Fixed–point theory and Green's functions for the solution of DEs: An iterative strategy

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Abstract

A recently developed iterative method for estimating the solution of ordinary and fractional boundary–value problems is described. The strategy is based on the construction of a tailored integral operator described in terms of the Green's function, which corresponds to the highest order linear derivative term. After then, the integral operator is subjected to a fixed–point scheme like Picard's, Mann's, or Ishikawa's. The convergence of the scheme is assessed. Numerical tests are used to assess the applicability and correctness of the approach.

Outline

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Problem Statement

The aim is to present fixed point iterative schemes for the solution of the following DE/FDE:

$$L[y] + N[y] = f(t, y),$$

where $L[y]$ is a linear operator in y, $N[y]$ is a nonlinear operator in y, and $f(t, y)$ is a linear or nonlinear function in y. The equation is supplemented with either ICs or BCs. An example of such a class is the FBVP:

$$y^{(\alpha)}(t) = f\left(t, y(t)\right),$$

$$y(0) = a, \;\; y(1) = b,$$

where $0 \leq t \leq 1$, $1 < \alpha \leq 2$.

Overview of Green's function

Consider the class of n^{th} order linear differential equations

$$L[u] \equiv a_1(t)\, u^{(n)}(t) \;+\; \dots \;+\; a_{n-1}(t)\, u'(t) \;+\; a_n(t)\, u(t) = f(t),$$

on [a,b] and complimented with n boundary conditions (BCs):

$$B_1[u(t)] = \alpha_1, \quad B_2[u(t)] = \alpha_2, \; \dots, \quad B_n[u(t)] = \alpha_n.$$

The Green's function is defined to be the solution for the equation:

$$-L[G(t|s)] = \delta(t-s).$$

where δ is the Kronecker Delta, and subject to the corresponding "homogeneous" boundary conditions.

For $t \neq s$, we need to solve $L[G(t|s)] = 0$, therefore

$$G(t|s) = \begin{cases} c_1 u_1 + c_2 u_2 + c_3 u_3 + \ldots + c_n u_n, & a < x < s \\ d_1 u_1 + d_2 u_2 + d_3 u_3 + \ldots + d_n u_n, & s < x < b \end{cases},$$

where $u_1, u_2, \ldots, u_n$ are linearly independent solutions of $L[u] = 0$.

The constants are to be found using the properties:

A. G satisfies the n homogeneous BCs:

$$B_1[G(t|s)] = B_2[G(t|s)] = ... = B_n[G(t|s)] = 0,$$

B. Continuity of $G,\ G',\ G'', ..., G^{(n-2)}$ at $t = s$. This results in the $n-1$ equations:

$$\begin{cases} c_1 u_1(s) + c_2 u_2(s) + ... + c_n u_n(s) = d_1 u_1(s) + d_2 u_2(s) + ... + d_n u_n(s), \\ c_1 u_1'(s) + c_2 u_2'(s) + ... + c_n u_n'(s) = d_1 u_1'(s) + d_2 u_2'(s) + ... + d_n u_n'(s), \\ ... \\ c_1 u_1^{(n-2)}(s) + ... + c_n u_n^{(n-2)}(s) = d_1 u_1^{(n-2)}(s) + ... + d_n u_n^{(n-2)}(s). \end{cases}$$

C. Jump discontinuity of $G^{(n-1)}$ at $t = s$:

$$d_1 u_1^{(n-1)}(s) + ... + d_n u_n^{(n-1)}(s) - c_1 u_1^{(n-1)}(s) - ... - c_n u_n^{(n-1)}(s) = \frac{1}{a_n(s)}.$$

Definition

The Caputo fractional derivative of order $m-1<\alpha\leq m$, of a function $g(t)$, is defined as

$$g^{(\alpha)}(t)=J^{m-\alpha}g^{(m)}(t),$$

when

$$J^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)\,ds, \quad \alpha>0,$$

for $m\in\mathbb{N}$, $t>0$ and $g\in C_{-1}^{m}$.

Lemma

The Laplace transform of Caputo fractional derivative for $m-1<\alpha\leq m, m\in\mathbb{N}$, can be determined in the form of:

$$\mathcal{L}[g^{(\alpha)}(x)]=\frac{s^mG(s)-s^{m-1}g(0)-s^{m-2}g'(0)-\ldots-g^{(m-1)}(0)}{s^{m-\alpha}}.$$

Example

Consider the FBVP:

$$y^{(\alpha)}(t) = f\left(t, y(t)\right),$$

$$y(0) = a, \ \ y(1) = b.$$

Here $0 \leq t \leq 1$ and $1 < \alpha \leq 2$.

The corresponding Green's function for this FBVP satisfies:

$$\begin{cases} -D^{\alpha} G(t,x) = \delta(t-x), \\ G(0,x) = 0, \ \ G(1,x) = 0. \end{cases}$$

Note that the Green's function satisfies the corresponding homogenous boundary conditions.

To obtain the Green's function explicitly, operate with Laplace transform.

$$\frac{s^2\mathcal{L}[G(t,x)] - sG(0,x) - G_t(0,x)}{s^{2-\alpha}} = -e^{-sx}.$$

Assume $G_t(0,x) = K$. Then

$$\mathcal{L}[G(t,x)] = \frac{K}{s^2} - \frac{1}{s^\alpha}e^{-sx}.$$

Laplace inverse yields

$$G(t,x) = Kt - \frac{1}{\Gamma(\alpha)}(t-x)^{\alpha-1}\mathcal{U}(t-x),$$

where $\mathcal{U}$ is the Unit Step function.

The constant K is found using the BC, $G(1,x) = 0$. We have

$$G(t,x) = \frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1}\, t - \frac{1}{\Gamma(\alpha)}(t-x)^{\alpha-1}\,\mathcal{U}(t-x).$$

Therefore

$$G(t,x) = \begin{cases} \dfrac{t(1-x)^{\alpha-1} - (t-x)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le x < t \le 1 \\[2ex] \dfrac{t(1-x)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t < x \le 1 \end{cases}.$$

Example

Consider the FBVP:

$$y^{(\alpha)}(t) - \lambda^2 y(t) = f\left(t, y(t)\right),$$

$$y(0) = a, \;\; y(1) = b.$$

Here $0 \leq t \leq 1$ and $1 < \alpha \leq 2$.

The corresponding Green's function for this FBVP satisfies:

$$\begin{cases} -D^{\alpha} G(t,x) + \lambda^2 G(t,x) \; = \; \delta(t-x), \\ G(0,x) = 0, \;\; G(1,x) = 0. \end{cases}$$

The Green's function for general value of α, where $1 < \alpha \leq 2$, is given by

$$G(t,x) = \begin{cases} \dfrac{t(1-x)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda^2(1-x)^{\alpha}\right) E_{\alpha,2}\left(\lambda^2 t^{\alpha}\right)}{E_{\alpha,2}\left(\lambda^2\right)} - (t-x)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda^2(t-x)^{\alpha}\right) & 0 \leq t < x, \\ \dfrac{t(1-x)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda^2(1-x)^{\alpha}\right) E_{\alpha,2}\left(\lambda^2 t^{\alpha}\right)}{E_{\alpha,2}\left(\lambda^2\right)}, & x < t \leq 1 \end{cases}.$$

Here $E_{a,b}$ is the Mittag–Leffler function.

The Green's function for the case $\alpha = 2$ reduces to:

$$G(t,x) = \begin{cases} \dfrac{\sinh(\lambda(1-x))}{\lambda \sinh(\lambda)} \sinh(\lambda t), & 0 \leq t < x \\ \dfrac{\sinh(\lambda(1-t))}{\lambda \sinh(\lambda)} \sinh(\lambda x), & x < t \leq 1 \end{cases}.$$

Example

Consider the FIVP:

$$y^{(\alpha)}(t) = f\left(t, y(t), y'(t)\right),$$

$$y(0) = a,\ \ y'(0) = b,\ \ y''(0) = b.$$

Here $0 \leq t \leq 1$ and $2 < \alpha \leq 3$.

The corresponding Green's function for this FIVP satisfies:

$$\begin{cases} -D^{\alpha} G(t, x) = \delta(t - x), \\ G(0, x) = 0,\ \ G_t(0, x) = 0,\ \ G_{tt}(0, x) = 0. \end{cases}$$

Overview of fixed–point iterative procedures

Let X be a normed linear space and $T : X \to X$ a given operator. Next, we list the most well–known fixed–point iterative schemes.

I. Picard's Iteration: $y_0 \in X$ and $\{y_n\}_{n=0}^{\infty}$ defined by:

$$y_{n+1} = T[y_n], \quad n = 0, 1, 2, ...$$

II. Krasnoselkij's Iteration: $y_0 \in X$, $\gamma \in [0, 1]$, $\{y_n\}_{n=0}^{\infty}$ defined by

$$y_{n+1} = (1 - \gamma)y_n + \gamma\, T[y_n], \qquad n = 0, 1, 2, ...$$

III. Mann's Iteration: $y_0 \in X$, $\{\alpha_n\} \subset [0,1]$, $\{y_n\}_{n=0}^{\infty}$ defined by:

$$y_{n+1} = (1-\alpha_n)y_n + \alpha_n T[y_n], \qquad n = 0, 1, 2, \ldots.$$

IV. Ishikawa's Iteration: $\{y_n\}_{n=0}^{\infty}$ is defined by:

$$\begin{cases} y_{n+1} = (1-\alpha_n)y_n + \alpha_n T[z_n], \\ z_n = (1-\beta_n)y_n + \beta_n T[y_n], \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset [0,1]$, and $y_0 \in X$ is arbitrary.

Method description

We present the iterative method for the FBVP:

$$y^{(\alpha)}(t) = f\left(t, y(t)\right),$$
$$y(0) = a, \;\; y(1) = b.$$

Here $0 \leq t \leq 1$, $1 < \alpha \leq 2$. Note that

$$y_p(t) = \int_0^1 G(t,x)\, f\left(x, y_p(x)\right)\, dx,$$

where y_p is a particular solution for the equation that satisfies the corresponding homogeneous boundary conditions.

An important note is that for the nonhomogeneous BCs, the particular solution contains terms outside the integral:

$$y_p(t) = \int_a^b G(t,x)f(x)\,dx + b\,G(t,1) + a\,\frac{\partial G}{\partial x}(t,0).$$

This latter term outside the integral will not be visible in the iterative scheme, as it is set to the value of the first iteration, i.e. y_0 is chosen as:

$$y_0(t) = b\,G(t,1) + a\,\frac{\partial G}{\partial x}(t,0).$$

This solution satisfies the nonhomogeneous boundary conditions.

Define the following integral operator:

$$L[y] \equiv \int_0^1 G(t,x)\, y^{(\alpha)}(x)\, dx.$$

Rewrite the equation as:

$$L[y] = \int_0^1 G(t,x) \left[y^{(\alpha)}(x) - f\left(x, y(x)\right) \right] dx + \int_0^1 G(t,x)\, f\left(x, y(x)\right) dx.$$

The equation reduces to:

$$L[y_p] = \int_0^1 G(t,x) \left[y_p^{(\alpha)}(x) - f\left(x, y_p(x)\right) \right] dx + y_p.$$

Applying Picard's iteration scheme to the operator $L[y]$, namely

$$y_{n+1} = L[y_n], \quad n = 0, 1, 2, \ldots,$$

we obtain the following iterative procedure:

$$y_{n+1} = y_n + \int_0^1 G(t, x) \left[y_n^{(\alpha)}(x) - f\left(x, y_n(x)\right) \right] dx.$$

Other well–known iterative procedures can be applied to the operator $L[y]$. For instance, applying Mann's procedure:

$$y_{n+1} = (1-\beta_n)y_n + \beta_n L[y_n], \qquad \forall n \geq 0,$$

to $L[y]$ yields the itertive scheme:

$$y_{n+1} = y_n + \beta_n \int_0^1 G(t,x)\left[y^{(\alpha)}(x) - f\left(x, y(x)\right)\right] dx.$$

Here β_n is a sequence between 0 and 1. Mann's iterative procedure can be used in some cases when Picard's scheme diverges.

If the sequence β_n is chosen correctly, the rate of convergence of the scheme will be accelerated and optimal values for β_n are obtained. One approach to find optimal values of β_n is by minimizing the $L^2[a,b]$–norm of the residual error, $R_n(x;\beta_n)$, of the n^{th} iteration y_n. For the first iterate, y_1, the L^2 norm of the residual error $R_1(x;\beta_1)$ is:

$$\|R_1(x;\beta_1)\|_{L^2}^2 = \int_a^b |R_1(x;\beta_1)|^2 dx,$$

needs to minimized for β_1. The other values of β_n can be acquired in a similar way.

The Mann's iterative scheme for the FBVP and subject to the BCs is given as follows:

$$y_{n+1} = y_n + \beta_n \int_0^t \left[\frac{t\,(1-x)^{\alpha-1} - (t-x)^{\alpha-1}}{\Gamma(\alpha)} \right] \left[y_n^{(\alpha)}(x) - f(x, y_n(x)) \right] dx$$

$$+ \beta_n \int_t^1 \left[\frac{t\,(1-x)^{\alpha-1}}{\Gamma(\alpha)} \right] \left[y_n^{(\alpha)}(x) - f(x, y_n(x)) \right] dx.$$

If $\beta_n = 1$, it reduces to Picard's scheme.

Consider the following differential equation:

$$L[u] + N[u] = f(t, u),$$

where $L[u]$ is a linear operator in u, $N[u]$ is a nonlinear operator in u, and $f(t, u)$ is a linear or nonlinear function in u. Applying Ishikawa fixed point iterative formula, yields the iterative scheme:

$$\begin{cases} w_n = v_n + \beta_n \int_a^b G(t,s) \left(L[v_n] + N[v_n] - f(s, v_n) \right) \, ds, \\ v_{n+1} = (1-\alpha_n) v_n + \alpha_n \left[w_n + \int_a^b G(t,s) \left(L[w_n] + N[w_n] - f(s, w_n) \right) \, ds \right]. \end{cases}$$

The special case $\alpha_n = 1$ and $\beta_n = 0$ results in Picard's scheme, while the case $\beta_n = 0$ and α_n yields Mann's scheme. The optimal values of the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ are found by minimizing the L^2-norm of the residual error, $R_n(t; \alpha_n, \beta_n)$, of the n^{th} iteration v_n:

$$\|R_n(t; \alpha_n)\|_{L^2}^2 = \frac{1}{b-a} \int_a^b R_n^2(t; \alpha_n, \beta_n)\, dt,$$

where for each n, $R_n(t; \alpha_n, \beta_n)$ is given by

$$R_n(t; \alpha_n) = L[u_n] + N[u_n] - f(t, u_n(t)).$$

Convergence Analysis

The convergence analysis is based on the contraction principle and Banach–Picard fixed point theorem. Consider the BVP:

$$u''(t) = f(t, u(t), u'(t)),$$
$$u(0) = A, \quad u(1) = B.$$

The Green's Ishikawa iterative procedure will take the form

$$\begin{cases} w_n &= v_n + \beta_n \int_0^1 G(t,s)\left(v_n''(s) - f(s, v_n, v_n')\right)\, ds, \\ v_{n+1} &= (1-\alpha_n)v_n + \alpha_n \left[w_n + \int_0^1 G(t,s)\left(w_n''(s) - f(s, w_n, w_n')\right)\, ds\right]. \end{cases}$$

The initial iterate v_0 satisfies the corresponding homogeneous linear equation $y'' = 0$ and the BCs. Thus, $v_0 = (B - A)t + A$.

Introduce the following operator, from the set of continuous functions on $[0,1]$ into itself, defined by

$$T_G(u) = u + \int_0^1 G(t,s)(u'' - f(s,u,u'))\,ds.$$

Then, the scheme becomes

$$\begin{cases} w_n &= (1-\beta_n)v_n + \beta_n T_G(v_n), \\ v_{n+1} &= (1-\alpha_n)v_n + \alpha_n T_G(w_n). \end{cases}$$

In the next theorem, we show that, under some hypothesis on the function f, our operator T_G is a contraction with respect to the supremum norm. In particular, T_G is a Zamfirescu operator. Therefore, we obtain that $(v_n)_n$ converges strongly to the fixed point of T_G. Here we have to assume that the sequence $(\alpha_n)_n$ satisfies the condition $\sum_{n\geq 0} \alpha_n = \infty$.

Theorem

Assume that the function f, which appears in the definition of the operator T_G, is such that

$$\frac{1}{4\sqrt{3}} \sup_{[0,1]\times\mathbb{R}^3} \left| \frac{\partial f}{\partial u} \right| < 1.$$

Then T_G is a contraction and hence, the Ishikawa iteration $(v_n)_n$ converges strongly to the fixed point of T_G.

A note on Calculus of Variation

Consider the differential equation

$$Lu + Nu = f(x),$$

where L and N are linear and nonlinear operators respectively, and $f(x)$ is the source inhomogeneous term defined on $[a, b]$. The Variational Iteration Method admits the use of a correction functional in the form:

$$u_{n+1}(x) = u_n(x) + \int_a^x \lambda(t)\left(Lu_n(t) + N\,\tilde{u}_n(t) - f(t)\right)\,dt.$$

Here λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which implies that $\delta\tilde{u}_n = 0$. Having λ determined, an iteration scheme is applied for the determination of the successive approximations $u_{n+1}(x),\ n \geq 0$, of the solution $u(x)$. The solution is constructed as follows:

$$u(x) = \lim_{n\to\infty} u_n(x).$$

It is important to note that the iterative scheme includes the left endpoint $x = a$ but not the right endpoint $x = b$, which is a setback when dealing with BVPs. The VIM is powerful and suitable for IVPs. Based on this drawback we will modify the correction functional for BVPs, to include both $x = a$ and $x = b$, as follows:

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + \int_a^x \lambda_1(t;x)\left[Lu_n(t) + N\,\tilde{u}_n(t) - f(t)\right]\,dt \\ &+ \int_x^b \lambda_2(t;x)\left[Lu_n(t) + N\,\tilde{u}_n(t) - f(t)\right]\,dt. \end{aligned}$$

Here $\tilde{u}_n$ is a restricted variation ($\delta\tilde{u}_n = 0$), $\lambda_1(t;x)$ and $\lambda_2(t;x)$ are two general Lagrange's multipliers defined on the intervals $[a,x]$ and $[x,b]$ respectively, that satisfy the corresponding homogeneous BCs at $x = b$ and $x = a$ respectively. As for the initial term or iterate, u_0, it is chosen to satisfy the given non–homogeneous BCs.

Numerical Results

Example. Consider the following differential equation:

$$y''(t) = -ay'^m(t) + y(t)\left(by^2(t) - \frac{3}{2}y(t) + \frac{1}{2}\right)$$

subject to

$$y(0) = 1, \quad y(1) = 2.$$

The initial iterate satisfies the linear differential operator y'' and the specified BCs. This gives $y_0 = 1 + t$.

The higher iterates are given by the following Ishikawa iterative procedure:

$$\begin{aligned} y_n &= x_n + \beta_n \int_0^t s(1-t)\left[x_n''(s) + a x_n'^m(s) - x_n(s)\left(b x_n^2(s) - \frac{3}{2}x_n(s) + \frac{1}{2}\right)\right] ds \\ &+ \beta_n \int_t^1 t(1-s)\left[x_n''(s) + a x_n'^m(s) - x_n(s)\left(b x_n^2(s) - \frac{3}{2}x_n(s) + \frac{1}{2}\right)\right] ds, \end{aligned}$$

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\left[y_n + \int_0^t s(1-t)\left(y_n''(s) + a y_n'^m(s) - y_n(s)\left(b y_n^2(s) - \frac{3}{2}y_n(s) + \frac{1}{2}\right)\right.\right. \\ &+ \left.\left. \int_t^1 t(1-s)\left(y_n''(s) + a y_n'^m(s) - y_n(s)\left(b y_n^2(s) - \frac{3}{2}y_n(s) + \frac{1}{2}\right)\right) ds\right], \end{aligned}$$

where $a = 1$, $b = 1$, $m = 2$. By minimizing the L^2-norm of the residual error, the optimal values for α_n and β_n are found to be $\alpha = 0.9345414427$ and $\beta = 0.8817524743$. Again, by minimizing the L^2-norm, the optimal value of α_n for Mann's is found to be $\alpha_n = 0.89091489$.

The results in the following table clearly show that Ishikawa approach is more accurate than both Picard and Mann strategies.

t	Ishikawa y_7	Ishikawa y_{15}	Picard y_{15}	Mann y_{15}
0.0	4.607507(−8)	2.962480(−17)	2.923141(−9)	6.824221(−11)
0.1	1.531930(−9)	1.293702(−18)	1.380092(−9)	5.875329(−11)
0.2	1.151114(−9)	3.315248(−19)	1.843756(−9)	4.317568(−12)
0.3	3.042586(−10)	1.247576(−20)	5.292063(−9)	7.459339(−11)
0.4	3.017951(−10)	7.714027(−20)	7.206256(−9)	1.420881(−10)
0.5	5.153632(−11)	1.941727(−20)	6.142973(−9)	1.594676(−10)
0.6	2.956696(−10)	8.764974(−21)	1.422278(−9)	9.797271(−11)
0.7	2.981282(−10)	3.892195(−20)	6.461737(−9)	4.990121(−11)
0.8	3.437264(−10)	1.106792(−20)	1.526312(−8)	2.581048(−10)
0.9	3.069509(−9)	4.225158(−19)	2.016841(−8)	4.464999(−10)
1.0	1.200573(−8)	2.885500(−18)	1.337204(−8)	4.518708(−10)

Comparison of the Residual Errors using Ishikawa scheme and that of Picard and Mann.

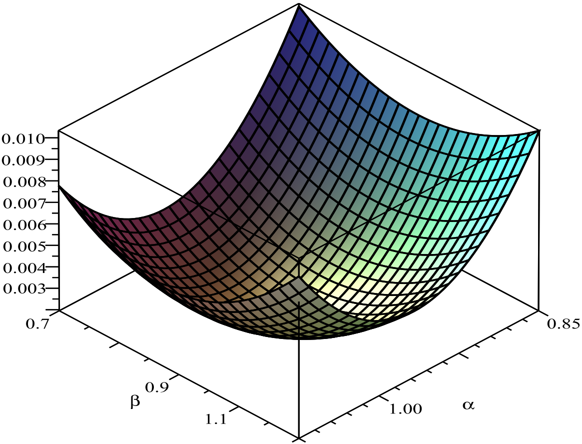

Figure. $L^2[R_1(t;\alpha,\beta)]$ versus α and β.

Example.

Consider the following Troesch's boundary layer problem:

$$u'' = \lambda \sinh \lambda u, \qquad \text{on} \quad 0 \leq t \leq 1,$$

subject to

$$u(0) = 0, \quad u(1) = 1.$$

Eigenvalues $\lambda > 1$: For this case, the difficulty of solving the Troesch's problem is due to the existence of the boundary layer. Therefore, we intend to convert the hyperbolic–type nonlinearity into polynomial-type via the variable transformation:

$$y(t) = \tanh\left(\frac{\lambda u(t)}{4}\right).$$

Then, the transformed Troesch's problem becomes

$$\left(1 - y^2\right) y'' + 2y \left(y'\right)^2 = \lambda^2 y \left(1 + y^2\right),$$

subject to the new boundary conditions:

$$y(0) = 0 \qquad y(1) = \tanh\left(\frac{\lambda}{4}\right),$$

and the solution to this transformed Troesch's problem is

$$u(t) = \frac{4}{\lambda} \tan^{-1}(y(t)).$$

Since the transformed problem has a dissimilar linear operator than in the original problem, the corresponding Green's function will be different. To implement the iteration method, we will decompose the differential equation into a linear and nonlinear terms, so it reads as follows: $Ly = Ny$. Here the linear operator is given by $Ly \equiv y'' - \lambda^2 y = 0$ while the nonlinear one is $Ny \equiv g(y, y', y'') = y^2 y'' - 2y\left(y'\right)^2 + \lambda^2 y^3$. The Green's function is:

$$G(t,s) = \begin{cases} \dfrac{\sinh(\lambda s)\sinh(\lambda(1-t))}{\lambda\sinh(\lambda)}, & 0 \leq s \leq t \\ \dfrac{\sinh(\lambda t)\sinh(\lambda(1-s))}{\lambda\sinh(\lambda)}, & t \leq s \leq 1 \end{cases}.$$

Applying Picard's based algorithm using this Green's function, we get the following iterative scheme:

$$\begin{aligned} y_{n+1} &= y_n + \int_0^t \frac{\sinh(\lambda s)\sinh(\lambda(1-t))}{\lambda \sinh(\lambda)} \left[(1-y_n^2)y_n'' + 2y_n(y_n')^2 - \lambda^2 y_n(1+y_n^2)\right] ds \\ &+ \int_t^1 \frac{\sinh(\lambda t)\sinh(\lambda(1-s))}{\lambda \sinh(\lambda)} \left[(1-y_n^2)y_n'' + 2y_n(y_n')^2 - \lambda^2 y_n(1+y_n^2)\right] ds. \end{aligned}$$

Using the PGEM, our results for different cases of $\lambda > 1$ at different t are reported in the Tables below.

λ	N	Numerical Solution U[N]	Err[N]	U[N]-U[N-1]
2000	2	$2.768(-90)$	$1.13(-1993)$	$1.45(-2907)$
1000	2	$1.488(-46)$	$4.24(-995)$	$5.54(-1408)$
200	2	$4.122(-11)$	$1.07(-196)$	$6.49(-239)$
150	2	$8.157(-9)$	$7.07(-147)$	$9.35(-171)$
100	2	$1.816(-6)$	$4.14(-97)$	$2.11(-105)$
80	2	$1.677(-5)$	$3.01(-77)$	$6.47(-79)$
40	2	$1.832(-3)$	$2.14(-42)$	$3.29(-38)$
20	2	$2.723(-2)$	$3.12(-25)$	$1.15(-19)$
10	6	$1.521(-1)$	$1.02(-21)$	$2.64(-21)$
5	7	$4.551(-1)$	$1.38(-12)$	$5.10(-18)$
2	16	$7.905(-1)$	$1.83(-17)$	$1.14(-19)$

Table: PGEM iteration at $t = 0.9$ for different λ.

t	U[2]	Err[2]	U[2]-U[1]
0.100	1.3428(−80)	4.1(−235)	3.7(−239)
0.200	6.515(−72)	1.1(−243)	5.1(−247)
0.300	3.161(−63)	6.8(−256)	1.2(−255)
0.400	1.534(−54)	8.4(−277)	2.4(−264)
0.500	7.440(−46)	2.1(−354)	4.9(−273)
0.600	3.610(−37)	8.1(−276)	4.8(−264)
0.700	1.751(−28)	8.2(−249)	2.3(−255)
0.800	8.497(−20)	9.4(−223)	1.0(−248)
0.900	4.122(−11)	1.1(−196)	6.5(−239)
0.999	2.306(−02)	6.2(−170)	3.0(−233)

Table: PGEM iteration for $\lambda = 200$.

t	PGEM U[6]	PGEM Err[6]	B-spline $y(t)$	Discont. Galerkin $y(t)$	VIM $y(t)$
0.100	0.00004211189927237319	5.01(−22)	4.209661158388(−5)	4.21118992(−5)	4.211189501276(−5)
0.200	0.00012996411582375519	2.41(−28)	1.299195124415(−4)	1.299641158(−4)	1.299641033085(−4)
0.300	0.00035897840138966156	1.47(−26)	3.588639886704(−4)	3.589784013(−4)	3.589783710236(−4)
0.400	0.00097790277180291363	4.01(−26)	9.776162458355(−4)	9.779027718(−4)	9.779027043800(−4)
0.500	0.00265902049035107778	1.09(−25)	2.658310470583(−3)	2.6590204903(−3)	2.659020349167(−3)
0.600	0.00722893121287760637	2.97(−25)	7.227189065535(−3)	7.2289312128(−3)	7.228930931326(−3)
0.700	0.01966406309701858931	8.15(−25)	1.965983675656(−2)	1.96640630970(−2)	1.966406256917(−2)
0.800	0.05373032935060024273	2.38(−24)	5.372021024854(−2)	5.37303293505(−2)	5.373032846396(−2)
0.900	0.15211407640471317805	1.02(−21)	1.520908055685(−1)	1.521140764047(−1)	1.521140752185(−1)
0.925	0.20200168378027548897	1.63(−20)	2.019922958185(−1)	–	2.020016825843(−1)
0.950	0.27626773384317687823	2.50(−19)	2.762369555536(−1)	–	2.762677326887(−1)
0.970	0.37226433277149032607	2.60(−18)	3.722343195635(−1)	–	3.722643317016(−1)
0.980	0.44823303866594251547	1.02(−17)	4.482026699135(−1)	–	4.482330376655(−1)
0.990	0.57407649980148049123	6.23(−17)	5.740488905704(−1)	–	5.740764989151(−1)
0.995	0.69011494478392945011	2.53(−16)	6.900982796199(−1)	–	6.901149440197(−1)
0.997	0.76576972840424519191	5.90(−16)	7.657602795389(−1)	–	7.657697277452(−1)
0.998	0.81803283021141076063	1.04(−15)	8.180272700747(−1)	–	8.180328296454(−1)
0.999	0.88899311815589450291	2.20(−15)	8.889905508685(−1)	–	8.889931177557(−1)

Table: Numerical solutions of the PGEM and other methods for $\lambda = 10$.

t	B-spline $y(t) - U[6]$	Discont. Galerkin $y(t) - U[6]$	VIM $y(t) - U[6]$
0.100	1.53(−8)	7.00(−14)	4.26(−12)
0.200	4.46(−8)	2.38(−14)	1.25(−11)
0.300	1.14(−7)	8.97(−14)	3.04(−11)
0.400	2.87(−7)	2.91(−15)	6.74(−11)
0.500	7.10(−7)	5.11(−14)	1.41(−10)
0.600	1.74(−6)	7.76(−14)	2.82(−10)
0.700	4.23(−6)	1.86(−14)	5.28(−10)
0.800	1.01(−5)	1.00(−13)	8.87(−10)
0.900	2.33(−5)	1.32(−14)	1.19(−9)
0.925	9.39(−6)	–	1.20(−9)
0.950	3.08(−5)	–	1.15(−9)
0.970	3.00(−5)	–	1.07(−9)
0.980	3.04(−5)	–	1.00(−9)
0.990	2.76(−5)	–	8.86(−10)
0.995	1.67(−5)	–	7.64(−10)
0.997	9.45(−6)	–	6.59(−10)
0.998	5.56(−6)	–	5.66(−10)
0.999	2.57(−6)	–	4.00(−10)

Table: Comparison of the PGEM with other methods for $\lambda = 10$.

t	PGEM U[2]	Err[2]	Discont. Galerkin $y(t)$	$y(t) - U[2]$
0.100	2.98993508907308104052803861176939335008 9(−9)	1.42(−27)	2.989864(−9)	7.11(−14)
0.200	2.24974418174610910376406603183756173378 4(−8)	4.89(−35)	2.2496907(−8)	5.35(−13)
0.300	1.66289622243078156968723001276569430695 9(−7)	3.55(−37)	1.66285667(−7)	3.96(−12)
0.400	1.22873075874737678541954401278294437536 8(−6)	2.62(−36)	1.228701537(−6)	2.92(−11)
0.500	9.07916151599995853569390670415905252483 9(−6)	1.94(−35)	9.078945592(−6)	2.16(−10)
0.600	6.70864363787063994255341177348362269853 2(−5)	1.43(−34)	6.7084840902(−5)	1.60(−09)
0.700	4.95706438365770122548966058905603420635 8(−4)	1.16(−33)	4.95694649225(−4)	1.18(−08)
0.800	3.66320476638083242755812525637169133188 4(−3)	2.78(−33)	3.663117627065(−3)	8.71(−08)
0.900	2.72316434702242221627553998687202576865 8(−2)	3.34(−27)	2.7230987802378(−2)	6.56(−07)

Table: Comparison of the PGEM with the Discontinuous Galerkin for $\lambda = 20$.

Figure 1 and Figure 2 show the PGEM iteration solutions of Troesch's problem for $\lambda = 0.5, 1, 2, 5, 10, 15, 20, 30$ and for $\lambda = 40, 60, 7100,$ respectively. The graphs illustrate that the thickness of the boundary layer decreases and becomes more evident as the eigenvalue increases.

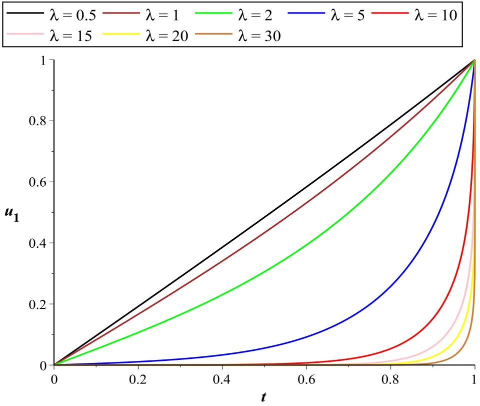

Figure 1. Numerical solution for $\lambda = 0.5, 1, 2, 5, 10, 15, 20, 30$.

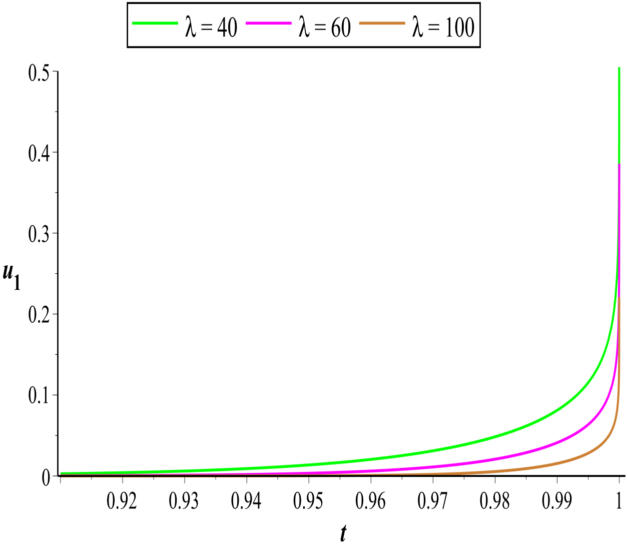

Figure 2. Numerical solution for $\lambda = 40, 60, 100$.

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Thank You