Subordination Principle, Stochastic Solutions and Feynman-Kac Formulae for Generalized Time-Fractional Evolution Equations

Yana Kinderknecht (Butko)

UNI KASSEL VERSITÄT

Seminar on Analysis, Differential Equations and Mathematical Physics
30. November, 2023

Classical Setting:

$(X_t)_{t\geqslant 0}$ is a time-homogeneous Markov process with state space $(Q, \mathcal{B}(Q))$ ($\Rightarrow$ no memory) $\Rightarrow$ transition kernel $P(t,x,dy) = \mathbb{P}(X_t \in dy \,|\, X_0 = x)$. Hence:

- $$(T_t)_{t\geqslant 0}, \quad T_t u_0(x) := \int_Q u_0(y) P(t,x,dy) \equiv \mathbb{E}^x\left[u_0(X_t)\right]$$

is an operator semigroup (i.e. $T_0 = \mathrm{Id}$, $T_t \circ T_s = T_{t+s}$).

- If $(T_t)_{t\geqslant 0}$ is a C_0-SG with generator $(L, \mathrm{Dom}(L))$ on a BS $\mathbf{X} \subset F(Q)$ then

$$u(t,x) := T_t u_0(x) \equiv \mathbb{E}^x\left[u_0(X_t)\right]$$

solves the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = Lu(t,x), \quad u(0,x) = u_0(x).$$

And

$$u(t,x) = u_0(x) + \int_0^t Lu(s,x)ds.$$

$(X_t)_{t\geqslant 0}$ is a time-homogeneous Markov process with state space $(Q, \mathcal{B}(Q))$ ($\Rightarrow$ no memory) $\Rightarrow$ transition kernel $P(t,x,dy) = \mathbb{P}(X_t \in dy \,|\, X_0 = x)$. Hence:

- $$(T_t)_{t\geqslant 0}, \quad T_t u_0(x) := \int_Q u_0(y) P(t,x,dy) \equiv \mathbb{E}^x\left[u_0(X_t)\right]$$

is an operator semigroup (i.e. $T_0 = \mathrm{Id}$, $T_t \circ T_s = T_{t+s}$).

- If $(T_t)_{t\geqslant 0}$ is a C_0-SG with generator $(L, \mathrm{Dom}(L))$ on a BS $\mathbf{X} \subset F(Q)$ then

$$u(t,x) := T_t u_0(x) \equiv \mathbb{E}^x\left[u_0(X_t)\right]$$

solves the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = Lu(t,x), \quad u(0,x) = u_0(x).$$

And

$$u(t,x) = u_0(x) + \int_0^t Lu(s,x)ds.$$

Basic pair: Heat equation $\leftrightsquigarrow$ Brownian motion.

$$u(t,x) = \mathbb{E}^x[u_0(B_t)] \quad \text{solves} \quad \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x)$$

Classical Setting

Basic couple: Heat equation $\leftrightsquigarrow$ Brownian motion.

$$u(t,x) = \mathbb{E}^x[u_0(B_t)] \quad \text{solves} \quad \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x)$$

$\Rightarrow$ **classical diffusion**

Basic features of classical diffusion:

- $Var(B_t) = t \quad \leftrightsquigarrow \quad \left\langle (\Delta x)^2 \right\rangle \sim \Delta t$
- Gaussian law

Anomalous diffusion

$$\left\langle (\Delta x)^2 \right\rangle \nsim \Delta t,$$

or / and:

NO Gaussian law

$\langle (\Delta x)^2 \rangle \nsim \Delta t$, or / and: NO Gaussian law

Anomalous diffusion has been observed in:

- **Subdiffusion:**
  - many complex self-organized systems in biology (e.g., motion of macromolecules in the cell cytoplasm and membrane);
  - charge carrier transport in amorphous semiconductors;
  - nuclear magnetic resonance (NMR) diffusometry in percolative and porous systems;
  - Rouse or reptation dynamics in polymeric systems;
  - transport on fractal geometries;
  - diffusion of a scalar tracer in an array of convection rolls;
  - dynamics of a bead in a polymeric network;
  - Sinai-type disorder;
  - ageing models....
- **Superdiffusion (incl. Lévy flights):**
  - Richardson's turbulent diffusion;
  - transport in turbulent plasma;
  - special domains of rotating flows;
  - quantum optics;
  - single molecule spectroscopy;
  - diffusion with search mechanisms (e.g., in cells)....

Some references:

Metzler R., Klafter J. (2000)
The random walk's guide to anomalous diffusion: a fractional dynamics approach
Phys. Rep. **339**, p.1-77.

Metzler R., Jeon J.H., Cherstvy A.G., Barkai E. (2014)
Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking
Phys. Chem. Chem. Phys. **16**, p. 24128-24164.

Sposini V., Chechkin A. V., Seno F., Pagnini G., Metzler R. (2018)
Random diffusivity from stochastic equations: comparison of two models for Brownian yet non-Gaussian diffusion
New J. Phys. **20**, 043044.

Anomalous diffusion

Heat equation $\rightsquigarrow$ time- and space-fractional heat equation:

$$\partial_t^\beta u(t,x) = -\left(-\frac{1}{2}\Delta\right)^\gamma u(t,x), \quad \beta \in (0,1], \quad \gamma \in (0,1],$$

i.e.

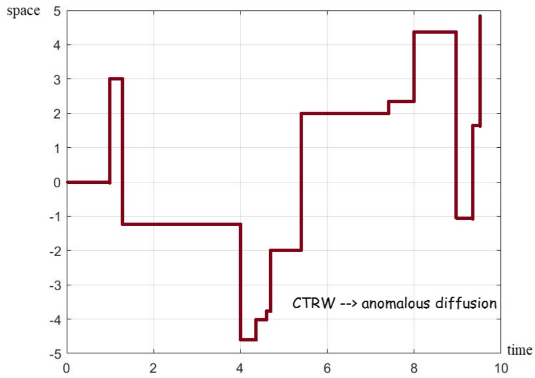
$$u(t,x) = u_0(x) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(-\frac{1}{2}\Delta\right)^\gamma u(s,x) ds.$$

Anomalous diffusion

Heat equation $\rightsquigarrow$ time- and space-fractional heat equation:

$$u(t,x) = u_0(x) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(-\frac{1}{2}\Delta\right)^\gamma u(s,x)dx, \quad \beta \in (0,1],\ \gamma \in (0,1].$$

This equation is a governing equation for scaling limits of CTRWs!

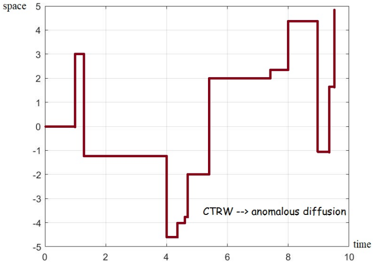


Anomalous diffusion

Heat equation $\rightsquigarrow$ time- and space-fractional heat equation:

$$u(t,x) = u_0(x) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(-\frac{1}{2}\Delta\right)^{\gamma} u(s,x)\,dx, \quad \beta \in (0,1],\ \gamma \in (0,1].$$

This equation is a governing equation for scaling limits of CTRWs!



The corresponding scaling limits of CTRWs are processes $(\xi_{\mathcal{E}_t})_{t\geqslant 0}$

Stochastic representations via inverse subordinators

Inverse subordinators are actively used to solve time-fractional evolution equations of the following form

$$u(t,x) = u_0(x) + \int_0^t k(t-s)Lu(s,x)ds, \qquad (**)$$

where $(L, \mathrm{Dom}(L))$ is the generator of a Markov process $(\xi_t)_{t\geqslant 0}$, and the convolution kernel k has Laplace transform $\frac{1}{h(\cdot)}$ for some BF h.
Then

$$h \quad \leftrightsquigarrow \quad (\eta^h_t)_{t\geqslant 0} \quad \leftrightsquigarrow \quad (\mathcal{E}^h_t)_{t\geqslant 0},$$

where $\mathbb{E}\left[e^{-\sigma\eta^h_t}\right] = e^{-th(\sigma)}$ and $\mathcal{E}^h_t := \inf\left\{s > 0 : \eta^h_s > t\right\}$. And

$$u(t,x) = \mathbb{E}^x\left[u_0\left(\xi_{\mathcal{E}^h_t}\right)\right]$$

provides a solution to (**).
NOTE: $(\mathcal{E}^h_t)_{t\geqslant 0}$ and hence $\left(\xi_{\mathcal{E}^h_t}\right)_{t\geqslant 0}$ are NOT Markov!

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geqslant 0$, $A \perp\!\!\!\perp G$, G is a Gaussian process.

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geqslant 0$, $A \perp\!\!\!\perp G$, G is a Gaussian process.

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{A_\beta} B_t^{\alpha/2},$$

where $\left(B_t^{\alpha/2}\right)_{t\geqslant 0}$ is a fractional Brownian motion with $H := \alpha/2$, $A_\beta \perp\!\!\!\perp B^{\alpha/2}$, $A_\beta \geqslant 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\beta n+1)}$.

Stochastic representations via RSGP

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{A_\beta} B_t^{\alpha/2},$$

where $\left(B_t^{\alpha/2}\right)_{t\geqslant 0}$ is a fractional Brownian motion with $H := \alpha/2$,
$A_\beta \perp\!\!\!\perp B^{\alpha/2}$, $A_\beta \geqslant 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\beta n+1)}$.

Recall: fractional Brownian motion (fBm) is a centered Gaussian process $(B_t^H)_{t\geqslant 0}$ that starts at zero and has the following covariance function:

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where $H \in (0,1)$ is called the **Hurst parameter** associated with the fBM.
The value of H determines what kind of process the fBm is:
if $H = 1/2$ then the process is in fact a Brownian motion;
if $H > 1/2$ then the increments of the process are positively correlated;
if $H < 1/2$ then the increments of the process are negatively correlated.

Def.: An $\mathbb{R}$–valued stoch. process $(X_t)_{t\geqslant 0}$ is called an H–SSSI process if

1. it is a **self-similar process** with Hurst parameter H, i.e. $(X_{at})_{t\geqslant 0}$ and $(a^H X_t)_{t\geqslant 0}$ have the same finite-dimensional distributions for any $a > 0$;
2. $(X_t)_{t\geqslant 0}$ has **stationary increments**, i.e. $X_t - X_s \sim X_{t+\tau} - X_{s+\tau}$ for all $t > s \geqslant 0$, $\tau > 0$.

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geqslant 0$, $A \perp\!\!\!\perp G$, G is a Gaussian process.

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{A_\beta} B_t^{\alpha/2},$$

where $\left(B_t^{\alpha/2}\right)_{t\geqslant 0}$ is a fractional Brownian motion with $H := \alpha/2$, $A_\beta \perp\!\!\!\perp B^{\alpha/2}$, $A_\beta \geqslant 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^\infty \frac{(-\lambda)^n}{\Gamma(\beta n+1)}$.

- $\left(X_t^{\alpha,\beta}\right)_{t\geqslant 0}$ is $\alpha/2$-SSSI

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geqslant 0$, $A \perp\!\!\!\perp G$, G is a Gaussian process.

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{A_\beta} B_t^{\alpha/2},$$

where $\left(B_t^{\alpha/2}\right)_{t\geqslant 0}$ is a fractional Brownian motion with $H := \alpha/2$, $A_\beta \perp\!\!\!\perp B^{\alpha/2}$, $A_\beta \geqslant 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^\infty \frac{(-\lambda)^n}{\Gamma(\beta n+1)}$.

- $\left(X_t^{\alpha,\beta}\right)_{t\geqslant 0}$ is $\alpha/2$-SSSI
- $u(t,x) := \mathbb{E}\left[u_0\left(x + X_t^{\alpha,\beta}\right)\right]$ solves

$$u(t,x) = u_0(x) + \frac{\alpha}{\beta\Gamma(\beta)} \int_0^t s^{\frac{\alpha}{\beta}-1}\left(t^{\frac{\alpha}{\beta}} - s^{\frac{\alpha}{\beta}}\right)^{\beta-1} \frac{1}{2}\frac{\partial^2 u(s,x)}{\partial x^2} ds,$$

Stochastic representations via RSGP

Another approach: **randomly scaled Gaussian processes (RSGP)**.

$$X_t := AG_t,$$

where $A \geqslant 0$, $A \perp\!\!\!\perp G$, G is a Gaussian process.

Example: Generalized Grey Brownian Motion (GGBM)

$$X_t^{\alpha,\beta} := \sqrt{A_\beta} B_t^{\alpha/2},$$

where $\left(B_t^{\alpha/2}\right)_{t\geqslant 0}$ is a fractional Brownian motion with $H := \alpha/2$,
$A_\beta \perp\!\!\!\perp B^{\alpha/2}$, $A_\beta \geqslant 0$ and $\mathbb{E}[\exp(-\lambda A_\beta)] = E_\beta(-\lambda) := \sum_{n=0}^\infty \frac{(-\lambda)^n}{\Gamma(\beta n+1)}$.

- $u(t,x) := \mathbb{E}\left[u_0\left(x + X_t^{\alpha,\beta}\right)\right]$ solves

$$u(t,x) = u_0(x) + \frac{\alpha}{\beta\Gamma(\beta)} \int_0^t s^{\frac{\alpha}{\beta}-1}\left(t^{\frac{\alpha}{\beta}} - s^{\frac{\alpha}{\beta}}\right)^{\beta-1} \frac{1}{2}\frac{\partial^2 u(s,x)}{\partial x^2} ds,$$

- Pagnini-Paradisi 2016: Yet another RSGP solves

$$\partial_t^\beta u(t,x) = -(-\Delta)^\gamma u(t,x), \quad \beta \in (0,1], \quad \gamma \in (0,1].$$

Natural questions:

We consider evolution equations of the form

$$u(t,x) = u_0(x) + \int_0^t k(t,s)Lu(s,x)ds, \qquad (*)$$

where L generates a Markov process with a C_0-SG on some BS $\mathbf{X}$.

Questions:

1. Which classes of stochastic processes $(X_t)_{t\geqslant 0}$ can be used to solve the evolution equation (*)?
2. What is in common between $X_t := \xi_{\mathcal{E}_t}$ and $X_t := AG_t$?
3. To which extent can we use the RSGPs?

Ch. Bender, M. Bormann, Ya.A. Butko (2022)
Subordination principle and Feynman-Kac formulae for generalized time-fractional evolution equations
Fract. Calc. Appl. Anal. 25 N4, 1818-1836 (2022).

Ch. Bender, Ya.A. Butko (2022)
Stochastic solutions of generalized time-fractional evolution equations
Frac. Calc. Appl. Anal. **25**(2), 2022.

Marie Bormann (2021)
Feynman-Kac-Formeln für Lösungen einer Klasse von Integro-Differentialgleichungen
Masterarbeit, Institut für Mathematische Stochastik, TU Braunschweig

Merten Mlinarzik (2023)
Stochastic calculus for randomly scaled Gaussian processes related to generalized time-fractional evolution equations
Masterarbeit, Institut für Mathematische Stochastik, TU Braunschweig

Generalized time-fractional evolution equations

Generalized time-fractional evolution equations

We consider

$$u(t) = u_0 + \int_0^t k(t,s) Lu(s) ds, \qquad (*)$$

where $(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$,

We consider

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

where $(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$, kernel k satisfies

Assumption 1: We consider a Borel-measurable kernel $k : (0,\infty) \times (0,\infty) \to \mathbb{R}$ satisfying the following condition: $\exists\, \alpha^* \in [0,1)$ and $\exists\, \varepsilon > 0$ such that for each $T > 0$

$$K_T := \sup_{0 < t \leqslant T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \| k(t,\cdot) \|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Generalized time-fractional evolution equations

We consider

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

where $(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t\geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$, kernel k satisfies

Assumption 1: We consider a Borel-measurable kernel $k : (0,\infty)\times(0,\infty) \to \mathbb{R}$ satisfying the following condition: $\exists\, \alpha^* \in [0,1)$ and $\exists\, \varepsilon > 0$ such that for each $T > 0$

$$K_T := \sup_{0<t\leqslant T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t,\cdot)\|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Question: Which processes $(X_t)_{t\geqslant 0}$ solve (*)?

We consider

$$u(t) = u_0 + \int_0^t k(t,s) L u(s)\,ds, \qquad (*)$$

where $(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t\geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$, kernel k satisfies

Assumption 1: $\exists\, \alpha^* \in [0,1)$, $\exists\, \varepsilon > 0$ such that $\forall\, T > 0$

$$K_T := \sup_{0<t\leqslant T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \| k(t,\cdot) \|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Thm. 1: Under Assm. 1 $\forall\, t \geqslant 0$, the function $\Phi(t,\cdot) : \mathbb{C} \to \mathbb{C}$ given by

$$\Phi(t,\lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n,$$

$$c_0(t) := 1,\ c_n(t) := \begin{cases} \int_0^t k(t,s) c_{n-1}(s)\,ds, & \forall\ t > 0, \\ 0 & t = 0, \end{cases} \qquad n \in \mathbb{N},$$

is well-defined and entire.

We consider

$$u(t) = u_0 + \int_0^t k(t,s) Lu(s) ds, \qquad (*)$$

where $(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t\geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$, kernel k satisfies

Assumption 1: $\exists\, \alpha^* \in [0,1)$, $\exists\, \varepsilon > 0$ such that $\forall\ T > 0$

$$K_T := \sup_{0<t\leqslant T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \| k(t,\cdot) \|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Thm. 1: $k \rightsquigarrow \Phi(t,\lambda) := \sum_{n=0}^\infty c_n(t)\lambda^n$, $c_n(t) := \int_0^t k(t,s) c_{n-1}(s) ds$.

We consider

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

where $(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$, kernel k satisfies

Assumption 1: $\exists\, \alpha^* \in [0,1)$, $\exists\, \varepsilon > 0$ such that $\forall\, T > 0$

$$K_T := \sup_{0<t\leqslant T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \| k(t,\cdot) \|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Thm. 1: $k \rightsquigarrow \Phi(t,\lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n$, $c_n(t) := \int_0^t k(t,s) c_{n-1}(s) ds$.

Assumption 2: $\Phi(t,-\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t\geqslant 0}$:

$$\int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t,-\lambda), \qquad \forall\, \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Assumption 1: $\exists\, \alpha^* \in [0,1)$, $\exists\, \varepsilon > 0$ such that $\forall\ T > 0$

$$K_T := \sup_{0 < t \leqslant T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \| k(t,\cdot) \|_{L^{1+\varepsilon}((0,t))} < \infty.$$

Thm. 1: $k \ \rightsquigarrow\ \Phi(t,\lambda) := \sum_{n=0}^{\infty} c_n(t) \lambda^n$, $c_n(t) := \int_0^t k(t,s) c_{n-1}(s) ds$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geqslant 0}$:

$$\int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \qquad \forall\, \lambda \in \mathbb{C}, \quad \mathrm{Re}\, \lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:
(i) For each $t \geqslant 0$, the operator $\Phi(t, L)$ given by the Bochner integral

$$\Phi(t,L)\varphi := \int_0^\infty T_a \varphi\, \mathcal{P}_{A(t)}(da), \qquad \varphi \in X, \qquad (1)$$

is well defined and it is a bounded linear operator on X.

$$u(t) = u_0 + \int_0^t k(t,s) L u(s)\,ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t\geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Assumption 1: $K_T := \sup\limits_{0<t\leqslant T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t,\cdot)\|_{L^{1+\varepsilon}((0,t))} < \infty.$

Thm. 1: $k \rightsquigarrow \Phi(t,\lambda) := \sum_{n=0}^\infty c_n(t)\lambda^n$, $c_n(t) := \int_0^t k(t,s) c_{n-1}(s)\,ds.$

Assumption 2: $\Phi(t,-\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t\geqslant 0}$:

$$\int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t,-\lambda), \qquad \forall\, \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:
(ii) For each $t > 0$ and each $u_0 \in \mathrm{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0\, \mathcal{P}_{A(t)}(da)$$

solves equation (*) and it holds $\lim_{t\searrow 0} u(t) = u_0$.

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Thm. 1: $k \rightsquigarrow \Phi(t,\lambda) := \sum_{n=0}^{\infty} c_n(t)\lambda^n$, $c_n(t) := \int_0^t k(t,s) c_{n-1}(s) ds$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geqslant 0}$:

$$\mathbb{E}\left[e^{-\lambda A(t)}\right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \qquad \forall\, \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:
(ii) For each $t > 0$ and each $u_0 \in \mathrm{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \, \mathcal{P}_{A(t)}(da) \equiv \mathbb{E}\left[T_{A(t)} u_0\right]$$

solves equation (*) and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Rem.: If T_t has stoch. repr. then $\Phi(t, L)$ too!

- $L := L_0 \leftrightsquigarrow \xi =$ Markov process, $\xi \perp\!\!\!\perp (A(t))_{t \geqslant 0}$:

$$T_t u_0(x) = \mathbb{E}^x\left[u_0(\xi_t)\right] \qquad \Rightarrow \qquad \Phi(t, L) u_0(x) = \mathbb{E}^x\left[u_0(\xi_{A(t)})\right]$$

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geqslant 0}$:

$$\mathbb{E}\left[e^{-\lambda A(t)}\right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \qquad \forall \lambda \in \mathbb{C}, \quad \mathrm{Re}\, \lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:
(ii) For each $t > 0$ and each $u_0 \in \mathrm{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \, \mathcal{P}_{A(t)}(da) \equiv \mathbb{E}\left[T_{A(t)} u_0\right]$$

solves equation (*) and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Rem.: If T_t has stoch. repr. then $\Phi(t, L)$ too!

- $L_0 \leftrightsquigarrow \xi =$ Markov process, $\xi \perp\!\!\!\perp (A(t))_{t \geqslant 0}$,
- $L := L_0 + V$, then $T_t u_0(x) = \mathbb{E}^x\left[u_0(\xi_t) e^{\int_0^t V(\xi_s) ds}\right]$,

$$\Phi(t, L) u_0(x) = \mathbb{E}^x\left[u_0(\xi_{A(t)}) e^{\int_0^{A(t)} V(\xi_s) ds}\right]$$

$$u(t) = u_0 + \int_0^t k(t,s) L u(s)\,ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t\geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Assumption 2: $\Phi(t,-\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t\geqslant 0}$:

$$\mathbb{E}\left[e^{-\lambda A(t)}\right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t,-\lambda), \qquad \forall\, \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:
(ii) For each $t > 0$ and each $u_0 \in \mathrm{Dom}(L)$, the function

$$u(t) := \Phi(t,L)u_0 := \int_0^\infty T_a u_0\, \mathcal{P}_{A(t)}(da) \equiv \mathbb{E}\left[T_{A(t)} u_0\right]$$

solves equation (*) and it holds $\lim_{t\searrow 0} u(t) = u_0$.

Rem.: If T_t has stoch. repr. then $\Phi(t,L)$ too!

- $L_0 \leftrightsquigarrow \xi =$ Markov process, $\xi \perp\!\!\!\perp (A(t))_{t\geqslant 0}$,
- $L := L_0 + V$, then $T_t u_0(x) = \mathbb{E}^x\left[u_0(\xi_t) e^{\int_0^t V(\xi_s)ds}\right]$,

$$\Phi(t,L)u_0(x) = \mathbb{E}^x\left[u_0(\xi_{A(t)}) e^{\int_0^{A(t)} V(\xi_s)ds}\right]$$

- $L :=$ generator of a subordinate semigroup; Schrödinger group......

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t\geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t\geqslant 0}$:

$$\mathbb{E}\left[e^{-\lambda A(t)}\right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \qquad \forall\, \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:
(ii) For each $t > 0$ and each $u_0 \in \mathrm{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \, \mathcal{P}_{A(t)}(da) \equiv \mathbb{E}\left[T_{A(t)} u_0\right]$$

solves equation (*) and it holds $\lim_{t\searrow 0} u(t) = u_0$.

Thm. 3: Under Assm. 1, Assm. 2 let k be **homogeneous** of $\theta - 1$ for some $\theta > 0$, i.e.

$$k(t, ts) = t^{\theta-1} k(1, s) \qquad \forall \quad t > 0, s \in (0,1).$$

Then one can choose $A(t) := At^\theta$, where A is a nonnegative RV such that

$$\int_0^\infty e^{-\lambda a} \mathcal{P}_A(da) = \Phi(1, -\lambda) \qquad \forall \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

$$u(t) = u_0 + \int_0^t k(t,s) L u(s)\, ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t\geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t\geqslant 0}$:

$$\mathbb{E}\left[e^{-\lambda A(t)}\right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \qquad \forall \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \mathrm{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \, \mathcal{P}_{A(t)}(da) \equiv \mathbb{E}\left[T_{A(t)} u_0\right]$$

solves equation (*) and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Thm. 3: $k(t, ts) = t^{\theta-1} k(1, s)$. Then $A(t) := A t^\theta$ is OK, $A \leftrightsquigarrow \Phi(1, -\cdot)$.

Example:

- $k(t,s) := \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$, $\beta \in (0,1]$. Then $\theta = \beta$; $\Phi(t, -\lambda) = E_\beta(-t^\beta \lambda)$=CMF

$$u(t) = u_0 + \int_0^t k(t,s) L u(s)\,ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t\geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t\geqslant 0}$:

$$\mathbb{E}\left[e^{-\lambda A(t)}\right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \qquad \forall\, \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \mathrm{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0\, \mathcal{P}_{A(t)}(da) \equiv \mathbb{E}\left[T_{A(t)} u_0\right]$$

solves equation (*) and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Thm. 3: $k(t, ts) = t^{\theta - 1} k(1, s)$. Then $A(t) := A t^\theta$ is OK, $A \leftrightsquigarrow \Phi(1, -\cdot)$.

Example: Time-fractional heat equation:

- $k(t,s) := \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$, $\beta \in (0,1]$. Then $\theta = \beta$; $\Phi(t, -\lambda) = E_\beta(-t^\beta \lambda)$=CMF
- $L := \frac{1}{2}\Delta \leftrightsquigarrow (B_t)_{t\geqslant 0}$=Brownian motion

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Assumption 2: $\Phi(t, -\cdot)$ is a CMF, i.e. $\exists$ nonnegative RVs $(A(t))_{t \geqslant 0}$:

$$\mathbb{E}\left[e^{-\lambda A(t)}\right] \equiv \int_0^\infty e^{-\lambda a} \mathcal{P}_{A(t)}(da) = \Phi(t, -\lambda), \qquad \forall \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0.$$

Thm. 2 (Subordination principle): Under Assm. 1, Assm. 2 holds:

(ii) For each $t > 0$ and each $u_0 \in \mathrm{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 := \int_0^\infty T_a u_0 \, \mathcal{P}_{A(t)}(da) \equiv \mathbb{E}\left[T_{A(t)} u_0\right]$$

solves equation (*) and it holds $\lim_{t \searrow 0} u(t) = u_0$.

Thm. 3: $k(t, ts) = t^{\theta - 1} k(1, s)$. Then $A(t) := A t^\theta$ is OK, $A \leftrightsquigarrow \Phi(1, -\cdot)$.

Example: Time-fractional heat equation. Solution: $u(t, x) = \mathbb{E}^x\left[u_0(X_t)\right]$

- $k(t,s) := \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$, $\beta \in (0, 1]$. Then $\theta = \beta$; $\Phi(t, -\lambda) = E_\beta(-t^\beta \lambda)$=CMF
- $L := \frac{1}{2}\Delta \leftrightsquigarrow (B_t)_{t \geqslant 0}$=Brownian motion

$$X_t: \quad B_{A(t)} \leftrightsquigarrow B_{At^\beta} \leftrightsquigarrow \sqrt{A} B_{t^\beta} \leftrightsquigarrow \sqrt{A} B_t^{\beta/2}$$

$$u(t) = u_0 + \int_0^t k(t,s) L u(s)\,ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Cor. 4: Under Assm. 1, Assm. 2. suppose $k(t,s) := \mathfrak{K}(t-s)$, where $\mathcal{L}\mathfrak{K} = 1/h$ for some Bernstein function h. Then

$$h \quad \leftrightsquigarrow \quad (\eta_t^h)_{t \geqslant 0} \quad \leftrightsquigarrow \quad (\mathcal{E}_t^h)_{t \geqslant 0},$$

where $\mathbb{E}\left[e^{-\sigma \eta_t^h}\right] = e^{-th(\sigma)}$ and $\mathcal{E}_t^h := \inf\left\{s > 0 : \eta_s^h > t\right\}$. Hence, one may take $A(t) := \mathcal{E}_t^h$.

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Cor. 4: Under Assm. 1, Assm. 2. suppose $k(t,s) := \mathfrak{K}(t-s)$, where $\mathcal{L}\mathfrak{K} = 1/h$ for some Bernstein function h. Then

$$h \quad \leftrightsquigarrow \quad (\eta_t^h)_{t \geqslant 0} \quad \leftrightsquigarrow \quad (\mathcal{E}_t^h)_{t \geqslant 0},$$

where $\mathbb{E}\left[e^{-\sigma \eta_t^h}\right] = e^{-th(\sigma)}$ and $\mathcal{E}_t^h := \inf\left\{s > 0 : \eta_s^h > t\right\}$. Hence, one may take $A(t) := \mathcal{E}_t^h$.

Example: Time-fractional heat equation. Solution: $u(t,x) = \mathbb{E}^x\left[u_0(X_t)\right]$

- $k(t,s) := \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}$. Then $h(\sigma) = \sigma^\beta$.
- $L := \frac{1}{2}\Delta \leftrightsquigarrow (B_t)_{t \geqslant 0}$=Brownian motion

$$X_t : \quad B_{\mathcal{E}_t^h} \quad \leftrightsquigarrow \quad B_{A(t)} \quad \leftrightsquigarrow \quad B_{At^\beta} \quad \leftrightsquigarrow \quad \sqrt{A} B_{t^\beta} \quad \leftrightsquigarrow \quad \sqrt{A} B_t^{\beta/2}$$

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds, \qquad (*)$$

$(L, \mathrm{Dom}(L))$ generates a C_0-SG $(T_t)_{t \geqslant 0}$ on a BS X, $u_0 \in \mathrm{Dom}(L)$.

Cor. 4: Under Assm. 1, Assm. 2. suppose $k(t,s) := \mathfrak{K}(t-s)$, where $\mathcal{L}\mathfrak{K} = 1/h$ for some Bernstein function h. Then one may take $A(t) := \mathcal{E}_t^h$.

Rem.: In the situation of Cor. 4, eq. (*) is equivalent to Cauchy problem

$$\mathcal{D}_t^h u(t,x) = Lu(t,x), \qquad u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d, \quad t > 0,$$

where $\mathcal{D}_t^h$ is a generalized time-fractional derivative of Caputo type:

$$\left(\mathcal{L}\left[\mathcal{D}_t^h v\right]\right)(\sigma) = h(\sigma)(\mathcal{L}v)(\sigma) - \frac{h(\sigma)}{\sigma} v(+0).$$

Special cases:

- Caputo derivative of order β: $h(\sigma) = \sigma^\beta$;
- a mixture of Caputo derivatives of orders β, $\beta_1, \ldots, \beta_m$: $h_1(\sigma) := \left(\sigma^{-\beta} + \sum_{j=1}^m b_j \sigma^{-\beta_j}\right)^{-1}$;
- distributed order derivative: $h(\sigma) := \int_0^1 \sigma^\beta \mu(d\beta)$.

Example 2 (Marichev-Saigo-Maeda kernels):

Example 2 (MSM kernels): Appell's third generalization F_3 of the Gauss hypergeometric function is defined by:

$$F_3\left(\alpha, \alpha', \beta, \beta', \gamma, x, y\right) = \sum_{m,n \geqslant 0} \frac{(\alpha)_m (\beta)_m (\alpha')_n (\beta')_n}{(\gamma)_{m+n} n! m!} x^m y^n, \qquad (1)$$

where $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$, $\gamma \notin -\mathbb{N}$, and the general Pochhammer symbol $(\lambda)_\nu$ is defined as follows:

$$(\lambda)_\nu := \begin{cases} 1, & \nu = 0, \quad \lambda \in \mathbb{C} \\ \lambda(\lambda - 1) \cdot \ldots \cdot (\lambda + n - 1), & \nu = n \in \mathbb{N}, \quad \lambda \in \mathbb{C}. \end{cases}$$

The series in (1) converges for $|y|, |x| < 1$ and can be analytically extended to reals $x, y < 1$.

Example 2 (MSM kernels): Appell's third generalization F_3 of the Gauss hypergeometric function is defined by:

$$F_3\left(\alpha,\alpha',\beta,\beta',\gamma,x,y\right)=\sum_{m,n\geqslant 0}\frac{(\alpha)_m(\beta)_m(\alpha')_n(\beta')_n}{(\gamma)_{m+n}n!m!}x^m y^n, \qquad (1)$$

where $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$, $\gamma\notin-\mathbb{N}$, and the general Pochhammer symbol $(\lambda)_\nu$ is defined as follows:

$$(\lambda)_\nu := \begin{cases} 1, & \nu=0, \quad \lambda\in\mathbb{C} \\ \lambda(\lambda-1)\cdot\ldots\cdot(\lambda+n-1), & \nu=n\in\mathbb{N}, \quad \lambda\in\mathbb{C}. \end{cases}$$

The series in (1) converges for $|y|,|x|<1$ and can be analytically extended to reals $x,y<1$.

For $b>0$, $a>0$, $\mu>-1$, and $\nu>\max\{-b,-a\mu\}$ we consider the **MSM kernel**

$$k(t,s):=\frac{a}{\Gamma(b/a)}(t^a-s^a)^{\frac{b}{a}-1}t^{a-\nu}s^{\nu-1}F_3\left(\frac{\nu}{a}-1,\frac{b}{a},1,\mu,\frac{b}{a},1-\left(\frac{s}{t}\right)^a,1-\left(\frac{t}{s}\right)^a\right)$$

Thm: Let $b > 0$, $a > 0$, $\mu > -1$, and $\nu > \max\{-b, -a\mu\}$. Consider the kernel

$$k(t,s) := \frac{a}{\Gamma(b/a)} (t^a - s^a)^{\frac{b}{a}-1} t^{a-\nu} s^{\nu-1} F_3\left(\frac{\nu}{a} - 1, \frac{b}{a}, 1, \mu, \frac{b}{a}, 1 - \left(\frac{s}{t}\right)^a, 1 - \left(\frac{t}{s}\right)^a\right)$$

where $0 < s < t$. Then the kernel k is homogeneous of degree $b - 1$ and satisfies $k(1,\cdot) \in L^{1+\varepsilon}((0,1))$ for some $\varepsilon > 0$. The corresponding Φ is

$$\Phi(1,z) = \Gamma(\lambda_2) E^{\lambda_3}_{\lambda_1,\lambda_2}(z),$$

where $\lambda_1 = \frac{b}{a}$, $\lambda_2 = \frac{\nu}{a} + \mu$, $\lambda_3 = 1 + \frac{\nu - a}{b}$, and $E^{\lambda_3}_{\lambda_1,\lambda_2}$ is the three parameter Mittag-Leffler (or Prabhakar) function

$$E^{\lambda_3}_{\lambda_1,\lambda_2}(z) := \sum_{n=0}^{\infty} \frac{(\lambda_3)_n}{\Gamma(\lambda_1 n + \lambda_2)\, n!} z^n.$$

Thm: Let $b \in (0,1]$, $a > 0$, $\mu > -1$, and $\nu > \max\{-b, -a\mu\}$. Consider the kernel

$$k(t,s) := \frac{a}{\Gamma(b/a)} (t^a - s^a)^{\frac{b}{a}-1} t^{a-\nu} s^{\nu-1} F_3\left(\frac{\nu}{a} - 1, \frac{b}{a}, 1, \mu, \frac{b}{a}, 1 - \left(\frac{s}{t}\right)^a, 1 - \left(\frac{t}{s}\right)^a\right)$$

where $0 < s < t$. Then the kernel k is homogeneous of degree $b - 1$ and satisfies $k(1,\cdot) \in L^{1+\varepsilon}((0,1))$ for some $\varepsilon > 0$. The corresponding Φ is:

$$\Phi(1,z) = \Gamma(\lambda_2) E^{\lambda_3}_{\lambda_1,\lambda_2}(z),$$

where $\lambda_1 = \frac{b}{a}$, $\quad \lambda_2 = \frac{\nu}{a} + \mu$, $\quad \lambda_3 = 1 + \frac{\nu - a}{b}$, $\quad$ and $E^{\lambda_3}_{\lambda_1,\lambda_2}$ is the three parameter Mittag-Leffler (or Prabhakar) function

$$E^{\lambda_3}_{\lambda_1,\lambda_2}(z) := \sum_{n=0}^{\infty} \frac{(\lambda_3)_n}{\Gamma(\lambda_1 n + \lambda_2)\, n!} z^n.$$

Sufficient conditions for $z \mapsto \Gamma(\lambda_2) E^{\lambda_3}_{\lambda_1,\lambda_2}(-z)$ to be CM are $0 < \lambda_1 \leqslant 1$, $0 < \lambda_3 \leqslant \frac{\lambda_2}{\lambda_1}$. This implies additional assumptions on parameters a, b, μ, ν:

$$b \leqslant a, \quad \nu > a - b, \quad \mu \geqslant \frac{b}{a} - 1.$$

Special cases:

(i) Let $\lambda_3 = 1$, i.e. $\nu = a$. Then $E^{\lambda_3}_{\lambda_1,\lambda_2}$ reduces to the two parameter Mittag-Leffler function E_{λ_1,λ_2} and we have

$$\Phi(t,z) = \Gamma(\mu+1)E_{\frac{b}{a},\mu+1}(zt^b).$$

Further, the corresponding kernel k simplifies to

$$k(t,s) = \frac{a}{\Gamma(b/a)}(t^a - s^a)^{b/a-1}t^{-a\mu}s^{a(\mu+1)-1}.$$

Special cases:

(i) Let $\lambda_3 = 1$, i.e. $\nu = a$. Then $E^{\lambda_3}_{\lambda_1,\lambda_2}$ reduces to the two parameter Mittag-Leffler function E_{λ_1,λ_2} and we have

$$\Phi(t,z) = \Gamma(\mu+1) E_{\frac{b}{a},\mu+1}(zt^b).$$

Further, the corresponding kernel k simplifies to

$$k(t,s) = \frac{a}{\Gamma(b/a)} (t^a - s^a)^{b/a-1} t^{-a\mu} s^{a(\mu+1)-1}.$$

(ii) Let $\lambda_2 = \lambda_3 = 1$, i.e. $\nu = a$ and $\mu = 0$. Then $E^{\lambda_3}_{\lambda_1,\lambda_2}$ reduces to the classical Mittag-Leffler function E_{λ_1} and we have

$$\Phi(t,z) = E_{\frac{b}{a}}(zt^b).$$

The corresponding kernel k simplifies to

$$k(t,s) = \frac{a}{\Gamma(b/a)} (t^a - s^a)^{b/a-1} s^{a-1}.$$

Let now $\beta \in (0,1]$, $\alpha \in (0,2)$. Choosing $b := \alpha$ and $a := \frac{\alpha}{\beta}$ we obtain the kernel of the governing equation of the GGBM and $\Phi(t,z) = E_\beta(zt^\alpha)$. If additionally $\alpha = \beta$, we get $k(t,s) = \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}$ and $\Phi(t,z) = E_\beta(zt^\beta)$.

Stochastic solutions with stationary increments

Stochastic solutions with stationary increments

Which equations can be solved by RSGPs and related SSSI-processes?

Stochastic solutions with stationary increments

Which equations can be solved by RSGPs and related SSSI-processes?

- For k, θ and A as in Thm. 3, $\mathcal{A} := A^{1/\gamma}\eta_1^h$, $h(\sigma) := \sigma^\gamma$, $c \leqslant 0$, $w \in \mathbb{R}^d$,

$$u(t,x) = \mathbb{E}\left[u_0\left(x + X_t^{\mathcal{A},\gamma,\theta} + \mathcal{A}wt^{\theta/\gamma}\right)e^{c\mathcal{A}t^{\theta/\gamma}}\right],$$

solves the evolution equation

$$u(t,x) = u_0(x) - \int_0^t k(t,s)\left(-\frac{1}{2}\Delta - w\nabla - c\right)^\gamma u(s,x)ds.$$

Here $X_t^{\mathcal{A},\gamma,\theta} := B_{\mathcal{A}t^{\theta/\gamma}}$ or $X_t^{\mathcal{A},\gamma,\theta} := \sqrt{\mathcal{A}}B_{t^{\theta/\gamma}}$, or, if $H := \frac{\theta}{2\gamma} \in (0,1)$, $X_t^{\mathcal{A},\gamma,\theta} := \sqrt{\mathcal{A}}B_t^H$, where $\left(B_t^H\right)_{t\geqslant 0}$ is a d-dim. fBm with Hurst parameter H which is independent from A and $(\eta_t^h)_{t\geqslant 0}$.

Stochastic solutions with stationary increments

- For k, θ and A as in Thm. 3,

$$u(t,x) = \mathbb{E}\left[u_0\left(g_\sigma\left(X_t^{\mathcal{A},\gamma,\theta}, x\right)\right) e^{\mathcal{A}t^{\theta/\gamma}\left(c-\frac{w^2}{2}\right)+wX_t^{\mathcal{A},\gamma,\theta}}\right]$$

solves the evolution equation

$$u(t,x) = u_0(x) - \int_0^t k(t,s)\left(-L_{(\sigma,w)} - c\right)^\gamma u(s,x)ds,$$

where
$$L_{(\sigma,w)}\varphi(x) := \frac{\sigma^2(x)}{2}\frac{d^2}{dx^2}\varphi(x) + \left(w + \frac{1}{2}\sigma'(x)\right)\sigma(x)\frac{d}{dx}\varphi(x),$$

$g_\sigma : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is the solution to the parametrized family of ODEs

$$\frac{\partial}{\partial y}g_\sigma(y,x) = \sigma(g_\sigma(y,x)), \qquad g_\sigma(0,x) = x,$$

and $X_t^{\mathcal{A},\gamma,\theta} := B_{\mathcal{A}t^{\theta/\gamma}}$ or $X_t^{\mathcal{A},\gamma,\theta} := \sqrt{\mathcal{A}}B_{t^{\theta/\gamma}}$, or, if $H := \frac{\theta}{2\gamma} \in (0,1)$, $X_t^{\mathcal{A},\gamma,\theta} := \sqrt{\mathcal{A}}B_t^H$, where $\left(B_t^H\right)_{t\geqslant 0}$ is a 1-dim. fBm with Hurst parameter H which is independent from $\mathcal{A}$.

Stochastic solutions with stationary increments

Which equations can be solved by RSGPs and related SSSI-processes?

• Similar results with linear fractional stable motion instead of fBm

Next step:

Stochastic calculus with respect to RSGP $\Rightarrow$ new processes and new equations!

Example: Consider a kernel $k(t,s)$ as in Assm.1 and Assm.2.
$k \rightsquigarrow X_t := \sqrt{A} B_t^H$, $H \in (0,1)$ as above. Define $Z_t := \int_0^t \nu(s)\delta X_s$. Then:

$$u(t,x) = \mathbb{E}\left[u_0\left(x + Z_t\right)\right]$$

solves evolution equation with the time-changed kernel

$$u(t,x) = u_0 + \int_0^t k(\sigma(t),\sigma(s))\sigma'(s)\frac{1}{2}\Delta u(s,x)ds,$$

where

$$\sigma(t) := \left\| M_-^H \mathbb{1}_{(0,1)}\nu \right\|_2^{1/H}.$$