality).

If the operator d*d* satisfies $(m_1*) - (m_3*)$ then the pair $(X, d*)$ is called a multiplicative metric space (MMS).

[3] A.E. Bashirov, E.M. Kurpınar and A. Ozyapici, Multiplicative calculus and its applications J. Math. Anal. Appl.,337 (2008), 36-48.

multiplicative metric...

$$\ln(\max\{a, b\}) = \max\{\ln a, \ln b\}$$

for all $a, b > 0a, b > 0$ as well as

$$e^{\max\{a,b\}} = \max\{e^a, e^b\}$$

for all $a, b \in \mathbb{R}$.

Theorem

$(X, d*)$ is an MMS if and only if $(X, \ln d*)$ is an S-MS, that is, (X, d) is an S-MS if and only if (X, e^d) is an MMS. [a]

[a] T.Dosenovic, M. Postolache and S. Radenovic, On multiplicative metric spaces: survey, Fixed Point Theory and Applications20162016:92

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Disadvantages...

1. Overlapping...
2. Competition
3. Results without motivation!
4. ...

Structure

Coupled Fixed Points

In 1987, Guo and Lakshmikantham introduced the notion of the coupled fixed point.

Definition

An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$x = F(x, y) \text{ and } y = F(y, x).$$

The concept of coupled fixed point was reconsidered by Gnana-Bhaskar and Lakshmikantham in 2006. In this paper, they proved the existence and uniqueness of a coupled fixed point of an operator $F : X \times X \to X$ on a partially ordered metric space under a condition called mixed monotone property.

Definition

Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y_2)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Gnana-Bhaskar and Lakshmikantham Theorem

Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a given mapping. Suppose that the following conditions hold:

(i) F has the mixed monotone property;

(ii) either F is continuous or X has the following properties:

- (X_1) if a nondecreasing sequence $\{x_n\}$ in X converges to a some point $x \in X$, then $x_n \preceq x$ for all n,
- (X_2) if a decreasing sequence $\{y_n\}$ in X converges to a some point $y \in X$, then $y_n \succeq y$ for all n;

(ii) there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$;

(iv) there exists a constant $k \in (0, 1)$ such that for all $(x, y), (u, v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)].$$

Then F has a coupled fixed point $(x^*, y^*) \in X \times X$.

Example to show the weakness of Gnana-Bhaskar and Lakshmikantham Theorem

Let $X = \mathbb{R}$ and $d : X \times X \to [0, \infty)$ be the Euclidean metric. Define a mapping $F : X \times X \to X$ as

$$F(x, y) = \frac{3x - y}{5} \quad \text{for all } x, y \in X.$$

It is clear that F is mixed monotone but we claim that it does not satisfy condition (iv). Indeed, assume that there exists k with $0 \leq k < 1$ such that

$$d(F(x, y), F(u, v)) \leq k \frac{d(x, u) + d(y, v)}{2} \tag{6}$$

holds for all $x \geq u$ and $v \geq y$. Let us take $x \neq u$, $y = v$ in the previous inequality. Hence, $t = |x - u| > 0$ and the inequality (6) turns into

$$\frac{3t}{5} = \frac{3|x - u|}{5} = d(F(x, y), F(u, v)) \leq \frac{|x - u|}{2} = k\frac{t}{2}. \tag{7}$$

Example

Recall that $0 \leq k < 1$ for any $t > 0$. Hence, the inequality (6) turns into

$$\frac{3t}{5} \leq k\frac{t}{2},$$

which is a contradiction. Hence, Gnana-Bhaskar and Lakshmikantham Theorem is not applicable for the operator F in order to prove that $(0,0)$ is the unique coupled fixed point of F.

Corrected version of Gnana-Bhaskar and Lakshmikantham Theorem

Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a given mapping. Suppose that the following conditions hold:

(i) F has the mixed monotone property;

(ii) either F is continuous or X has the following properties:

- (X_1) if a nondecreasing sequence $\{x_n\}$ in X converges to a some point $x \in X$, then $x_n \preceq x$ for all n,
- (X_2) if a decreasing sequence $\{y_n\}$ in X converges to a some point $y \in X$, then $y_n \succeq y$ for all n;

(ii) there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$;

(iv) there exists a constant $k \in (0, 1)$ such that for all $(x, y), (u, v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(x, u) + d(y, v)].$$

Then F has a coupled fixed point $(x^*, y^*) \in X \times X$.

Remark

Moreover, if for all $(x, y), (u, v) \in X \times X$ there exists $(z_1, z_2) \in X \times X$ such that $(x, y) \preceq_2 (z_1, z_2)$ and $(u, v) \preceq_2 (z_1, z_2)$, we have uniqueness of the coupled fixed point and $x^* = y^*$.

Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and $F : X \times X \to X$ be a given mapping. We endow the product set $X \times X$ with the partial order:

$$(x, y), (u, v) \in X \times X, \ (x, y) \preceq_2 (u, v) \Longleftrightarrow x \preceq u, \ y \succeq v.$$

Let $Y = X \times X$. It is easy to show that the mappings $\eta, \delta : Y \times Y \to [0, \infty)$ defined by

$$\begin{aligned} \eta((x, y), (u, v)) &= d(x, u) + d(y, v); \\ \delta((x, y), (u, v)) &= \max\{d(x, u), d(y, v)\}, \end{aligned}$$

for all $(x, y), (u, v) \in Y$, are metrics on Y.
Now, define the mapping $T : Y \to Y$ by

$$T(x, y) = (F(x, y), F(y, x)), \text{ for all } (x, y) \in Y.$$

It is easy to show that

Lemma

The following properties hold:

(a) *if* (X, d) *is complete, then* (Y, η) *and* (Y, δ) *are complete;*

(b) F *has the mixed monotone property if and only if* T *is monotone nondecreasing with respect to* $\preceq_2$*;*

(c) $(x, y) \in X \times X$ *is a coupled fixed point of* F *if and only if* (x, y) *is a fixed point of* T.

Theorem

Gnana-Bhaskar and Lakshmikantham Theorem follows from Ran and Reurings Theorem and Nieto and López Theorem .

From (iv), for all $(x, y), (u, v) \in X \times X$ with $x \succeq u$ and $y \preceq v$, we have

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)]$$

and

$$d(F(v, u), F(y, x)) \leq \frac{k}{2}[d(x, u) + d(y, v)].$$

This implies that for all $(x, y), (u, v) \in X \times X$ with $x \succeq u$ and $y \preceq v$,

$$d(F(x, y), F(u, v)) + d(F(v, u), F(y, x)) \leq k[d(x, u) + d(y, v)],$$

that is,

$$\eta(T(x, y), T(u, v)) \leq k\eta((x, y), (u, v)),$$

for all $(x, y), (u, v) \in Y$ with $(x, y) \succeq_2 (u, v)$. From Lemma , since (X, d) is complete, (Y, η) is also complete. Since F has the mixed monotone property, T is a nondecreasing mapping with respect to $\preceq_2$. From (iii), we have $(x_0, y_0) \preceq_2 T(x_0, y_0)$.

Now,if F is continuous, then T is continuous. In this case, applying RR-Theorem, we get that T has a fixed point, which implies from Lemma that F has a coupled fixed point. If conditions (X_1) and (X_2) are satisfied, then Y satisfies the following property: if a nondecreasing (with respect to $\preceq_2$) sequence $\{u_n\}$ in Y converges to a some point $u \in Y$, then $u_n \preceq_2 u$ for all n. Applying NL-Theorem, we get that T has a fixed point, which implies that F has a coupled fixed point. If in addition, we suppose that for all $(x, y), (u, v) \in X \times X$ there exists $(z_1, z_2) \in X \times X$ such that $(x, y) \preceq_2 (z_1, z_2)$ and $(u, v) \preceq_2 (z_1, z_2)$, from the last part of Theorems RR and NL, we obtain the uniqueness of the fixed point of T, which implies the uniqueness of the coupled fixed point of F. Now, let $(x^*, y^*) \in X \times X$ be the unique coupled fixed point of F. Since (y^*, x^*) is also a coupled fixed point of F, we get $x^* = y^*$.

Structure

Definition

An ordered MS $(X, d, \preccurlyeq)$ is said to have the *sequential monotone property* if it verifies:

(i) If $\{x_m\}$ is a non-decreasing sequence and $\{x_m\} \xrightarrow{d} x$, then $x_m \preccurlyeq x$ for all m.

(ii) If $\{y_m\}$ is a non-increasing sequence and $\{y_m\} \xrightarrow{d} y$, then $y_m \succcurlyeq y$ for all m.

Henceforth, fix a partition $\{\mathrm{A}, \mathrm{B}\}$ of $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $\mathrm{A} \cup \mathrm{B} = \Lambda_n$ and $\mathrm{A} \cap \mathrm{B} = \varnothing$ such that A and B are non-empty sets. We will denote:

$$\Omega_{\mathrm{A,B}} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(\mathrm{A}) \subseteq \mathrm{A} \text{ and } \sigma(\mathrm{B}) \subseteq \mathrm{B}\} \quad \text{and}$$
$$\Omega'_{\mathrm{A,B}} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(\mathrm{A}) \subseteq \mathrm{B} \text{ and } \sigma(\mathrm{B}) \subseteq \mathrm{A}\}.$$

If $(X, \preccurlyeq)$ is a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, we will use the following notation:

$$x \preccurlyeq_i y \quad \Leftrightarrow \quad \begin{cases} x \preccurlyeq y, & \text{if } i \in \mathrm{A}, \\ x \succcurlyeq y, & \text{if } i \in \mathrm{B}. \end{cases}$$

Consider on the product space X^n the following partial order: for $\mathrm{X} = (x_1, x_2, \ldots, x_n), \mathrm{Y} = (y_1, y_2, \ldots, y_n) \in X^n$,

$$\mathrm{X} \sqsubseteq \mathrm{Y} \quad \Leftrightarrow \quad x_i \preccurlyeq_i y_i, \text{ for all } i. \tag{8}$$

We say that two points X and Y are *comparable* if $\mathrm{X} \sqsubseteq \mathrm{Y}$ or $\mathrm{X} \sqsupseteq \mathrm{Y}$.

Proposition

If $X \sqsubseteq Y$, it follows that

$$(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \sqsubseteq (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) \quad \text{if } \sigma \in \Omega_{A,B},$$
$$(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \sqsupseteq (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) \quad \text{if } \sigma \in \Omega'_{A,B}.$$

Let $F : X^n \to X$ be a mapping.

Definition

Let $(X, \preccurlyeq)$ be a partially ordered space. We say that F has the *mixed monotone property (w.r.t. $\{\mathrm{A}, \mathrm{B}\}$)* if F is monotone non-decreasing in arguments of A and monotone non-increasing in arguments of B, i.e., for all $x_1, x_2, \ldots, x_n, y, z \in X$ and all i,

$$y \preccurlyeq z \quad \Rightarrow \quad F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preccurlyeq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$$

Henceforth, let $\sigma_1, \sigma_2, \ldots, \sigma_n : \Lambda_n \to \Lambda_n$ be n mappings from Λ_n into itself and let Υ be the n-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. The main aim of this paper is to study the following class of multidimensional fixed points.

Definition

(Roldan et al.) A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a *Υ-fixed point of the mapping F* if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = x_i \qquad \text{for all } i. \tag{9}$$

If one represents a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, i.e., $\sigma = (\sigma(1), \sigma(2), ..., \sigma(n))$, then

- Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n = 2$, $\sigma_1 = \tau = (1, 2)$ and $\sigma_2 = (2, 1)$;
- Berinde and Borcut's tripled fixed points are associated to $n = 3$, $\sigma_1 = \tau = (1, 2, 3), \sigma_2 = (2, 1, 2)$ and $\sigma 2 = (3, 2, 1)$;
- Berzig and Samet's multidimensional fixed points are given when $\mathrm{A} = \{1, 2, \ldots, m\}$ and $\mathrm{B} = \{m + 1, m + 2, \ldots, n\}$.

For more details see Roldan et al. JMAA 2012.

Lemma

Let (X,d) *be a MS and define* $D_n, \Delta_n : X^n \times X^n \to [0,\infty)$*, for all* $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n) \in X^n$*, by*

$$D_n(A,B) = \max_{1\le i\le n} d(a_i, b_i) \quad \textit{and} \quad \Delta_n(A,B) = \sum_{i=1}^{n} d(a_i, b_i).$$

Then D_n *and* Δ_n *are complete metrics on* X^n*.*

Actually, both metrics are equivalent since $D_n \le \Delta_n \le nD_n$.

Theorem

Let $(X, d, \preccurlyeq)$ *be a partially ordered MS and let* $F : X^n \to X$ *be a mapping. Let* $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *be a n-tuple of mappings from* $\{1, 2, \ldots, n\}$ *into itself verifying* $\sigma_i \in \Omega_{\mathrm{A,B}}$ *if* $i \in \mathrm{A}$ *and* $\sigma_i \in \Omega'_{\mathrm{A,B}}$ *if* $i \in \mathrm{B}$. *Define* $F_\Upsilon : X^n \to X^n$, *for all* $x_1, x_2, \ldots, x_n \in X$, *by*

$$F_\Upsilon(x_1, x_2, \ldots, x_n) = (F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_n(n)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \ldots, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})).$$

1. *If* F *has the mixed monotone property, then* F_Υ *is monotone nondecreasing w.r.t. the partial order* $\sqsubseteq$ *on* X^n *given by (8).*
2. *If* F *is continuous (w.r.t.* D_n *or* Δ_n*), then* F_Υ *is also continuous (w.r.t.* D_n *or* Δ_n*).*
3. *A point* $(x_1, x_2, \ldots, x_n) \in X^n$ *is a* Υ*-fixed point of* F *if, and only if,* $(x_1, x_2, \ldots, x_n)$ *is a fixed point of* F_Υ.

Throughout this section, let $(X, d, \preccurlyeq)$ be an ordered MS, let $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself such that $\sigma_i \in \Omega_{\mathrm{A,B}}$ if $i \in \mathrm{A}$ and $\sigma_i \in \Omega'_{\mathrm{A,B}}$ if $i \in \mathrm{B}$, and let $F : X^n \to X$ be a mapping. Consider the following conditions:

(I) (X, d) is complete.

(II) F has the mixed monotone property.

(III) F is continuous or $(X, d, \preccurlyeq)$ has the sequential monotone property.

(IV) There exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying

$$x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}) \quad \text{for all } i.$$

In some cases, we will also use an additional hypothesis.

(V) For all i, the mapping σ_i is a permutation of $\{1, 2, \ldots, n\}$.

This last condition implies that, for all i and all $\mathrm{X}, \mathrm{Y} \in X^n$,

$$\max_{1\leq j\leq n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)}) = \max_{1\leq j\leq n} d(x_j, y_j) = D_n(\mathrm{X}, \mathrm{Y}) \quad \text{and}$$
$$\sum_{j=1}^{n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)}) = \sum_{j=1}^{n} d(x_j, y_j) = \Delta_n(\mathrm{X}, \mathrm{Y}). \tag{10}$$

Roldán, Martínez and Roldán's multidimensional fixed point results.

In 2012, Roldán *et al.* proved the following theorem in order to show sufficient conditions to ensure the existence of Φ-coincidence points (we particularize it in the case of Υ-coincidence points taking τ as the identity mapping on $\{1, 2, \ldots, n\}$ and g as the identity mapping on X).

Theorem (Roldán *et al.*)

Under hypothesis (I)-(IV), assume that there exists $k \in [0, 1)$ verifying:

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq k \max_{1 \leq i \leq n} d(x_i, y_i) \qquad (11)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

We shall prove the following result.

Theorem

Theorem 45 follows from Theorems Ran-Reuring and Nieto &

Uniqueness of the fixed point also follows from Theorems **??** and **??**.

Theorem

Under the hypothesis of Theorem 45, assume that for all $A, B \in X^n$ there exists $U \in X^n$ such that $A \sqsubseteq U$ and $B \sqsubseteq U$. Then F has a unique Υ-fixed point.

In order to assure the uniqueness of the fixed point, notice that the previous result shows a sufficient condition which is only related to the partial order $\preccurlyeq$ on X (and its extension $\sqsubseteq$ to X^n). It is not difficult to prove that a similar property of uniqueness may be included in the results we will present throughout this paper. However, for brevity, we will not write this part.

Bhaskar and Lakshmikantham's coupled fixed point results

We present the following multidimensional extension of the main result of Bhaskar and Lakshmikantham 2006NA using a similar argument of which we have showed in the previous subsection.

Corollary

Under hypothesis (I)-(IV), assume that there exists $k \in [0,1)$ verifying:

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{k}{n} \sum_{i=1}^{n} d(x_i, y_i)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Berinde's coupled fixed point results

We extend [Theorem 3, Berinde NA 2011] in the following sense.

Theorem

Under hypothesis (I)-(IV), assume that there exists $k \in [0,1)$ *verifying:*

$$\sum_{i=1}^{n} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})) \leq k \sum_{j=1}^{n} d(x_j, y_j)$$

for which $x_i \preccurlyeq_i y_i$ *for all* i*. Then* F *has, at least, one* Υ*-fixed point.*

Notice that this result implies that [Theorem 3, Berinde NA 2011] follows from Theorems Ran & Reuring and Nieto & Rodriguerz-Lopez.

Berzig and Samet's multidimensional fixed point results and Berinde and Borcut's tripled fixed point results

Berzig and Samet 2012CAMWA extended the main result of Berinde and Borcut (2011 NA) in the setting of multidimensional mappings.
Both results are special cases of the following theorem, which was also established in [Corollary 16, Roldan et al. JMAA 2012].

Theorem

Under hypothesis (I)-(IV), assume that there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1)$ *such that* $\alpha_1 + \alpha_2 + \ldots + \alpha_n < 1$ *verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \sum_{j=1}^{n} \alpha_j d(x_j, y_j)$$

for which $x_i \preccurlyeq_i y_i$ *for all* i. *Then* F *has, at least, one* Υ*-fixed point.*

Ćirić, Cakić, Rajović and Ume's multidimensional fixed point results

The following theorem is a multidimensional version of Theorem 2.2 in Ćirić *et al.*

Theorem

Under hypothesis (I)-(IV), assume that there exists $\phi \in \Phi_3$ *verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \phi\left(\max_{1 \leq i \leq n} d(x_i, y_i)\right)$$

for which $x_i \preccurlyeq_i y_i$ *for all* i. *Then* F *has, at least, one* Υ*-fixed point.*

Corollary

Under hypothesis (I)-(IV), assume that there exists $\phi \in \Phi_3$ verifying:

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \phi\left(\frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i)\right)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Corollary

Under hypothesis (I)-(IV), assume that there exists $\phi \in \Phi_3$ verifying:

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{1}{n}\,\phi\left(\sum_{i=1}^{n} d(x_i, y_i)\right)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

We shall prove the following result as a generalization of the main result of Lakshmikantham and Ćirić 2009NA.

Theorem

Under hypothesis (I)-(V), assume that there exists $\psi \in \Phi_2$ *verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \psi\left(\frac{1}{n}\sum_{j=1}^{n} d(x_j, y_j)\right)$$

for which $x_i \preccurlyeq_i y_i$ *for all* i*. Then* F *has, at least, one* Υ*-fixed point.*

Harjani, López and Sadarangani's coupled fixed point results

We shall prove the following result which is an extension of the main result of Harjani *et al.*2011NA.

Theorem

Under hypothesis (I)-(V), assume that there exist $\psi, \varphi \in \Phi_1$ *verifying:*

$$\psi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \leq \psi\left(\max_{1 \leq i \leq n} d(x_i, y_i)\right) - \varphi\left(\max_{1 \leq i \leq n} d(\right.$$

for which $x_i \preccurlyeq_i y_i$ *for all* i*. Then* F *has, at least, one* Υ*-fixed point.*

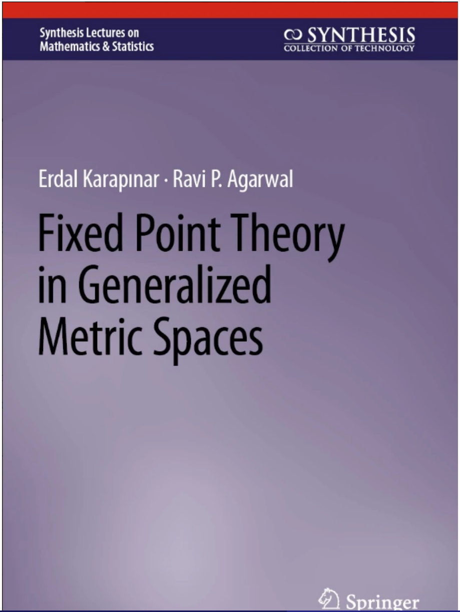
Structure

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

G.H. Hardy (1887-1977)

"A century later the multitude is vaster and the utility little greater."

Brailey Sims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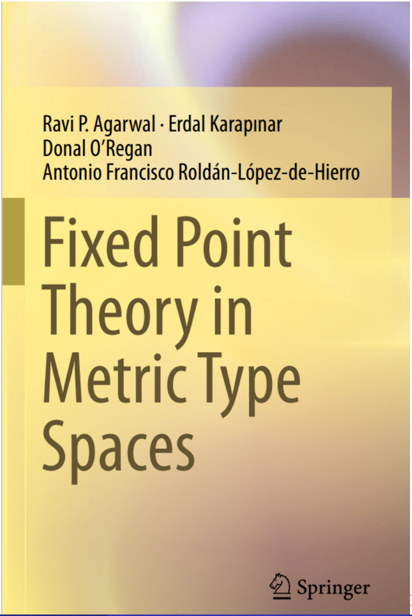
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