A theory of reproducing Hardy and Bergman spaces in octonionic settings

Sören Kraußhar (Universität Erfurt)
joint work with F. Colombo and I. Sabadini (Politecnico di Milano)

International biweekly online seminar on Analysis, Differential Equations and Mathematical Physics, 23/03/2023

1. Rudiments of octonionic function theory

Octonions $\mathbb{O}$ elements of the form:

$$z = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7$$

where $e_4 = e_1 e_2$, $e_5 = e_1 e_3$, $e_6 = e_2 e_3$ and $e_7 = e_4 e_3 = (e_1 e_2) e_3$.

Multiplication is explained by the table:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_5	$-e_2$	$-e_3$	$-e_7$	e_6
e_2	$-e_4$	-1	e_6	e_1	e_7	$-e_3$	$-e_5$
e_3	$-e_5$	$-e_6$	-1	$-e_7$	e_1	e_2	e_4
e_4	e_2	$-e_1$	e_7	-1	$-e_6$	e_5	$-e_3$
e_5	e_3	$-e_7$	$-e_1$	e_6	-1	$-e_4$	e_2
e_6	e_7	e_3	$-e_2$	$-e_5$	e_4	-1	$-e_1$
e_7	$-e_6$	e_5	$-e_4$	e_3	$-e_2$	e_1	-1

Not anymore associative but alternative. We have the Moufang relations and composition property. No zero-divisors - still a normed division algebra.

We recall

Definition 1. *(Dentoni, Sce, Nono, Xingmin Li, Li-Zhong Peng) Let $U \subseteq \mathbb{O}$ be open. $f : U \to \mathbb{O}$ is left octonionic monogenic if $\mathcal{D}f = 0$. Here, $\mathcal{D} := \frac{\partial}{\partial x_0} + \sum_{i=1}^{7} e_i \frac{\partial}{\partial x_i}$. If f satisfies $\overline{\mathcal{D}}f = 0$ we call f left octonionic anti-monogenic.*

Important contrast to Clifford analysis: Left (right) octonionic monogenic functions **do neither form a right nor a left $\mathbb{O}$-module**.

[Xingmin-Li, Li-Zhong Peng L2000]: No direct analogue of Stokes' formula. Even if $\mathcal{D}f = 0$ and $g\mathcal{D} = 0$ we don't have

$$\int_{\partial G} g(x)\ (d\sigma(x) f(x)) = 0 \quad \text{nor} \quad \int_{\partial G} (g(x) d\sigma(x))\ f(x) = 0.$$

But, following Xingmin-Li, Li-Zhong Peng, Qian Tao 2008:

$$\begin{aligned}\int_{\partial G} g(x)\ (d\sigma(x) f(x)) &= \int_{G} \Big(g(x)(\mathcal{D}f(x)) + (g(x)\mathcal{D}) f(x) \\ &- \sum_{j=0}^{7} [e_j, \mathcal{D}g_j(x), f(x)] \Big) dV,\end{aligned}$$

where $[a, b, c] := (ab)c - a(bc)$.

Proposition 1. Nono, XL2002*(Cauchy's integral formula).* *Let $U \subseteq \mathbb{O}$ open and $G \subseteq U$ be an 8-D compact oriented manifold with a strongly Lipschitz boundary ∂G. If $f : U \to \mathbb{O}$ is left octonionic monogenic, then for all $x \in G$*

$$f(x) = \frac{3}{\pi^4} \int_{\partial G} q_{\mathbf{0}}(y-x)(d\sigma(y)f(y)).$$

Note: Putting the parenthesis differently, leads to different formula

$$\begin{aligned} \frac{3}{\pi^4} \int_{\partial G} (q_{\mathbf{0}}(y-x)d\sigma(y))f(y) &= f(x) \\ &+ \int_{G} \sum_{i=0}^{7} [q_{\mathbf{0}}(y-x), \mathcal{D}f_i(y), e_i] dy_0 \cdots dy_7. \end{aligned}$$

Slice monogenic octonionic functions associated with the classical book structure

Basic theory: Struppa, Gentili (2010), Perotti, Ghiloni (2011)

Consider

$$\mathbb{S} = \{x = \sum_{k=1}^{7} x_k e_k \ \text{ such that } \ \sum_{k=1}^{7} x_k^2 = 1\}.$$

If $I \in \mathbb{S}$, then $I^2 = -1$ so all elements of $\mathbb{S}$ are called imaginary units and for any $I \in \mathbb{S}$ we can consider the complex plane $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ containing 1 and I.

Each $x \in \mathbb{O}$ can be written as $x = u + Iv$ with uniquely defined $I \in \mathbb{S}$ if $x \notin \mathbb{R}$. By $[x]$ we denote the sphere associated with x:

$$[x] = \{y = u + Jv \ \text{ where } J \in \mathbb{S}, x = u + Iv\}.$$

Definition 2. *Let Ω be a domain in $\mathbb{O}$. A real differentiable function $f : \Omega \to \mathbb{O}$ is called slice monogenic if, for every $I \in \mathbb{S}$, its restriction f_I to $\mathbb{C}_I$ is holomorphic on $\Omega \cap \mathbb{C}_I$, i.e.*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u + Iv) = 0,$$

on $\Omega \cap \mathbb{C}_I$.

$\mathcal{SM}(\Omega)$: set of slice monogenic functions on Ω.

Example. Polynomials with coefficients written on the right are (left) slice monogenic functions.

By writing the components in terms of four complex valued functions, one obtains

Theorem 1. *(Splitting Lemma) If f is a slice monogenic function on the open set $\Omega \subseteq \mathbb{O}$, then for every $I_1 \in \mathbb{S}$, we can find I_2 and I_4 in $\mathbb{S}$, such that there are four holomorphic functions F_1, F_2, G_1, G_2 from $B_8(0,r) \cap \mathbb{C}_{I_1}$ to $\mathbb{C}_{I_1}$ such that for any $z = u + vI_1$, it is*

$$f_{I_1}(z) = F_1(z) + F_2(z)I_2 + (G_1(z) + G_2(z)I_2)I_4.$$

This result is the key ingredient for proving

Theorem 2. *The function $f : B_8(0,r) \to \mathbb{O}$ is slice monogenic, if and only if it has a series expansion of the form*

$$f(x) = \sum_{n=0}^{\infty} x^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0).$$

We also recall

Theorem 3. *(Representation Formula)*
Let Ω be an axially symmetric slice domain. Then for every $I, J \in \mathbb{S}$:

$$f(x+Iy) = \frac{1}{2}\left[f(x+yJ)+f(x-yJ)\right]+\frac{1}{2}I\left[J(f(x-yJ)-f(x+yJ))\right].$$

A Cauchy formula was proved by Ghiloni, Perotti, Recupero in 2017. As in the monogenic setting it requires a careful use of the parentheses:

Theorem 4. *Let Ω be an axially symmetric bounded set in $\mathbb{O}$, $I \in \mathbb{S}$ and let $\Omega \cap \mathbb{C}_I$ have a smooth boundary. Let f be slice monogenic in Ω. Then for every $x \in \Omega$:*

$$f(x) = \frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_I)} S_L^{-1}(s,x)\left[I^{-1}ds f(s)\right],$$

where $S_L^{-1}(s,x) = -(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}(x-\bar{s})$ is the slice monogenic Cauchy kernel.

2. Rudiments of octonionic RKHS

Throughout this talk let $\Omega \subset \mathbb{O}$: simply-connected orientable with strongly Lipschitz boundary, say $\Sigma = \partial\Omega$, where $n(y)$ exists a.e. on $\partial\Omega$.

Before starting, recall from Goldstine, Horwitz (1964) and Huo, Ren (2021)

Definition 3. *An octonionic Hilbert space H is a left $\mathbb{O}$-module with $\mathbb{O}$-valued inner product $(\cdot,\cdot) : H \times H \to \mathbb{O}$ such that $(H, \langle\cdot,\cdot\rangle_0)$ is a real Hilbert space where $\langle\cdot,\cdot\rangle_0 := \Re(\cdot,\cdot)$. The $\mathbb{O}$-valued inner product is supposed to satisfy $f, g, h \in H$, all $\alpha \in \mathbb{O}$ and all $r \in \mathbb{R}$:*

(i) $(f+g,h)=(f,h)+(g,h)$

(ii) $(g,f)=\overline{(f,g)}$

(iii) $(f,f)\in\mathbb{R}^{\geq 0}$ *and* $(f,f)=0$ *iff* $f=0$.

(iv) $(fr,g)=(f,g)r$ *(*$r\in\mathbb{R}$*)*

(v) $(f\alpha,f)=(f,f)\alpha$

(vi) $\langle f\alpha,g\rangle_0=\Re\{(f\alpha,g)\}=\Re\{(f,g)\alpha\}$ *(*$\mathbb{O}$-**para-linearity***)*

Now consider $\mathbb{O}$-valued functionals

$$\mathcal{T} : H \to \mathbb{O}, \ \mathcal{T}(f) := (f, g).$$

However, the requirement of $\mathcal{T}$ being $\mathbb{O}$-linear in the classical sense that $\mathcal{T}(f\alpha) = \mathcal{T}(f)\alpha$ is too strong for establishing a powerful theory of RKHS - $\mathbb{O}$-linearity should be **replaced by the weaker notion** of $\mathbb{O}$-para-linearity, see also Ren et al. 2022:

We call $\mathcal{T}$ $\mathbb{O}$-para-linear if $\Re([\alpha, f, \mathcal{T}]) = 0$ where $[\alpha, f, \mathcal{T}] := \mathcal{T}(f\alpha) - (\mathcal{T}(f))\alpha$ is the **second associator**.

Following Ren et al. (2022) this condition can be interpreted equivalently as $\Re(f\alpha, g) = \Re((f, g)\alpha)$.

3. Hardy spaces of octonionic monogenic functions.

$H^2(\partial\Omega, \mathbb{O})$: closure of $L^2(\partial\Omega)$-octonion-valued functions — octonionic monogenic functions inside of Ω with continuous extension to $\partial\Omega$.

Being inspired by Wang, Li 2018 and Wang, Li 2020 we first consider:

Definition 4. *For $f, g \in L^2(\partial\Omega)$ one defines the $\mathbb{R}$-linear $\mathbb{O}$-valued inner product*

$$(f, g)_{\partial\Omega} := \frac{3}{\pi^4} \int_{\partial\Omega} (\overline{n(x)g(x)})\,(n(x)f(x))dS(x)$$

Note: $(\cdot,\cdot)$ only $\mathbb{R}$-linear not $\mathbb{O}$-linear, but $\mathbb{O}$-para-linear.

Consequence: No Cauchy-Schwarz inequality nor direct analogue of the Fischer-Riesz representation theorem using this inner product.

But it is an octonionic Hilbert space in the sense of the previous definition!

Reproduction properties have to be considered component-wisely on the level of the real parts with respect to the real-inner valued products and then if possible lifted again to an octonionc function, see [ConKra2021] for details.

Formulas for the reproducing Szegö kernel of octonionic monogenic functions.

- Szegö kernel for $B_8(0,1)$ (Wang, Li 2018)

$$S_{S_7}(x,y) = \frac{1-\overline{x}y}{|1-\overline{x}y|^8}.$$

- Szegö kernel for $H^+(\mathbb{O})$ (K., MMAS 2021)

$$S_{H^+}(x,y) = \frac{\overline{x}+y}{|\overline{x}+y|^8}$$

- Let $d > 0$. The octonionic monogenic Szegö kernel of the strip domain $T := \{x \in \mathbb{O} \mid 0 < x_0 < d\}$ reads, cf. K. MMAS 2021,

$$S_T(x,y) = \sum_{n=-\infty}^{+\infty} (-1)^n \frac{\overline{x} + y + 2dn}{|\overline{x} + y + 2dn|^8}.$$

Proof of the formula for the unit ball.
For any function $f \in H^2(S_7)$ the octonionic Cauchy formula yields

$$\begin{aligned} f(y) &= \frac{3}{\pi^4} \int\limits_{S_7} \frac{\overline{x-y}}{|x-y|^8} \cdot (x \cdot f(x)) dS(x) \\ &= \frac{3}{\pi^4} \int\limits_{S_7} \frac{\overline{x} - \overline{y}(x\overline{x})}{|x||1-\overline{x}y|^8} \cdot (x \cdot f(x)) dS(x). \end{aligned}$$

Octonionic calculation rules imply $\overline{y}(x\overline{x}) = (\overline{y}x)x$. Moreover, $|x| = 1$,

$$\begin{aligned} f(y) &= \frac{3}{\pi^4} \int\limits_{S_7} \frac{\overline{x} - (\overline{y} \cdot x) \cdot \overline{x}}{|1-\overline{x}y|^8} \cdot (x \cdot f(x)) dS(x) \\ &= \frac{3}{\pi^4} \int\limits_{S_7} \Big(\frac{\overline{1-\overline{x}y}}{|1-\overline{x}y|^8} \cdot \overline{x}\Big) \cdot \Big(x \cdot f(x)\Big) dS(x) \end{aligned}$$

Assume that $\mathcal{T}$ is $\mathbb{O}$-para-linear and representable with a global kernel function $k(x, y)$ in the way

$$[\mathcal{T}f](x, y) = \langle f(\cdot), k(x, \cdot)\rangle_0.$$

Then there is a uniquely defined adjoint operator $\mathcal{T}^*$ acting on the corresponding dual space satisfying

$$\langle \mathcal{T}f, g\rangle_0 = \langle f, \mathcal{T}^* g\rangle_0.$$

In the setting of $\langle f, g\rangle_0$ we can talk about self-adjointness of an $\mathbb{O}$-para-linear octonionic integral operator $\mathcal{T}$ when $\langle \mathcal{T}f, g\rangle_0 = \langle f, \mathcal{T}g\rangle_0$.

Self-adjointness w.r.t. $\langle \cdot, \cdot\rangle_0$ happens when the associated kernel function of $\mathcal{T}^*$ is the conjugate of the kernel function of $\mathcal{T}$ in view of compatibility condition $\langle uv, w\rangle = \langle v, \overline{u}w\rangle$. In this context, we may speak about orthogonality.

Notion of self-adjointness for full octonionic inner product $(\cdot, \cdot)$ in the sense of usual $\mathbb{O}$-linearity is problematic. Due to lack of associativity we cannot expect in general $(\mathcal{T}f, g) = (f, \mathcal{T}g)$ for all $f, g \in L^2(\partial\Omega)$.

4. Bergman spaces of octonionic monogenic functions

Amazing feature: Also Bergman setting requires **intrinsic weight factor** in the definition of the inner product!!

Let us generally assume that $\Omega \subset \mathbb{O}$ is a simply connected orientable domain with strongly Lipschitz boundary $\partial\Omega$, where the exterior unit normal field $n(y)$ exists at almost every $y \in \partial\Omega$.

Now let $\omega(x)$ be a weight factor satisfying $|\omega(x)| = 1$ **a.e.** in Ω and $\omega(x) = 0$ **at most** in a Lebesgue-null set of Ω. Then define

$$(f,g)_{\Omega,\omega} = (f,g)_{\Omega} = \int_{\Omega} (\overline{\omega(x)g(x)})(\omega(x)f(x))dV(x).$$

Definition 5. *Let Ω be a convex domain with a smooth oriented $\partial\Omega$. If x is an interior point of Ω such that there is a uniquely defined point $\tilde{x} \in \partial\Omega$ satisfying*

$$|x - \tilde{x}| < |x - y| \ \ \forall y \in \partial\Omega \backslash \{\tilde{x}\},$$

then we may set $\omega(x) := n(\tilde{x})$, where $n(\tilde{x})$ indicates the exterior normal unit vector at $\tilde{x} \in \partial\Omega$. For all other interior points $x \in \Omega$ not having a uniquely defined nearest point in the boundary put $\omega(x) := 0$.

Assumption. Rest of the section: Suppose Ω to be that the Lebesgue measure of all points $x \in \Omega$ which do not have a uniquely defined nearest point on the boundary is zero.

Examples.

In $B_8(0,1)$ we have

$$\omega(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here, $n(\tilde{x}) = \frac{x}{|x|}$ for all $x \neq 0$.

In $H^+(\mathbb{O})$ we have $\omega(x) \equiv 1$ for all $x \in H^+(\mathbb{O})$.

In $T = \{x \in \mathbb{O} \mid 0 < \mathrm{Re}(x) < d\}$ we have

$$\omega(x) = \begin{cases} -1 & 0 < \mathrm{Re}(x) < \frac{d}{2} \\ 0 & \mathrm{Re}(x) = \frac{d}{2} \\ 1 & \frac{d}{2} < \mathrm{Re}(x) < d \end{cases}$$

Note that the set of $x \in T$ with $\mathrm{Re}(x) = \frac{d}{2}$ has in fact Lebesgue measure zero.

The next definition thus makes sense and encompasses all previously introduced particular definitions for $B_8(0,1), H^+(\mathbb{O})$ and T (also includes classical case $\omega(x) \equiv 1$).

Note that in the associative case these weight factors will be cancelled out, since $\overline{\omega(x)}\omega(x) = 1$.

Definition 6. *Let $\Omega \subset \mathbb{O}$ be a special domain and $\omega(x)$ be a weight factor with the properties stated above.*

For $f, g \in L^2(\Omega, \mathbb{O})$ one defines the following $\mathbb{R}$-linear $\mathbb{O}$-valued inner product

$$(f,g)_\Omega := \frac{3}{\pi^4} \int_\Omega (\overline{\omega(x) g(x)})\,(\omega(x) f(x)) dV(x)$$

The space of octonion-valued functions

$$\mathcal{B}^2(\Omega) := L^2(\Omega) \cap \{f : \Omega \to \mathbb{O} | \mathcal{D}f(x) = 0 \ \ \forall x \in \Omega\}$$

is called the Bergman space of left octonionic monogenic functions.

This inner product induces the usual $L^2(\Omega)$-norm, i.e.

$$(f, f)_\Omega = \frac{3}{\pi^4} \int_\Omega (\overline{\omega(x) f(x)}) \ (\omega(x) f(x)) dV(x) = \frac{3}{\pi^4} \int_\Omega |f(x)|^2 dV(x),$$

We can prove

Proposition 2. *The set* $(\mathcal{B}^2(\Omega), (\cdot, \cdot)_\Omega)$ *is an octonionic Hilbert space in the sense of Definition 3.*

Formulas for the octonionic monogenic Bergman kernels.

- Bergman kernel on $B_8(0,1)$: Wang, Li, 2018

$$B_{B_8}(x,y) = \frac{6(1-|x|^2y^2)(1-\overline{x}y)}{|1-\overline{x}y|^{10}} + \frac{2(1-\overline{x}y)(1-\overline{x}y)}{|1-\overline{x}y|^{10}}$$

- Bergman kernel on $H^+(\mathbb{O})$: Wang, Li, 2020

$$B_{H+}(x,y) = -2\frac{\partial}{\partial x_0}\left(\frac{\overline{x}+y}{|\overline{x}+y|^8}\right)$$

- Bergman kernel on T: K. 2021.

$$B_T(x,y) = (-2)\sum_{n=-\infty}^{+\infty}\frac{\partial}{\partial x_0}\left(\frac{\overline{x}+y+2dn}{|\overline{x}+y+2dn|^8}\right).$$

5. Hardy spaces of octonionic slice monogenic functions

5.1 General geometrical definition of Hardy spaces of octonionic slice monogenic functions

From now on assume that $\Omega \subset \mathbb{O}$ such that $\Omega \cap \mathbb{C}_I$ is an axially symmetric simply connected, orientable domain with strongly Lipschitz boundary.

Definition 7. *Now we define* $\mathbf{H}^2(\Omega, \mathbb{O})$ *as the closure of the set of slice monogenic functions on* Ω *belonging to* $L^2(\partial\Omega \cap \mathbb{C}_I)$ *for all* $I \in \mathbb{S}$*, namely*

$$\int_{\partial\Omega\cap\mathbb{C}_I} |f(z)|^2 |ds(z)| < \infty,$$

where $|ds(z)|$ *is the arc element.*

Also here we need to implement an **intrinsic weight factor** of norm 1!

We write the arc element on $\partial\Omega \cap \mathbb{C}_I$ as $t(z)|ds(z)|$ where

$$|t(z)ds(z)| = |ds(z)| \quad \text{and} \quad |t(z)|^2 = 1 = \overline{t(z)}t(z).$$

We equip $\mathbf{H}^2(\Omega, \mathbb{O})$ with the $\mathbb{O}$-valued $\mathbb{R}$-bilinear form

$$[f,g]_I := \int_{\partial\Omega\cap\mathbb{C}_I} \overline{(t(z)g(z))}\, |ds(z)|\, (t(z)f(z)),\ z = u + Iv \qquad (1)$$

for some fixed, but arbitrary $I \in \mathbb{S}$. This bilinear form depends on Ω and on choice of $I \in \mathbb{S}$.

We shall use the notation $\{\cdot\}_i$ for i-part and write $[f,g]_I = \sum_{i=0}^{7}\{[f,g]_I\}_i e_i$ where, in particular, $\{[f,g]_I\}_0 = \mathrm{Re}[f,g]_I$.

We proved

Proposition 3. *The set* $(\mathbf{H}^2(\Omega, \mathbb{O}), [\cdot,\cdot]_I)$ *is an octonionic Hilbert space in the sense of Definition 3.*

5.2. The octonionic unit ball case

Additional feature: we can provide a sequential characterization in terms of the Taylor coefficients.

Recall: If $f, g \in \mathcal{SM}(B_8(0,1))$, then

$$f(x) = \sum_{n \geq 0} x^n a_n \quad \text{and} \quad g(x) = \sum_{n \geq 0} x^n b_n.$$

We can now define an $\mathbb{O}$-valued inner product

$$[f, g] := \sum_{n \geq 0} \overline{b_n} a_n. \tag{2}$$

First we observe

Proposition 4. *The inner product (2) is* $\mathbb{R}$*-linear and defines the norm*

$$\|f\| = [f, f]^{1/2} = \left(\sum_{n\geq 0} |a_n|^2\right)^{1/2}. \qquad (3)$$

Definition 8. *The octonionic Hardy space of slice monogenic functions in the unit ball* $\mathbf{H}^2(B_8(0,1), \mathbb{O})$ *is defined as the subset of* $\mathcal{SM}(B_8(0,1))$ *consisting of functions* $f(x) = \sum_{n\geq 0} x^n a_n$ *for which* $\sum_{n\geq 0} |a_n|^2 < \infty$ *and equipped with the inner product (2).*

Proposition 5. $(\mathbf{H}^2(B_8(0,1),\mathbb{O}),[\cdot,\cdot])$ *is an octonionic Hilbert space in the sense of Definition 3.*

Next we provide the counterpart of the Szegö kernel in this framework,

$$\begin{aligned}\mathcal{S}(y,x) &= (1-2\mathrm{Re}(x)y+|x|^2y^2)^{-1}(1-yx) \\ &= (1-\bar{y}\bar{x})(1-2\mathrm{Re}(y)\bar{x}+|y|^2\bar{x}^2)^{-1}\end{aligned} \tag{4}$$

This kernel is well-defined for $1-2\mathrm{Re}(x)y+|x|^2y^2\neq 0$, namely for $x\notin[y^{-1}]$. It satisfies

$$\overline{\mathcal{S}(y,x)}=\mathcal{S}(x,y).$$

The kernel is called the octonionic slice monogenic Szegö kernel of the unit ball.

It is left slice monogenic in y and right slice monogenic in $\bar{x}$.

For $y, x \in B_8(0,1)$ we have $\mathcal{S}(y,x) = \sum_{n\geq 0} y^n \bar{x}^n$ and that

$\mathcal{S}(\cdot, x) \in \mathbf{H}^2(B_8(0,1), \mathbb{O})$ in fact $|\mathcal{S}(y,x)|^2 \leq \sum_{n\geq 0} |x|^{2n} < \infty$.

Also here we can prove

Theorem 5. $\mathbf{H}^2(B_8(0,1), \mathbb{O})$ *has a uniquely defined reproducing kernel satisfying:*

$$[f, \mathcal{S}(\cdot, x)] = f(x),$$

for any $f \in \mathbf{H}^2(B_8(0,1), \mathbb{O})$.

Proof. Let $f(x) = \sum_{n\geq 0} x^n a_n$. We immediately have

$$[f(y), \mathcal{S}(y,x)] = \left[\sum_{n\geq 0} y^n a_n, \sum_{n\geq 0} y^n \bar{x}^n\right] = \sum_{n\geq 0} x^n a_n = f(x).$$

□

Now we want to develop analogous **geometric characterization** of the Hardy space like in the monogenic case.

Consider functions $\mathcal{SM}(B_8(0,1)$ and adapt (1) to this case, so that $z = e^{I\theta}$, $t(z) = Ie^{I\theta}$, $|ds(z)| = d\theta$, $\theta \in [0, 2\pi]$:

$$[f,f]_I := \frac{1}{2\pi}\int_0^{2\pi}\left(\overline{Ie^{I\theta}f(e^{I\theta})})d\theta(Ie^{I\theta}f(e^{I\theta}))\right) < \infty \qquad (5)$$

for some $I \in \mathbb{S}$. We need the constant $1/2\pi$ to have $[1,1]_I = 1$.

We have:

Proposition 6. *(i)* $f \in \mathbf{H}^2(B_8(0,1), \mathbb{O})$ *if and only if* $[f,f]_I < \infty$ *for some* $I \in \mathbb{S}$.

(ii) Let $I, J \in \mathbb{S}$. *Then* $[f,f]_I < \infty$ *iff* $[f,f]_J < \infty$.

From this result we now obtain:
Corollary 1. *The Hardy space* $\mathbf{H}^2(B_8(0,1),\mathbb{O})$ *consists of the subset of* $\mathcal{SM}(B_8(0,1))$ *such that* $[f,f]_I$ *is finite for some* I *(and so for all) in* $\mathbb{S}$*.*

Remark: To obtain a description of the kernel according to the inner product $[\cdot,\cdot]_I$ we consider $x,y\in\mathbb{C}_I$ and $[f(y),\mathcal{S}(y,x)]_I$ where $\mathcal{S}(y,x)$ is defined in (4) and the inner product is (1). We have:

$$\begin{aligned}[f(y),\mathcal{S}(y,x)]_I &= \frac{1}{2\pi}\int_{S^7\cap\mathbb{C}_I}\overline{(t(y)\mathcal{S}(y,x))}\,|ds(z)|\,(t(y)f(y))\\ &= \frac{1}{2\pi}\int_{S^7\cap\mathbb{C}_I}(\overline{\mathcal{S}(y,x)}\;\overline{t(y)})\,|ds(z)|\,(t(y)f(y))\end{aligned}$$

and using Artin's theorem we obtain that the latter expression equals

$$\frac{1}{2\pi}\int_{S^7\cap\mathbb{C}_I}\mathcal{S}(x,y)\,|ds(z)|\,f(y)$$

$$= \frac{1}{2\pi}\int_{S^7\cap\mathbb{C}_I} \mathcal{S}(x,y)\,|ds(z)|\,(F_1(y) + F_2(y)I_2 + (G_1(y) + G_2(y)I_2)I_4) = f(x)$$

where we used the Splitting Lemma to write $f(y)$, again Artin's theorem and applied the reproducing property to the components of f. Note that

The reproducing property of the kernel holds on each complex plane!

The values of the function f can be computed at any point in the unit ball using the Representation Formula!

6. Further results

We also managed to address succesfully

- Definition of the Hardy spaces of slice monogenic functions in the setting of $H^+(\mathbb{O})$ and strip domain T including explicit formulas for the reproducing kernel functions

- Analogous theory for Bergman spaces of slice monogenic functions in the settings of the unit ball (including sequential characterizations), the right half-space and the strip domain T together with explicit representation formulas od the reproducing kernel function

- Kerzman-Stein theory for the monogenic setting

7. Main references:

[KraMMAS] R.S. Kraußhar. Recent and new results on octonionic Bergman and Szegö kernels, *Mathematical Methods in the Applied Sciences* (2022), https://doi.org/10.1002/mma.7316, 14pp.

[ConKra2021] D. Constales and R.S. Kraußhar. Octonionic Kerzman-Stein Operators, *Complex Analysis and Operator Theory* **15**: 104 (2021), 23pp.

[CKS3] F. Colombo, R.S. Kraußhar and I. Sabadini. Octonionic monogenic and slice monogenic Hardy and Bergman spaces, under review. Preprint February 2023.