# Well-posedness of the governing equations for quasi-linear viscoelastic model with pressure-dependent moduli in which both stress and strain appear linearly

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# Outline

1. Motivation
   - Implicit material response linear in strain
   - Implicit constitutive model linear in both stress and strain
   - Regularization by thresholding to prevent finite time blow-up
   - Analytical solution under uniform triaxial loading
2. Quasi-linear viscoelastic theory
   - Linear hereditary integrals with aging or convolution memory
   - Viscoelastic boundary value problem
   - Variational formulation of the problem
   - Solution on coercive and maximal monotone graph
3. Discussion
   - The corresponding nonlinear elastic problem
   - Elastic solution on coercive and maximal monotone graph
   - The correspondence principle
   - Semi-analytical solution for isotropic extension or compression
4. Conclusion

The implicit material response[1] of elastic bodies between the Cauchy stress $\boldsymbol{\sigma}$ and the deformation gradient $\mathbf{F}$ accounting for the mass density $\rho$:

$$\mathfrak{F}(\rho, \boldsymbol{\sigma}, \mathbf{F}) = \mathbf{0}$$

cannot be inverted to express stress as a function of strain, and vice versa

The Cayley–Hamilton theorem for isotropic $\mathfrak{F}$ and the left Cauchy–Green deformation tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^\top$ reveals the constitutive equation

$$\alpha_0 \mathbf{1} + \alpha_1 \boldsymbol{\sigma} + \alpha_2 \mathbf{B} + \alpha_3 \boldsymbol{\sigma}^2 + \alpha_4 \mathbf{B}^2 + \alpha_5(\boldsymbol{\sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\sigma}) + \alpha_6(\boldsymbol{\sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\sigma}^2) + \alpha_7(\boldsymbol{\sigma}\mathbf{B}^2 + \mathbf{B}^2\boldsymbol{\sigma}) + \alpha_8(\boldsymbol{\sigma}^2\mathbf{B}^2 + \mathbf{B}^2\boldsymbol{\sigma}^2) = \mathbf{0}$$

where $\mathbf{1}$ is the matrix of ones, $\alpha_i(\rho)$ depend on the traces $\mathrm{tr}\boldsymbol{\sigma}$, $\mathrm{tr}\mathbf{B}$, $\mathrm{tr}\boldsymbol{\sigma}^2$, $\mathrm{tr}\mathbf{B}^2$, $\mathrm{tr}\boldsymbol{\sigma}^3$, $\mathrm{tr}\mathbf{B}^3$, $\mathrm{tr}(\boldsymbol{\sigma}\mathbf{B})$, $\mathrm{tr}(\boldsymbol{\sigma}^2\mathbf{B})$, $\mathrm{tr}(\boldsymbol{\sigma}\mathbf{B}^2)$, $\mathrm{tr}(\boldsymbol{\sigma}^2\mathbf{B}^2)$

For small $\nabla\mathbf{u}$ and the linearized strain $\boldsymbol{\epsilon} = (\nabla\mathbf{u} + \nabla\mathbf{u}^\top)/2 \approx (\mathbf{B}^\top - \mathbf{I})/2$ yields

$$\beta_0\boldsymbol{\epsilon} + (\beta_1 + \beta_2\boldsymbol{\epsilon})\mathbf{I} + (\beta_3 + \beta_4\boldsymbol{\epsilon})\boldsymbol{\sigma} + (\beta_5 + \beta_6\boldsymbol{\epsilon})\boldsymbol{\sigma}^2 + \beta_7(\boldsymbol{\sigma}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\boldsymbol{\sigma}) + \beta_8(\boldsymbol{\sigma}^2\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\boldsymbol{\sigma}^2) = \mathbf{0}$$

where $\mathbf{I}$ is identity, $\beta_i(\rho)$ depend on the invariants: $\mathrm{tr}\boldsymbol{\sigma}$, $(\mathrm{tr}^2\boldsymbol{\sigma} - \mathrm{tr}\boldsymbol{\sigma}^2)/2$, $\det\boldsymbol{\sigma}$

The model is linear in strain!

[1] K.R. Rajagopal, On implicit constitutive relations, *Appl. Math.* **48** (2003), 279–319

## Nonlinear constitutive model linear in strain

If $\beta_4 = \ldots = \beta_8 = 0$, then a subclass of the material response[2]:

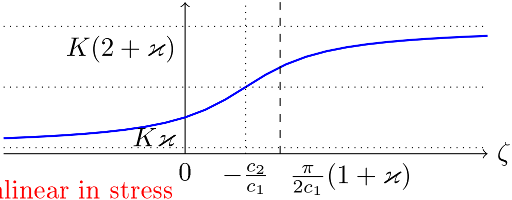
$$(\beta_0 + \beta_2)\boldsymbol{\epsilon} + \beta_1 \mathbf{I} + \beta_3 \boldsymbol{\sigma} = 0 \qquad (MR)$$

can be inverted for strain in term of stress e.g. deviatoric and spherical parts:

$$\boldsymbol{\epsilon} = \frac{1}{2\mu}\boldsymbol{\sigma}^* + \frac{\mathrm{tr}\boldsymbol{\sigma}}{9B(\mathrm{tr}\boldsymbol{\sigma})}\mathbf{I}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \frac{\mathrm{tr}\boldsymbol{\sigma}}{3}\mathbf{I}$$

where Lamé constant $\mu = E/(2(1+\nu))$, Young modulus $E > 0$, Poisson ratio $\nu \in (0, 0.5)$, bulk modulus depends on the mean stress $\mathrm{tr}\boldsymbol{\sigma}$, e.g. for polymers:

$$B(\zeta) = K\big(1 + \varkappa + \frac{2}{\pi}\arctan(c_1\zeta + c_2)\big), \quad K, c_1 > 0$$




The model is nonlinear in stress

[2] K.R. Rajagopal, G. Saccomandi, The mechanics and mathematics of the effect of pressure on the shear modulus of elastomers, *Proc. R. Soc. A* **465** (2009), 3859–3874

## Constitutive model linear in both stress and strain

A subclass of the constitutive relation $(MR)$ which is also linear in stress[3]:

$$(1+\lambda_3 \mathrm{tr}\boldsymbol{\sigma})\boldsymbol{\epsilon} = \frac{1+\lambda_1 \mathrm{tr}\boldsymbol{\epsilon}}{2\mu}\boldsymbol{\sigma}^* + \frac{1+\lambda_4 \mathrm{tr}\boldsymbol{\epsilon}}{9K}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} \qquad (LBSS)$$

with $\lambda_1$, $\lambda_3$, $\lambda_4$ free moduli, on taking the trace:

$$\mathrm{tr}\boldsymbol{\epsilon} - \frac{1}{3K}\mathrm{tr}\boldsymbol{\sigma} + \Big(\lambda_3 - \frac{\lambda_4}{3K}\Big)(\mathrm{tr}\boldsymbol{\epsilon})\mathrm{tr}\boldsymbol{\sigma} = 0$$

For $\lambda_1 = \lambda_3 = 0$ the explicit relations:

- using $\mathrm{tr}\boldsymbol{\epsilon} = \mathrm{tr}\boldsymbol{\sigma}/(3K - \lambda_4 \mathrm{tr}\boldsymbol{\sigma})$ invert for strain in term of stress:

$$\boldsymbol{\epsilon} = \frac{1}{2\mu}\boldsymbol{\sigma}^* + \frac{1}{3(3K - \lambda_4 \mathrm{tr}\boldsymbol{\sigma})}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} \qquad \text{(polymer)}$$

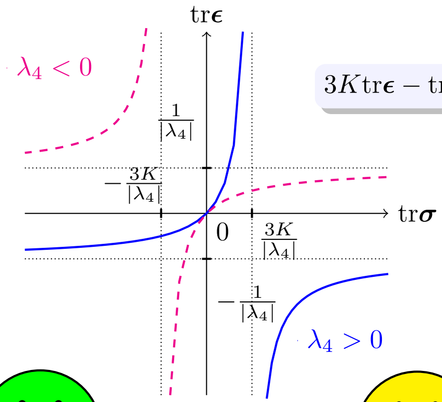
- using $1/(1+\lambda_4 \mathrm{tr}\boldsymbol{\epsilon}) \approx 1 - \lambda_4 \mathrm{tr}\boldsymbol{\epsilon}$ invert for stress in term of strain:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon}^* + K(1 - \lambda_4 \mathrm{tr}\boldsymbol{\epsilon})(\mathrm{tr}\boldsymbol{\epsilon})\mathbf{I} \qquad \text{(porous)}$$

The balance of mass: $\rho_R = \rho(\det \mathbf{F})$ when linearized: $\rho_R = \rho(1 + \mathrm{tr}\boldsymbol{\epsilon})$ makes the response of porous materials when $\mathrm{tr}\boldsymbol{\epsilon}$ is replaced by $\rho_R/\rho - 1$ for density

[3] K.R. Rajagopal, An implicit constitutive relation for describing the small strain response of porous elastic solids whose material moduli are dependent on the density, *Math. Mech. Solids* **26** (2021), 1138–1146

## Implicit constitutive model linear in both stress and strain




$$3K\mathrm{tr}\boldsymbol{\epsilon} - \mathrm{tr}\boldsymbol{\sigma} - \lambda_4(\mathrm{tr}\boldsymbol{\epsilon})\mathrm{tr}\boldsymbol{\sigma} = 0$$

mechanical engineer

mathematician

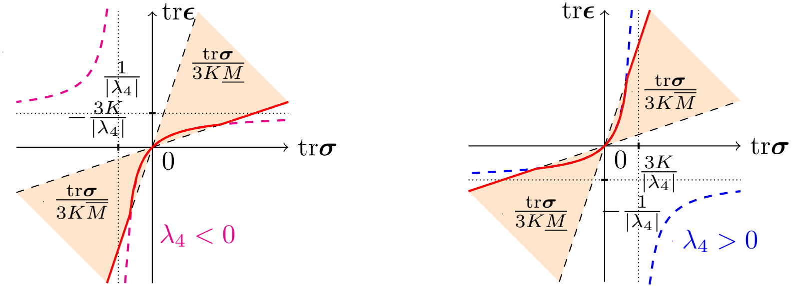
## Regularization by thresholding

For lower and upper thresholds $0 < \underline{M} \leq 1 \leq \overline{M}$ the cut-off function:

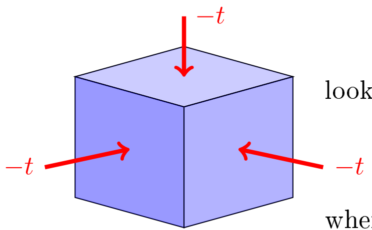
$$B(\zeta) := K \begin{cases} \underline{M}, & \text{if } 1 - \lambda_4/(3K)\zeta < \underline{M} \\ 1 - \lambda_4/(3K)\zeta, & \text{if } \underline{M} \leq 1 - \lambda_4/(3K)\zeta \leq \overline{M} \\ \overline{M}, & \text{if } 1 - \lambda_4/(3K)\zeta > \overline{M} \end{cases}$$

the thresholding equation: $\mathrm{tr}\boldsymbol{\epsilon} = \mathrm{tr}\boldsymbol{\sigma}/(3B(\mathrm{tr}\boldsymbol{\sigma}))$ and

$$\boldsymbol{\epsilon} = \frac{1}{2\mu}\boldsymbol{\sigma}^* + \frac{\mathrm{tr}\boldsymbol{\sigma}}{9B(\mathrm{tr}\boldsymbol{\sigma})}\mathbf{I}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \frac{\mathrm{tr}\boldsymbol{\sigma}}{3}\mathbf{I} \qquad (TE)$$



## Analytical solution under uniform triaxial loading




The cube $\Omega = (0,1)^3$ is loaded by the force:

$$\boldsymbol{\sigma} = t\mathbf{I} \quad \text{on } \partial\Omega$$

look for the displacement with unknown $u$:

$$\mathbf{u} = \frac{1}{3}u\mathbf{x} + R\mathbf{x}, \quad \boldsymbol{\epsilon} = \frac{1}{3}u\mathbf{I}$$

where $R\mathbf{x}$ is rigid displacement for $R \in \mathbb{R}^{3\times 3}_{\text{skew}}$

- the nonlinear elastic equation for stress:

$$\boldsymbol{\epsilon} = \frac{\boldsymbol{\sigma}}{3K - \lambda_4 \text{tr}\boldsymbol{\sigma}} \;\Rightarrow\; u = \frac{t}{K - \lambda_4 t}$$

- the thresholding equation:

$$\boldsymbol{\epsilon} = \frac{\boldsymbol{\sigma}}{3B(\text{tr}\boldsymbol{\sigma})} \;\Rightarrow\; u = \frac{t}{B(3t)} = \begin{cases} t/(K\underline{M}), & \text{if } 1 - \lambda_4/Kt < \underline{M} \\ t/(K - \lambda_4 t), & \text{if } \underline{M} \le 1 - \lambda_4/Kt \le \overline{M} \\ t/(K\overline{M}), & \text{if } 1 - \lambda_4/Kt > \overline{M} \end{cases}$$

Graphs of the dilatation $u$ versus the loading $t$:

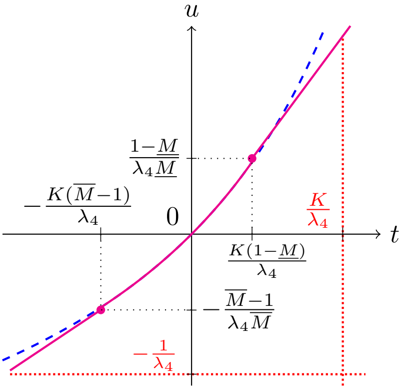



Figure: case $\lambda_4 > 0$

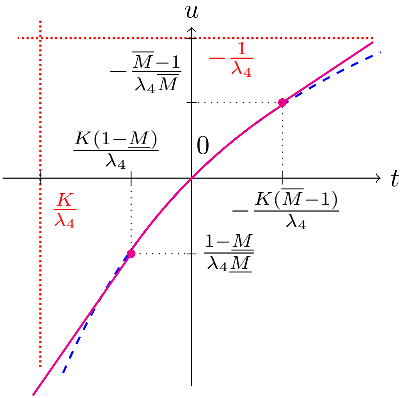



Figure: case $\lambda_4 < 0$

## Quasi-linear viscoelastic model

For the constitutive equation given by nonlinear operator $\mathcal{F}$ e.g.:

$$\boldsymbol{\epsilon} = \mathbf{E}_1(\|\boldsymbol{\sigma}^*\|)\boldsymbol{\sigma}^* + \mathbf{E}_2(\mathrm{tr}\boldsymbol{\sigma})(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} := [\mathcal{F}]\boldsymbol{\sigma} \qquad (CE)$$

consider the quasi-linear viscoelastic equation with linear operator $\mathcal{I}$:

$$\boldsymbol{\varepsilon}(t) = [\mathcal{I}(t)]\boldsymbol{\epsilon}, \quad t \geq 0$$

given e.g. by nonlinear in time hereditary integral with aging memory[4]:

$$[\mathcal{I}(t)]\boldsymbol{\epsilon} = \mathbf{J}(t,t)\boldsymbol{\epsilon}(t) - \int_0^t \Big(\frac{\partial}{\partial s}\mathbf{J}(t,s)\Big)\boldsymbol{\epsilon}(s)\,ds \qquad (HE)$$

where $\mathbf{J}$ isotropic tensorial kernel and scalar $J$ implies the Volterra equation:

$$[\mathcal{I}(t)]\boldsymbol{\epsilon} = J(t,t)\boldsymbol{\epsilon}(t) - \int_0^t \Big(\frac{\partial}{\partial s}J(t,s)\Big)\boldsymbol{\epsilon}(s)\,ds$$

the integral operator $\mathcal{I}$ cannot be inverted

[4] A.S. Wineman, Nonlinear viscoelastic solids- a review, *Math. Mech. Solids* **14** (2009), 300–366

- the time-shift approach implies the constitutive response:

$$[\mathcal{I}(t)]\boldsymbol{\epsilon} = J_0\boldsymbol{\epsilon}(t) + \int_0^t J\Big(\int_s^t \Big(\frac{\tau}{s_0}\Big)^\mu d\tau\Big)\Big(\frac{s}{s_0}\Big)^\mu \boldsymbol{\epsilon}(s)\, ds$$

- convolution equation for the Prony's series with $J_1, \dots, J_N$, $\tau_1, \dots \tau_N \geq 0$:

$$[\mathcal{I}(t)]\boldsymbol{\epsilon} = J_0\boldsymbol{\epsilon}(t) + \sum_{n=1}^{N} \frac{J_n}{\tau_n} \int_0^t e^{-(t-s)/\tau_n} \boldsymbol{\epsilon}(s)\, ds \qquad (PS)$$

on differentiating $(PS)$ as $N = 1$ and $J_0 = 0$ implies Kelvin–Voigt model:

$$\tau_1 \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon} = J_1 \boldsymbol{\epsilon}$$

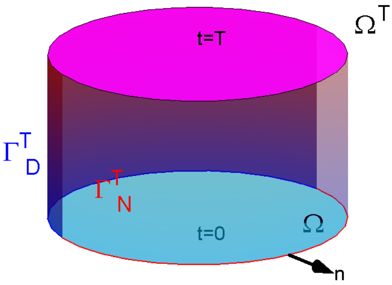
twice differentiating as $N = 2$ yields the Burgers material:

$$\tau_1 \ddot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}} = J_0 \tau_1 \ddot{\boldsymbol{\epsilon}} + \Big[J_0 + \tau_1 \Big(\frac{J_1}{\tau_1} + \frac{J_2}{\tau_2}\Big)\Big] \dot{\boldsymbol{\epsilon}} + \frac{J_1}{\tau_1} \boldsymbol{\epsilon}$$

- the fractional derivative $\nu \in (0, 1)$ using the Liouville–Weyl integral:

$$[\mathcal{I}(t)]\boldsymbol{\epsilon} = J_0\boldsymbol{\epsilon}(t) + c \int_{-\infty}^t \frac{\boldsymbol{\epsilon}(s)}{(t-s)^{1-\nu}}\, ds$$

## Geometric configuration




The solid body occupies the bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ and normal $\mathbf{n} = (n_1, \dots, n_d)$ comprising two parts $\Gamma_{\rm N}$, $\Gamma_{\rm D}$

For $t \in (0, T)$, $\mathbf{x} = (x_1, \dots, x_d)$ cylinder $\Omega^T = (0, T) \times \Omega$ with side surfaces $\Gamma^T_{\rm N} = (0, T) \times \Gamma_{\rm N}$ and $\Gamma^T_{\rm D} = (0, T) \times \Gamma_{\rm D}$

Given:
- body force $\mathbf{f} = (f_1, \dots, f_d)(t, \mathbf{x}) \in C([0, T]; L^2(\Omega : \mathbb{R}^d))$ in $\Omega^T$
- boundary force $\mathbf{g} = (g_1, \dots, g_d)(t, \mathbf{x}) \in C([0, T]; L^2(\Gamma_{\rm N}; \mathbb{R}^d))$ at $\Gamma^T_{\rm N}$
- continuous response multi-valued function $\mathcal{F} : \mathbb{R}^{d\times d}_{\rm sym} \rightrightarrows \mathbb{R}^{d\times d}_{\rm sym}$
- continuous creep function $\mathcal{I} \in C([0, T])$, $\mathcal{I}(t) : \mathbb{R}^{d\times d}_{\rm sym} \rightrightarrows \mathbb{R}^{d\times d}_{\rm sym}$

## Governing viscoelastic relations

The symmetric stress tensor $\boldsymbol{\sigma}(t, \mathbf{x}) = (\sigma_{ij})_{i,j=1}^{d} \in \mathbb{R}^{d \times d}_{\text{sym}}$ and displacement $\mathbf{u}(t, \mathbf{x}) = (u_1, \dots, u_d)$ build the linearized strain $\boldsymbol{\varepsilon}(t, \mathbf{x}) = (\varepsilon_{ij})_{i,j=1}^{d} \in \mathbb{R}^{d \times d}_{\text{sym}}$:

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla^{\top} \mathbf{u}) \qquad (LS)$$

satisfy the equilibrium equation:

$$-\nabla \cdot \boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega^T \qquad (EE)$$

the viscoelastic constitutive equation:

$$\boldsymbol{\varepsilon}(t) = [\mathcal{I}(t) \circ \mathcal{F}]\boldsymbol{\sigma} \quad \text{for } t \in (0, T) \qquad (VE)$$

and the mixed homogeneous Dirichlet–Neumann boundary condition:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}^T, \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{N}}^T \qquad (BC)$$

## Variational formulation of the problem

The variational solution:

$$\mathbf{u} \in C([0,T]; H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^d)), \quad \boldsymbol{\sigma} \in C([0,T]; L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \qquad (VS)$$

defined in the Sobolev space:

$$H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^d) = \left\{ \mathbf{v}(\mathbf{x}) = (v_1,\dots,v_d) \in H^1(\Omega;\mathbb{R}^d) \middle| \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_\mathrm{D} \right\}$$

satisfies the variational equation for all test functions $\mathbf{v} \in H^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^d)$:

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\Gamma_\mathrm{N}} \mathbf{g}\cdot\mathbf{v}\,dS_{\mathbf{x}} \qquad (VEE)$$

where dot stands for the scalar product of tensors: $\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} = \sum_{i,j=1}^d \sigma_{ij}\varepsilon_{ij}$

and the viscoelastic constitutive equation $(VE)$ given in the form of selection:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u})) \in \mathfrak{G}(t), \quad t \in [0,T] \qquad (S)$$

on the time-dependent graph between stress and strain:

$$\mathfrak{G}(t) = \left\{ (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \in (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2 \middle| \quad \boldsymbol{\varepsilon} = [\mathcal{I}(t)\circ\mathcal{F}]\boldsymbol{\sigma} \right\} \qquad (G)$$

## Theorem on existence of variational solution

Assume continuous $\mathcal{I}(t) \circ \mathcal{F}$, $t \in [0, T]$ in $(G)$: (i) The graph includes the origin:

$$(\mathbf{0}, \mathbf{0}) \in \mathfrak{G}(t)$$

(ii) The graph is coercive with uniform estimate for all $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \in \mathfrak{G}(t)$:

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} \geq M_1 \|\boldsymbol{\sigma}\|^2 + M_2 \|\boldsymbol{\varepsilon}\|^2, \quad M_1, M_2 > 0,\ M_1 M_2 \leq 1/4$$

(iii) The graph is monotone for all pairs $(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1)$ and $(\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2) \in \mathfrak{G}(t)$:

$$(\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^2) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \geq 0$$

(iv) The graph is maximal monotone: for $(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1) \in (\mathbb{R}^{d \times d}_{\mathrm{sym}})^2$

$$\text{if } (\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^2) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \geq 0 \quad \text{for all } (\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2) \in \mathfrak{G}(t), \quad \text{then } (\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1) \in \mathfrak{G}(t)$$

Then there exists solution $(\boldsymbol{\sigma}, \mathbf{u})$ to the viscoelasticity problem $(VEE)$ and $(S)$

based on Galerkin approximation and Browder–Minty fixed point theorem[5]

[5] M. Bulíček, J. Málek, E. Süli, Existence of global weak solutions to implicitly constituted kinetic models of incompressible homogeneous dilute polymers, *Commun. Part. Diff. Eq.* **38** (2013), 882–924

## The corresponding nonlinear elastic problem

The elastic stress $\boldsymbol{\sigma}^{\mathrm{e}}$ and displacement $\mathbf{u}^{\mathrm{e}}$ satisfy the equilibrium equation:

$$-\nabla \cdot \boldsymbol{\sigma}^{\mathrm{e}} = \mathbf{f} \quad \text{in } \Omega^T$$

and the mixed homogeneous Dirichlet–Neumann boundary condition:

$$\mathbf{u}^{\mathrm{e}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}^T, \quad \boldsymbol{\sigma}^{\mathrm{e}}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{N}}^T$$

expressed by the variational equation for all test functions $\mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d)$:

$$\int_\Omega \boldsymbol{\sigma}^{\mathrm{e}} \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{v}\, dS_{\mathbf{x}}$$

The constitutive equation:

$$\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{e}}) = [\mathcal{F}]\boldsymbol{\sigma}^{\mathrm{e}}$$

holds in the form of selection:

$$(\boldsymbol{\sigma}^{\mathrm{e}}, \boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{e}})) \in \mathfrak{G}$$

on the graph:

$$\mathfrak{G} = \left\{(\boldsymbol{\sigma}, \boldsymbol{\epsilon}) \in (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2 | \quad \boldsymbol{\epsilon} = [\mathcal{F}]\boldsymbol{\sigma}\right\}$$

## Theorem on existence of variational solution

Assume $\mathcal{F}$ is continuous such that: (i) The graph $\mathfrak{G}$ includes the origin:

$$(\mathbf{0}, \mathbf{0}) \in \mathfrak{G}$$

(ii) The graph is coercive with uniform estimate for all $(\boldsymbol{\sigma}, \boldsymbol{\epsilon}) \in \mathfrak{G}$:

$$\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} \geq M_1 \|\boldsymbol{\sigma}\|^2 + M_2 \|\boldsymbol{\epsilon}\|^2, \quad M_1, M_2 > 0,\ M_1 M_2 \leq 1/4$$

where the Frobenius norm $\|\boldsymbol{\sigma}\| = \sqrt{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}}$

(iii) The graph is monotone for all pairs $(\boldsymbol{\sigma}^1, \boldsymbol{\epsilon}^1)$ and $(\boldsymbol{\sigma}^2, \boldsymbol{\epsilon}^2) \in \mathfrak{G}$:

$$(\boldsymbol{\epsilon}^1 - \boldsymbol{\epsilon}^2) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \geq 0$$

(iv) The graph is maximal monotone: for $(\boldsymbol{\sigma}^1, \boldsymbol{\epsilon}^1) \in (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2$

$$\text{if } (\boldsymbol{\epsilon}^1 - \boldsymbol{\epsilon}^2) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \geq 0 \quad \text{for all } (\boldsymbol{\sigma}^2, \boldsymbol{\epsilon}^2) \in \mathfrak{G}, \quad \text{then } (\boldsymbol{\sigma}^1, \boldsymbol{\epsilon}^1) \in \mathfrak{G}$$

Then there exists the variational solution to the nonlinear elasticity problem:

$$\mathbf{u}^{\mathrm{e}} \in C([0,T]; H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d)), \quad \boldsymbol{\sigma}^{\mathrm{e}} \in C([0,T]; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$$

For the thresholding equation $(TE)$:

$$\boldsymbol{\epsilon} = \frac{1}{2\mu}\boldsymbol{\sigma}^* + \frac{\mathrm{tr}\boldsymbol{\sigma}}{9B(\mathrm{tr}\boldsymbol{\sigma})}\mathbf{I}$$

(i) The graph $\mathfrak{G}$ includes the origin: $(\mathbf{0},\mathbf{0}) \in \mathfrak{G}$

(ii) For all $(\boldsymbol{\sigma},\boldsymbol{\epsilon}) \in \mathfrak{G}$ the graph is coercive:

$$\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma} \geq M_1\|\boldsymbol{\sigma}\|^2 + M_2\|\boldsymbol{\epsilon}\|^2$$

where the factors are

$$M_1 = \frac{1}{2}\min\Big(\frac{1}{2\mu}, \frac{d}{9K\overline{M}^2}\Big), \quad M_2 = \frac{1}{2}\min\Big(2\mu, \frac{9K\underline{M}^4}{\overline{M}^2 d}\Big)$$

(iii) For all pairs $(\boldsymbol{\sigma}^1,\boldsymbol{\epsilon}^1),(\boldsymbol{\sigma}^2,\boldsymbol{\epsilon}^2) \in \mathfrak{G}$ the graph is monotone:

$$(\boldsymbol{\epsilon}^1-\boldsymbol{\epsilon}^2)\cdot(\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^2) \geq 0$$

(iv) For $(\boldsymbol{\sigma}^1,\boldsymbol{\epsilon}^1) \in (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2$ the graph is maximal monotone:

$$\text{if } (\boldsymbol{\epsilon}^1-\boldsymbol{\epsilon}^2)\cdot(\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^2) \geq 0 \text{ for all } (\boldsymbol{\sigma}^2,\boldsymbol{\epsilon}^2) \in \mathfrak{G}, \quad \text{then } (\boldsymbol{\sigma}^1,\boldsymbol{\epsilon}^1) \in \mathfrak{G}$$

## The correspondence principle

Let there exist solution $(\boldsymbol{\sigma}^{\mathrm{e}}, \mathbf{u}^{\mathrm{e}})$ to the nonlinear elasticity problem:

$$\int_\Omega \boldsymbol{\sigma}^{\mathrm{e}} \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{v}\, dS_{\mathbf{x}}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{e}}) = [\mathcal{F}]\boldsymbol{\sigma}^{\mathrm{e}}$$

If operator $\mathcal{I}(t)$ commutes with the strain tensor $\boldsymbol{\varepsilon}$:

$$[\mathcal{I}(t)]\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}\big([\mathcal{I}(t)]\mathbf{u}\big) \qquad (C)$$

then solution to the corresponding viscoelastic problem is given by formula:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{e}}, \quad \mathbf{u} = [\mathcal{I}(t)]\mathbf{u}^{\mathrm{e}}$$

since in this case: $\boldsymbol{\varepsilon}\big([\mathcal{I}(t)]\mathbf{u}^{\mathrm{e}}\big) = [\mathcal{I}(t)]\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{e}}) = [\mathcal{I}(t) \circ \mathcal{F}]\boldsymbol{\sigma}^{\mathrm{e}}$

$(C)$ holds for arbitrary scalar operator $\mathcal{I}(t)$ prescribing the Volterra equation:

$$[\mathcal{I}(t)]\boldsymbol{\epsilon} = J(t,t)\boldsymbol{\epsilon}(t) - \int_0^t \Big(\frac{\partial}{\partial s} J(t,s)\Big)\boldsymbol{\epsilon}(s)\, ds$$

## Semi-analytical solution for isotropic extension or compression

For $d = 3$ consider isotropic extension or compression independent of $\mathbf{x}$:

$$\boldsymbol{\sigma}^* = \boldsymbol{\epsilon}^* = \mathbf{0}, \quad \boldsymbol{\sigma} = -p(t)\mathbf{I}, \quad \boldsymbol{\epsilon} = u(t)/3\,\mathbf{I}$$

the Volterra convolution equation:

$$\mathrm{tr}\boldsymbol{\varepsilon}(t) = [\mathcal{I}(t)]u = \int_0^t \frac{J_1}{\tau_1} e^{-(t-s)/\tau_1} u(s)\,ds$$

on the grid $0 = t_0 < t_1 < \ldots < t_M = T$ the piecewise-affine approximation:

$$u_M(t) = u(t_{k-1}) + (t - t_{k-1})\delta u_k \quad \text{as } t \in [t_{k-1}, t_k]$$

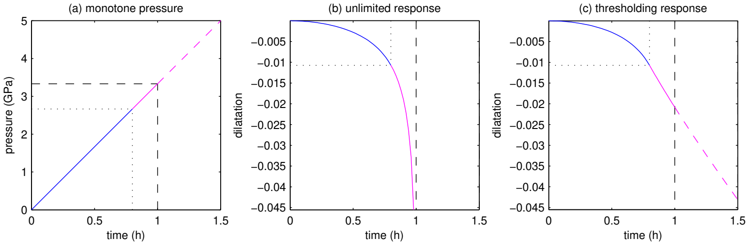
$$\text{where} \quad \delta u_k := \frac{u(t_k) - u(t_{k-1})}{\delta t_k}, \quad \delta t_k = t_k - t_{k-1}, \quad k = 1, \ldots, M$$

the numerical quadrature:

$$\mathrm{tr}\boldsymbol{\varepsilon}_M(t) = [\mathcal{I}(t)]u_M = \sum_{k=1}^{M} I_k^M, \quad I_k^M := \int_{t_{k-1}}^{t_k} \frac{J_1}{\tau_1} e^{-(t-s)/\tau_1} u_M(s)\,ds$$

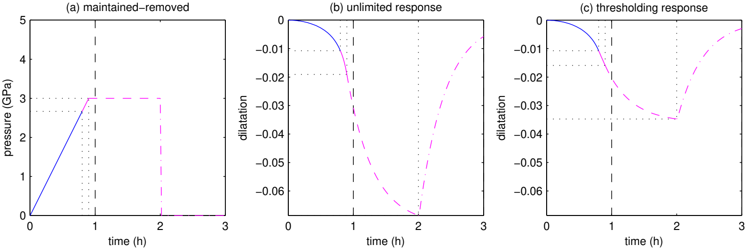
$$\text{where} \quad I_k^M/J_1 = \big(u(t_k) - \tau_1\delta u_k\big)e^{(t_k - t)/\tau_1} - \big(u(t_{k-1}) - \tau_1\delta u_k\big)e^{(t_{k-1} - t)/\tau_1}$$

# Creep test by monotone loading and maintained-removed pressure




Creep test by monotone loading-
-maintained-removed pressure:

| $p(t) = 3.3t$ | $p(t) = 2.9$ | $p(t) = 0$ |
|---|---|---|
| $t \in [0, 0.9]$ | $t \in [0.9, 2]$ | $t \in [2, 3]$ |



## Conclusion

- The constitutive relation stems from implicit material response between the histories of the stress and the deformation gradient of a body
- A-priori thresholding is enforced through the mean pressure that ensures that the solution does not blow-up in finite time
- Well-posedness for the resulting mixed variational problem is established within the theory of coercive and maximal monotone graphs
- The quasi-linear viscoelastic constitutive model is prescribed by tensorial hereditary integrals with aging or convolution memory kernels
- For scalar Volterra equation, the correspondence principle provides formula of viscoelastic solution from the nonlinear elastic problem
- For isotropic extension or compression, numerical solution is given for monotone loading and creep test by maintained-removed pressure

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