On the norm of the Riesz projection from L^∞ to L^p

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The one-dimensional case

The talk is based on our joint paper with Hervé Queffelléc, Eero Saksman, and Kristian Seip.

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The talk is based on our joint paper with Hervé Queffelléc, Eero Saksman, and Kristian Seip.
Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. We consider the normalized Lebesgue measure on $\mathbb{T}$: $d\mu = dx/(2\pi)$. Thus, $\mu(\mathbb{T}) = 1$. For $p \in [1, \infty)$, let $L^p(\mathbb{T})$ be the set of integrable functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_p^p := \int_{\mathbb{T}} |f|^p d\mu < \infty.$$

Also, let $L^\infty(\mathbb{T})$ be the space of essentially bounded functions $f : \mathbb{T} \to \mathbb{C}$. Next, for every function $f \in L(\mathbb{T}) := L^1(\mathbb{T})$ we define its trigonometric Fourier series

$$f \sim \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx},$$

where

$$\hat{f}(k) = \int_{\mathbb{T}} f(x) e^{-ikx} d\mu.$$

For a function $f \in L(\mathbb{T})$ we can try to define

$$P^+ f = \sum_{k \in \mathbb{Z}_+} \hat{f}(k) e^{ikx},$$

The function $P^+ f$ is well-defined for $f \in L^p(\mathbb{T})$, $p > 1$. Actually, $P^+ f$ can be defined for any $f \in L(\mathbb{T})$, but in general it is not an integrable function. If $1 < p < \infty$, then, by the classical theorem of M. Riesz, the operator P^+ is bounded as an operator $L^p \to L^p$.

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Clearly, for $p = 2$ the norm of this operator is 1. B. Hollenbeck and I.E. Verbitsky (2000) proved that for any $p \in (1, \infty)$ the norm is $1/\sin(\pi/p)$. We observe that this is greater than 1 if $p \neq 2$.

For $2 \leq p, q \leq \infty$ we denote by $\|P^+\|_{q,p}$ the norm of the operator $P^+ : L^q(\mathbb{T}) \to L^p(\mathbb{T})$. It is easy to see that $\|P^+\|_{q,p} = \infty$ if $p > q$ or $p = q = \infty$. We will assume that $2 \leq p \leq q$ and $p < \infty$. Since $\|f\|_q \geq \|f\|_p$, we get

$$\|P^+\|_{q,p} \leq \|P^+\|_{p,p} < \infty.$$

In particular, $\|P^+\|_{q,2} \leq 1$. Since always $\|P^+\|_{q,p} \geq 1$, we conclude that $\|P^+\|_{q,2} = 1$ for any $q \geq 2$.

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Although we know that $\|P^+\|_{p,p} > 1$ for any $p > 2$, it can happen that $\|P^+\|_{q,p} = 1$ for some $q > p > 2$. It was a nice question for which $p \geq 2$ the equality $\|P^+\|_{\infty,p} = 1$ holds. The answer was given by J. Marzo and K. Seip (2011). They showed that $\|P^+\|_{\infty,p} = 1$ for $p \leq 4$ and $\|P^+\|_{\infty,p} > 1$ for $p > 4$.

Let $p = 4$. We write $P^- = I - P^+$ where I is the identity operator. Thus,

$$P^- f = \sum_{-k \in \mathbb{N}} \hat{f}(k) e^{ikx},$$

We have $(P^+ f)^2 \perp (P^- f)^2$ whenever f is a bounded function on $\mathbb{T}$. Therefore,

$$\|P^+ f\|_4^4 = \|(P^+ f)^2\|_2^2 \le \|(P^+ f)^2 - (P^- f)^2\|_2^2$$

$$= \|f(P^+ f - P^- f)\|_2^2 \le \|f\|_\infty^2 \|f\|_2^2 \le \|f\|_\infty^4.$$

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This estimate implies that $\|P^+\|_{\infty,4} = 1$. Since the mapping $p \to \|P^+\|_{\infty,p}$ is nondecreasing, we conclude that $\|P^+\|_{\infty,p} = 1$ for $p \le 4$.

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We have to show that $\|P^+\|_{\infty,p} > 1$ for $p > 4$.

Let $0 < \varepsilon < 1/2$. J.Marzo and K. Seip define the function

$$f(x) = (1 - \varepsilon e^{ix})^2 / |1 - \varepsilon e^{ix}|^2.$$

Clearly, $\|f\|_p = 1$ It is possible to show that $P^+ f(x) = 1 - \varepsilon^2 - \varepsilon e^{ix}$. Next, one can use the power series expansion:

$$\|P^+ f\|_p^p = 1 + \left(\frac{p^2}{4} - p \right) \varepsilon^2 + O(\varepsilon^4)$$

when $\varepsilon \to 0+$. This gives $\|P^+ f\|_p > 1$ by choosing ε sufficiently small.

It is possible to get the equality $\|P^+\|_{q,p} = 1$ for some other pairs (p, q), $q > p > 2$. Indeed, we know that $\|P^+\|_{2,2} = 1$ and $\|P^+\|_{\infty,4} = 1$. Denote $q_0 = 2, p_0 = 2, q_1 = \infty, p_1 = 4$. Using the Riesz–Thorin interpolation theorem, we conclude that for any $\alpha \in (0, 1)$ and p, q given by the equalities

$$\frac{1}{p} = \alpha\frac{1}{p_0} + (1-\alpha)\frac{1}{p_1} = \frac{1+\alpha}{4},$$

$$\frac{1}{q} = \alpha\frac{1}{q_0} + (1-\alpha)\frac{1}{q_1} = \frac{\alpha}{2},$$

we have $\|P^+\|_{q,p} \le 1$ implying $\|P^+\|_{q,p} = 1$. In particular,

$$\|P^+\|_{4,8/3} = 1. \tag{1}$$

The multidimensional case

Also, J. Marzo and K. Seip (2011) study functions of n variables defined on $\mathbb{T}^n$. For $\mathbf{x} \in \mathbb{T}^n$ we write $\mathbf{x} = (x_1, \ldots, x_n)$. We use similar nonation for other variables, e.g. $\mathbf{k} = (k_1, \ldots, k_n)$. Also, denote $d\mu_j = dx_j/(2\pi)$. We define the n-dimensional probability measure on $\mathbb{T}^n$:

$$d\mu^{(n)} = dx_1 \ldots dx_n/(2\pi)^n.$$

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The spaces $L^p(\mathbb{T}^n)$ and the decomposition of functions $f \in L(\mathbb{T}^n)$ into the trigonometric Fourier series are defined in a natural way. For a function $f \in L(\mathbb{T}^n)$ we can try to define

$$P_n^+ f = \sum_{\mathbf{k} \in \mathbb{Z}_+^n} \hat{f}(\mathbf{k}) e^{i\mathbf{kx}},$$

The function $P_n^+ f$ is well-defined for $f \in L^p(\mathbb{T}^n)$, $p > 1$ since the theorem of M. Riesz can be extended to $\mathbb{T}^n$.

Again, for $2 \leq p \leq q$, $p < \infty$, we denote by $\|P_n^+\|_{q,p}$ the norm of the operator $P_n^+ : L^q(\mathbb{T}^n) \to L^p(\mathbb{T}^n)$. As we have noticed for the case $n = 1$, $1 \leq \|P_n^+\|_{q,p} < \infty$.

Again, for $2 \leq p \leq q$, $p < \infty$, we denote by $\|P_n^+\|_{q,p}$ the norm of the operator $P_n^+ : L^q(\mathbb{T}^n) \to L^p(\mathbb{T}^n)$. As we have noticed for the case $n = 1$, $1 \leq \|P_n^+\|_{q,p} < \infty$.
We observe that if we restrict the operator P_n^+ on the subspace of $L^q(\mathbb{T}^n)$ formed by the functions not depending on x_n then we get an operator with the norm $\|P_{n-1}^+\|_{q,p}$. This shows that the mapping $n \to \|P_n^+\|_{q,p}$ is nondecreasing.

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Also, the mapping $p \to \|P_n^+\|_{q,p}$ is nondecreasing. Moreover, it is not difficult to deduce from Hőlder's inequality that the last mapping is continuous (and even locally Lipshitzian).

J. Marzo and K. Seip consider the number

$$p_n = \sup\{p : \|P_n^+\|_{\infty,p} = 1\}.$$

This number was called the critical exponent by T. Fiegel, T. Iwaniec, and A. Pelchynski (1984). Clearly, $p_n \geq 2$.

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It follows from the previous discussion that $p_n \leq p_{n-1}$ for $n > 1$. We know that $p_1 = 4$. Hence, $p_n \leq 4$ for any n. Since the mapping $p \to \|P_n^+\|_{\infty,p}$ is nondecreasing and continuous, the critical exponent has the following property: $|P_n^+\|_{\infty,p} = 1$ if $2 \leq p \leq p_n$ and $|P_n^+\|_{\infty,p} > 1$ if $p > p_n$.

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For $n > 1$ the exact value of p_n is unknown, and the gap between lower and upper estimates is rather large.

J. Marzo and K. Seip (2011) established the following inequality:

$$p_n \geq 2 + 2/(2^n - 1).$$

Thus, $\|P_n^+\|_{\infty,p} = 1$ for $p = 2 + 2/(2^n - 1)$. This equality was proved by induction on n. For simplicity we will discuss the case $n = 2$. For larger n the proof (of the induction step) is essentially the same.

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$$P_{n,j}^+ f = \sum_{\mathbf{k} \in \mathbb{Z}^n, k_j \geq 0} \hat{f}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}},$$

The routine technique using Fubini's theorem gives

$$\|P_{n,j}^+\|_{q,p} = \|P_1^+\|_{q,p}. \tag{2}$$

The Riesz operator P_2^+ is a composition of two partial Riesz operators $P_{2,1}^+$ and $P_{2,2}^+$. By (2) and the one-dimensional result of J. Marzo and K. Seip,

$$\|P_{2,1}^+\|_{\infty,4} = \|P_1^+\|_{\infty,4} = 1.$$

Next, by (2) and (1),

$$\|P_{2,2}^+\|_{4,8/3} = \|P_1^+\|_{4,8/3} = 1.$$

Therefore,

$$\|P_2^+\|_{\infty,8/3} \le \|P_{2,1}^+\|_{\infty,4}\|P_{2,2}^+\|_{4,8/3} = 1,$$

as required.

The infinite–dimensional case

We will consider functions of countably many variables. Let μ_∞ denote Haar measure normalized so that $\mu_\infty(\mathbb{T}^\infty) = 1$, and L^p be the corresponding L^p space. Again, any function $f \in L(\mathbb{T}^\infty)$ has the Fourier expansion

$$f \sim \sum_{\mathbf{k}} \hat{f}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}},$$

where now the sum is taken over all $\mathbf{k} = (k_1, k_2, \dots,)$ with integers $k_1, k_2, \dots,$ such that only finitely many of them are nonzero.

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One of the reasons to study infinite-dimensional trigonometric series is their connection with generalized Dirichlet series. There is a natural way to associate with such an infinitely dimensional trigonometric series a Dirichlet series

$$\sum_{r \in \mathbb{Q},\, r>0} a_r r^{it}.$$

Again, we define the Riesz operator P_∞^+, the norms $\|P_\infty^+\|_{q,p}$ and the critical exponent

$$p_\infty = \sup\{p : \ \|P_\infty^+\|_{\infty,p} = 1\}.$$

Clearly, $\|P_\infty^+\|_{q,2} = 1$ for any $q \geq 2$. However, now we do not claim that $\|P_\infty^+\|_{q,p} < \infty$ for $2 < p \leq q$, and we will see soon that this is not true.

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J. Marzo and K. Seip (2011) proved that

$$p_\infty = \lim_{n\to\infty} p_n.$$

If it turns out that $p_\infty > 2$, then, due to the Riesz interpolation theorem, there are p and q, $2 < p < q < \infty$, such that $\|P_\infty^+\|_{q,p} = 1$. But J. Marzo and K. Seip (2011) established that if $p_\infty = 2$, then $\|P_\infty^+\|_{q,p} = \infty$ for any $p > 2$.

Our main result

S.K. , H. Queffeléc, E. Saksman, and K. Seip (2022) proved the following theorem.

Theorem

$$\lim_{n\to\infty} p_n = 2.$$

We know, that this theorem implies $p_\infty = 2$. So, the Riesz projection P_∞^+ on the infinite-dimensional torus is not bounded from L^q to L^p when $2 < p < q \le \infty$.
Let me discuss briefly the sketch of the proof.

A key ingredient is the estimate of the Dirichlet kernel associated with a finite–dimensional Euclidean ball. Let $d \geq 2$, $R > 0$. We consider the Dirichlet kernel defined on $\mathbb{T}^d$

$$D_(R,d)(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d,\, \|\mathbf{k}\|_2 \leq R} e^{i\mathbf{k}\mathbf{x}},$$

where $\|\mathbf{k}\|_2$ denotes the Euclidean norm of a vector $\mathbf{k} \in \mathbb{Z}^d$. K.I. Babenko (1971, 2008) proved that for any $d \geq 2$ and $R > 0$

$$\|D_{R,d}\|_1 \geq C(d) R^{(d-1)/2}, \tag{3}$$

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where $C(d) > 0$.
For any function $f \in L(\mathbb{T}^d)$ we can define its spherical partial sum of order R

$$S_R(f)(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d,\, \|\mathbf{k}\|_2 \leq R} \hat{f}(\mathbf{k}) e^{i\mathbf{kx}}.$$

Thus, for any $q \geq 1$ we have the operator $S_R : L^q(\mathbb{T}^d) \to L(\mathbb{T}^d)$.

Let $q > 1$, $R \geq 1$. Consider a function

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d,\, \max_j |k_j| \leq R} e^{i\mathbf{kx}}.$$

It is easy to see that

$$\|f\|_q \leq C(d,q) R^{(q-1)d/q}.$$

Since $S_R(f) = D_{R,d}$, we conclude from the last inequality and (3) that

$$\|S_R\|_q^{(d)} \geq C(d) C(q,d)^{-1} R^{(d-1)/2-(q-1)d/q},$$

where $\|S_R\|_q^{(d)}$ denotes the norm of the operator
$S_R : L^q(\mathbb{T}^d) \to L(\mathbb{T}^d)$.

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where $\|S_R\|_q^{(d)}$ denotes the norm of the operator
$S_R: L^q(\mathbb{T}^d) \to L(\mathbb{T}^d)$. Let $q \in (1,2)$. Fix d with $d > 2q/(2-q)$.
Then $(d-1)/2 - (q-1)d/q > 0$. For sufficiently large $R = R_{q,d}$ we have

$$\|S_R\|_q^{(d)} > 1. \tag{4}$$

Another important part of the proof is to show that inequality (4) implies

$$\|P_n^+\|_{q,1} > 1$$

for n depending on q and $R_{q,d}$. It is possible to write an explicit estimate of n, but we did not try to do it.

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Let $p > 2$, $1/p + 1/q = 1$. By duality arguments,

$$\|P_n^+\|_{\infty,p} = \|P_n^+\|_{q,1} > 1.$$

Hence, $p_n < p$. Such a number n exists for any $p > 2$ as required.

THANK YOU!