A Simple Wiener-Hopf factorization method for pricing options with barriers in Lévy-driven models

Kudryavtsev Oleg

Russian Customs Academy, Rostov branch, koe@donrta.ru
Southern Federal University, koe@sfedu.ru
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Outline

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Pricing options under Lévy processes

Option valuation under Lévy processes has been dealt with by a host of researchers.
However, pricing path-dependent options in exponential Lévy models still remains a computational challenge.

Path-dependent options: barrier options

A double barrier option is a contract which pays the specified amount $G(S_T)$ at the terminal date T, provided during the lifetime of the contract, the price of the stock does not cross specified constant barriers D from above and U from below. When at least one of the barriers is crossed, the option expires worthless or the option owner is entitled to some *rebate*.
If $U = +\infty$, we obtain a down-and-out single barrier option.
If $D = -\infty$, we deal with an up-and-out single barrier option.

Historical background

Methods for pricing barrier options: drawbacks

- Monte Carlo methods: *slow*
- Finite difference schemes: *application entails a detailed analysis of the underlying Lévy process*
- Wiener-Hopf factorization methods:
  - Single barriers: *non trivial approximate formulas are needed in general case*
  - Double barriers: *application involves an iterative solution to the pair of coupled WH-integral equations or matrix factorization*

The main goal

The goal of the current paper is to suggest a new, easy, and effective method to price double barrier options under pure non-Gaussian Lévy processes with jumps of finite variation. The main advantage of the approach is applying explicit Wiener-Hopf factorization formulas

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Lévy processes: a short reminder

General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$.

The characteristic exponent of Lévy process

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y 1_{|y|\le 1})F(dy),$$

where σ^2 is the variance of the Gaussian component, and the Lévy measure $F(dy)$ satisfies $\int_{\mathrm{R}\setminus\{0\}} \min\{1, y^2\}F(dy) < +\infty$.
If $F(dx) = \pi(x)dx$, $\pi(x)$ – Lévy density.

Examples of Lévy processes, $F(\mathbb{R}) < \infty$

Jump diffusion

$X_t = \gamma_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$, where W_t – Brownian motion, N_t – Poisson process with intensity λ, and Y_i – i.i.d of jumps.

Kou model

The Lévy density $\pi(x)$, is of the form

$$\pi(x) = (1-p)\lambda\Lambda_- e^{\Lambda_- x} 1_{\{x<0\}} + p\lambda\Lambda_+ e^{-\Lambda_+ x} 1_{\{x>0\}}.$$

where $\Lambda_- > 0, \Lambda_+ > 1, 0 < p < 1, \lambda > 0$.
If we set $c_+ = (1-p)\lambda\Lambda_-$, $c_- = p\lambda\Lambda_+$, $\lambda_+ = \Lambda_-$, $\lambda_- = -\Lambda_+$, then

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $\sigma > 0$, $\mu = \gamma_0 - \int_{-1}^{1} xF(dx)$, $c_\pm > 0$ and $\lambda_- < -1 < 0 < \lambda_+$.

Examples of Lévy processes, $F(\mathbb{R}) = \infty$

A Lévy process of finite variation

$$\psi(\xi) = -i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y}) F(dy), \int_{\mathbb{R}\setminus\{0\}} \min\{1, |y|\} F(dy) < +\infty.$$

Tempered stable Lévy processes (TSL) of finite variation

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] +$$
$$c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}],$$

where $\nu_+, \nu_- \in (0,1)$, $c_+, c_- > 0$, $\mu \in \mathbb{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. If $c_- = c_+ = c$ and $\nu_- = \nu_+ = \nu$, then we obtain a KoBoL (CGMY) model.

$$\pi(x) = c_+ e^{\lambda_+ x} |x|^{-\nu_+ - 1} 1_{\{x<0\}} + c_- e^{\lambda_- x} |x|^{-\nu_- - 1} 1_{\{x>0\}}.$$

In the CGMY parametrization: $C = c$, $Y = \nu$, $G = \lambda_+$, $M = -\lambda_-$.

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Definition

Direct Fourier transform $\mathcal{F}_{x\to\xi}$:

$$\hat{g}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} g(x)dx.$$

Inverse Fourier transform $\mathcal{F}^{-1}_{\xi\to x}$:

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \hat{g}(\xi)d\xi.$$

Some properties

- $\mathcal{F}_{x\to\xi}\mathcal{F}^{-1}_{\xi\to x} = I$ and $\mathcal{F}^{-1}_{\xi\to x}\mathcal{F}_{x\to\xi} = I$
- $\mathcal{F}_{x\to\xi}(g * f) = \overline{\mathcal{F}_{x\to\xi}(g)} \cdot \mathcal{F}_{x\to\xi}(f) = \mathcal{F}_{x\to\xi}(g) \cdot \overline{\mathcal{F}_{x\to\xi}(f)}$, where $(g * f)(x) = \int_{-\infty}^{+\infty} g(x+y)f(y)dy = \int_{-\infty}^{+\infty} g(z)f(z-x)dz$.

Wiener-Hopf factorization for $E[e^{i\xi X_{T_q}}]$

Let $q > 0$, X_t be a Lévy process with characteristic exponent $\psi(\xi)$, $T_q \sim \operatorname{Exp} q$, $\bar{X}_t = \sup_{0\leqslant s\leqslant t} X_s$ and $\underline{X}_t = \inf_{0\leqslant s\leqslant t} X_s$ – supremum and infimum processes.

$$\phi_q^+(\xi) = E[e^{i\xi \bar{X}_{T_q}}], \quad \phi_q^-(\xi) = E[e^{i\xi \underline{X}_{T_q}}], \quad \frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi).$$

Wiener-Hopf factorization for operators: $\mathcal{E} = \mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+$

$$\mathcal{E}_q g(x) = E[g(x + X_{T_q})] = \int_{-\infty}^{+\infty} g(x+y)P(y)dy,$$

$$\mathcal{E}_q^+ g(x) = E[g(x + \bar{X}_{T_q})] = \int_{-\infty}^{+\infty} g(x+y)P_+(y)dy,$$

$$\mathcal{E}_q^- g(x) = E[g(x + \underline{X}_{T_q})] = \int_{-\infty}^{+\infty} g(x+y)P_-(y)(dy),$$

where $P(y)$, $P_\pm(y)$ are probability densities with $P_\pm(y) = 0, \forall \pm y < 0$.

Pseudo-differential operator (PDO)

A PDO $A = a(D)$ with the symbol $a(\xi)$ acts as follows ($D = -i\frac{d}{dx}$):

$$Ag(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi)\hat{g}(\xi)d\xi.$$

In short, $Ag(x) = \mathcal{F}^{-1}_{\xi\to x} a(\xi)\mathcal{F}_{x\to\xi} g(x)$

$\mathcal{E}$ and $\mathcal{E}^{\pm}$ as PDO

$$\mathcal{E}_q g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} q(q+\psi(\xi))^{-1}\hat{g}(\xi)d\xi,$$

$$\mathcal{E}^{\pm}_q g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \phi^{\pm}_q(\xi)\hat{g}(\xi)d\xi.$$

WHF in an operator form: $\mathcal{E} = \mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+$.

Useful facts and relations

Let X_t – Lévy process, and $T_q \sim \text{Exp}\, q$. Then

- $\underline{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ – independent;
- $\bar{X}_{T_q}$ and $X_{T_q} - \underline{X}_{T_q}$ – identically distributed.

A trivial factorization: $\bar{X}_{T_q} = X_{T_q}$

We have $\underline{X}_{T_q} = 0$, and we obtain:

$$\phi_q^+(\xi) = \frac{q}{q + \psi(\xi)}, \quad \phi_q^-(\xi) = 1.$$

A trivial factorization: $\underline{X}_{T_q} = X_{T_q}$

We have $\bar{X}_{T_q} = 0$, and we obtain:

$$\phi_q^-(\xi) = \frac{q}{q + \psi(\xi)}, \quad \phi_q^+(\xi) = 1.$$

Explicit WHF: Gaussian Lévy process

Let $X_t = \gamma_0 t + \sigma W_t$, then

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\gamma\xi.$$

For $q > 0$,the equation $q + \psi(\xi) = 0$ has two roots $-i\beta_-$ and $-i\beta_+$, where $\beta_- < 0$ and $\beta_+ > 0$.

The function $q(q + \psi(\xi))^{-1}$ admits WHF with

$$\phi_q^+(\xi) = \frac{\beta_+}{\beta_+ - i\xi}, \quad \phi_q^-(\xi) = \frac{-\beta_-}{-\beta_- + i\xi}.$$

The functions ϕ_q^- and ϕ_q^+ are chf of exponential distributions on negative ans positive half-lines, respectively:

$$P_q^-(dx) = -\beta_- e^{-\beta_- x} 1_{(-\infty;0]}(x)dx, \, P_q^+(dx) = \beta_+ e^{-\beta_+ x} 1_{[0;+\infty)}(x)dx.$$

Explicit WHF: Kou model

In Kou model

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi}.$$

For $q > 0$,the equation $q + \psi(\xi) = 0$ has four roots $-i\beta_1^-$, $-i\beta_0^-$, $-i\beta_0^+$ and β_1^+, where $\beta_1^- < \lambda_- < \beta_0^- < 0 < \beta_0^+ < \lambda_+ < \beta_1^+$.

The function $q(q + \psi(\xi))^{-1}$ admits WHF with

$$\phi_q^+(\xi) = \frac{\lambda_+ - i\xi}{\lambda_+} \prod_{j=0,1} \frac{\beta_j^+}{\beta_j^+ - i\xi},$$

$$\phi_q^-(\xi) = \frac{-\lambda_- + i\xi}{-\lambda_-} \prod_{j=0,1} \frac{-\beta_j^-}{-\beta_j^- + i\xi}.$$

Approximate Wiener-Hopf factorization

Formulas for WH-factors

For a wide class of Lévy processes X_t, the following integral representations for $\phi_q^+(\xi)$, $\phi_q^-(\xi)$ are valid (see details in Boyarchenko and Levendorskii (2002)):

$$\phi_q^+(\xi) = \exp\left[(2\pi i)^{-1}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta\right];$$

$$\phi_q^-(\xi) = \exp\left[-(2\pi i)^{-1}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+}\frac{\xi\ln(q+\psi(\eta))}{\eta(\xi-\eta)}d\eta\right],$$

where $\omega_- < 0 < \omega_+$ with ω_-, ω_+ depending on the Lévy process X_t parameters.

The direct computation of $\phi_q^+(\xi)$ and $\phi_q^-(\xi)$ require $O(NM)$ operations, where N is a number of ξ-points and M is a number of points for numerical integration

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The problem for double barrier options

Let T, K, D, U be the maturity, strike, the lower barrier, and the upper barrier, and the stock price $S_t = De^{X_t}$ be an exponential Lévy process under a chosen risk-neutral measure ($\psi(-i) + r = 0$) which has no diffusion component ($\sigma = 0$) and only jumps of finite variation.

Let us introduce $h = \ln U/D$. We consider options, whose payoff at maturity date T depends on $(X_T, \underline{X}_T, \bar{X}_T)$.

Consider

$$V(T, x) = E^x \left[e^{-rT} 1_{\underline{X}_T > 0} 1_{\bar{X}_T < h} G(X_T) \right],$$

where

time 0 is the beginning of a period under consideration,

T – the final date,

0 and h are the absorbing barriers,

$G(X_T)$ – the payoff function at time T provided the barriers has't been crossed.

Time randomization and Laplace transform

$$\hat{V}(q,x) = \int_0^{+\infty} e^{-qt} E^x\left[e^{-rt} G(X_t) 1_{\underline{X}_t>0} 1_{\bar{X}_t<h}\right] dt$$
$$= E^x \left[\int_0^{+\infty} e^{-(q+r)t} G(X_t) 1_{\underline{X}_t>0} 1_{\bar{X}_t<h} dt\right].$$

$$v_n(q,x) := \frac{(-1)^{n-1} q^n}{(n-1)!} \partial_q^{n-1} \hat{V}(q,x)$$
$$= \int_0^{+\infty} \frac{q^n t^{n-1}}{(n-1)!} e^{-(q+r)t} E^x\left[G(X_t) 1_{\underline{X}_t>0} 1_{\bar{X}_t<h}\right] dt$$
$$= E^x \left[\frac{G(X_{\mathrm{T}(n,q+r)})}{(1+r/q)^n} 1_{\underline{X}_{\mathrm{T}(n,q+r)}>0} 1_{\bar{X}_{\mathrm{T}(n,q+r)}<h}\right],$$

$\mathrm{T}(n,q)$ is a Gamma random variable.

Numerical Laplace transform inversion

Post-Widder formula

If $f(\tau)$ is a function of a nonnegative real variable τ and the Laplace transform $\tilde{f}(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\, d\tau$ is known, the approximate Post-Widder formula for $f(\tau)$ can be written as

$$f(\tau) = \lim_{N\to\infty} f_N(\tau); \quad f_N(\tau) := \frac{(-1)^{N-1}}{(N-1)!} \left(\frac{N}{\tau}\right)^N \tilde{f}^{(N-1)}\left(\frac{N}{\tau}\right),$$

where $\tilde{f}^{(N)}(\lambda)$ – Nth derivative of the Laplace transform $\tilde{f}$ at λ.
Set $q = T/N$. The convergence $v_N(T/N, x)$ to $V(T, x)$ as $N \to \infty$ is of order N^{-1}.

Single barrier options in Lévy models, Post-Widder formula

Kudryavtsev, O., "An efficient numerical method to solve a special class of integro-differential equations relating to the Levy models" // *Mathematical Models and Computer Simulations*, 2011, V.3., N.6.

Iterative scheme

Taking into account that $\mathrm{T}(n, q+r) \sim \mathrm{T}(n-1, q+r) + T_{q+r}$, we can represent $X_{\mathrm{T}(n,q+r)} = X_{T^1_{q+r}} + \ldots + X_{T^n_{q+r}}$, where $T^1_{q+r}, \ldots, T^n_{q+r}$ are consecutive time increments being independent exponentially distributed random variables with the parameter $q+r$.

Using the relations

$$1_{\underline{X}_{\mathrm{T}(n,q+r)}>0} = 1_{\underline{X}_{\mathrm{T}(n-1,q+r)}>0} 1_{X_{\mathrm{T}(n-1,q+r)}+\underline{X}_{T^n_{q+r}}>0},$$

$$1_{\bar{X}_{\mathrm{T}(n,q+r)}<h} = 1_{\bar{X}_{\mathrm{T}(n-1,q+r)}<h} 1_{X_{\mathrm{T}(n-1,q+r)}+\bar{X}_{T^n_{q+r}}<h},$$

we obtain that for $n = 1, 2, \ldots$

$$v_n(q,x) = E^x\left[\frac{v_{n-1}(q, X_{T_{q+r}})}{(1+r/q)} 1_{\underline{X}_{T_{q+r}}>0} 1_{\bar{X}_{T_{q+r}}<h}\right],$$

where $v_0(q,x) = G(x) 1_{(0,h)}(x)$.

Convergence

Theorem 1

Let N be a suffciently large natural number. Set $q = T/N$, $v_0(q,x) = G(x)1_{(0,h)}(x)$, and for $n = 1, 2, \ldots$ define

$$v_n(q,x) = E^x\left[\frac{v_{n-1}(q, X_{T_{q+r}})}{(1 + r/q)} 1_{\underline{X}_{T_{q+r}}>0} 1_{\bar{X}_{T_{q+r}}<h}\right],$$

where the random time $T_{q+r} \sim \mathrm{Exp}\,(q + r)$.
For a fixed x, $v_N(T/N, x)$ converges to $V(T, x)$ as $N \to \infty$.

The state-of-art implementation of the Wiener-Hopf method

Pricing single barrier options

One needs to factorize $(q+r)/(q+r+\psi(\xi))$.Then one can calculate the sequence $v_n(q,x)$ with $q = N/T$ and $h = +\infty$ as follows: for $n = 1, \ldots, N$

$$\begin{aligned} v_n(q,x) &= \frac{1}{(1+r/q)} \cdot E^x[v_{n-1}(q, \underline{X}_{T_{q+r}} + \bar{X}_{T_{q+r}}) 1_{\underline{X}_{T_{q+r}}>0}] \\ &= \frac{1}{(1+r/q)} \mathcal{E}^-_{q+r} 1_{(0,+\infty)} \mathcal{E}^+_{q+r} v_{n-1}(q,x). \end{aligned}$$

Unfortunately, in the case of double barrier options (with $h < +\infty$), such formulas are not available for general Lévy models. Instead one needs to factorize the following matrix:

$$\begin{pmatrix} \exp(i\xi h) & 0 \\ (q+r)/(q+r+\psi(\xi)) & \exp(-i\xi h) \end{pmatrix}$$

Wiener-Hopf factorization for pricing single barrier options

- In Kudryavtsev and Levendorskiǐ (2009) the fast, accurate and universal numerical method for pricing single barrier option under Lévy models was developed;
- Eberlein et al. (2011) derive expressions for the analytically extended characteristic function of the supremum and the infimum of a Lévy process at the final time moment;

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Wiener-Hopf factorization for pricing double barrier options

- Boyarchenko and Levendorskiĭ (2011) suggest the method that involves iterative solving the pair of coupled Wiener-Hopf integral equations for the prices under Laplace transform.
- Phelan et al. (2018) suggest the method that involves iterative solving the pair of coupled Wiener-Hopf integral equations for required characteristic functions related to the distribution of Levy process postion, its supremum and infimum.

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Outline

Splitting

A subordinator is a Lévy process with sample paths being almost surely non-decreasing. It has no diffusion component, only positive jumps of finite variation and non-negative drift.

The first step

- We represent $X_t = X_t^+ - (-X_t^-)$, where X_t^+ and $-X_t^-$ are subordinators. Denote by $\psi_+(\xi)$ and $\psi_-(\xi)$ the characteristic exponents of X_t^+ and X_t^-, respectively.
- If the drift $\mu \geqslant 0$:
  $\psi_+(\xi) = -i\mu\xi + \int_0^{+\infty}(1 - e^{i\xi y})F(dy)$,
  $\psi_-(\xi) = \int_{-\infty}^0(1 - e^{i\xi y})F(dy)$.
- If the drift $\mu < 0$:
  $\psi_+(\xi) = \int_0^{+\infty}(1 - e^{i\xi y})F(dy)$,
  $\psi_-(\xi) = -i\mu\xi + \int_{-\infty}^0(1 - e^{i\xi y})F(dy)$.

Approximation. Key ideas

Introduce $X_t^{+,1} \sim X_t^+$ and $X_t^{+,2} \sim X_t^+$.
It follows that $X_t \sim X_{t/2}^{+,1} + X_t^- + X_{t/2}^{+,2}$.

For a fixed $t > 0$ the current position of X_t with starting point x has the same distribution as the final position of the discrete-time process Y_j^t with the following dynamics:

- $Y_0^t = x$;
- $Y_1^t = Y_0^t + X_{t/2}^{+,1}$ – an upward movement;
- $Y_2^t = Y_1^t + X_t^-$ – a downward movement;
- $Y_3^t = Y_2^t + X_{t/2}^{+,2}$ – an upward movement.

Let a natural number N be sufficiently large and $q = N/T$.
We approximate $X_{T_{q+r}}$ with $Y^{T_{q+r}}$, since the randomized time T_{q+r} converges in quadratic mean to 0 as $N \to +\infty$.

Approximate Wiener-Hopf factorization

Introduce the following operators:

$$\mathcal{E}^+ u(x) = E[u(x + \bar{X}^+_{T_{q+r/2}})], \mathcal{E}^- u(x) = E[u(x + \underline{X}^-_{T_{q+r}})].$$

$$\begin{aligned} v_n(q,x) &\approx \frac{1}{(1+r/q)} E[v_{n-1}(q, Y_3^{T_{q+r}}) 1_{\underline{Y}^{T_{q+r}}>0} 1_{\overline{Y}^{T_{q+r}}<h}] \\ &= \frac{1_{(0,h)}(x)}{(1+r/q)} \cdot \\ &\quad E^x[v_{n-1}(q, X^{+,1}_{T_{q+r/2}} + X^-_{T_{q+r}} + X^{+,2}_{T_{q+r/2}}) 1_{X^{+,1}_{T_{q+r/2}}<h} \\ &\quad \cdot 1_{X^{+,1}_{T_{q+r/2}} + X^-_{T_{q+r}} > 0} 1_{X^{+,1}_{T_{q+r/2}} + X^-_{T_{q+r}} + X^{+,2}_{T_{q+r/2}} < h}] \\ &= \frac{1_{(0,h)}(x)}{(1+r/q)} \mathcal{E}^+ 1_{(-\infty,h)} \mathcal{E}^- 1_{(0,+\infty)} \mathcal{E}^+ 1_{(-\infty,h)} v_{n-1}(q,x). \end{aligned}$$

Pricing barrier options

$$\mathcal{E}_q^+ 1_{(-\infty,h)} = 1_{(-\infty,h)} \mathcal{E}_q^+ 1_{(-\infty,h)};\, \mathcal{E}_q^- 1_{(0,+\infty)} = 1_{(0,+\infty)} \mathcal{E}_q^- 1_{(0,+\infty)}.$$

Single barrier case

$$\begin{aligned} v_0(q,x) &= G(x) 1_{(0,+\infty)}(x);\, q = N/T, \\ v_n(q,x) &= \frac{1_{(0,+\infty)}(x)}{(1+r/q)} \mathcal{E}^+ \mathcal{E}^- 1_{(0,+\infty)} \mathcal{E}^+ v_{n-1}(q,x),\, n = 1, \ldots, N. \end{aligned}$$

Double barrier case

$$\begin{aligned} v_0(q,x) &= G(x) 1_{(0,h)}(x);\, q = N/T, \\ v_n(q,x) &= \frac{1_{(0,h)}(x)}{(1+r/q)} \mathcal{E}^+ 1_{(0,h)} \mathcal{E}^- 1_{(0,h)} \mathcal{E}^+ v_{n-1}(q,x),\, n = 1, \ldots, N. \end{aligned}$$

Approximate Wiener-Hopf factorization

Notice that $X^+_{T_{q+r/2}}$ and $X^-_{T_{q+r}}$ admit trivial factorizations. Set

$$\phi_+(\xi) = E[e^{i\xi \bar{X}^+_{T_{q+r/2}}}] = E[e^{i\xi \bar{X}^+_{T_{2(q+r)}}}], \phi_-(\xi) = E[e^{i\xi \underline{X}^-_{T_{q+r}}}].$$

$$\phi_+(\xi) = \frac{2(q+r)}{2(q+r)+\psi_+(\xi)}, \phi_-(\xi) = \frac{q+r}{q+r+\psi_-(\xi)}.$$

Now, we can rewrite the operators $\mathcal{E}^+$ and $\mathcal{E}^-$ as follows

$$\begin{aligned}
\mathcal{E}^+ u(x) &= E[u(x+\bar{X}^+_{T_{q+r/2}})] \\
&= (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi} 2(q+r)/(2(q+r)+\psi_+(\xi))\hat{u}(\xi)d\xi \\
\mathcal{E}^- u(x) &= E[u(x+X^-_{T_{q+r}})] \\
&= (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{ix\xi}(q+r)/(q+r+\psi_-(\xi))\hat{u}(\xi)d\xi.
\end{aligned}$$

Outline

Platform of numerical methods for computational finance

Premia

The program platform Premia (www.premia.fr) developed by the "MathRisk" team at INRIA (the French national institute for research in computer science and control) and financially supported by a consortium of French banks (Credit Agricole Corporate and Investment Bank, Natixis and others).
Premia is a software designed for option pricing, hedging and financial model calibration. It is provided with it's C/C++ source code and an extensive scientific documentation.

PNL

PNL is a numerical library for C and C++ programmers. It is free software under the GNU LGPL. This library is currently used by the PREMIA software. Available at https://github.com/pnlnum/pnl

Simple Wiener-Hopf factorization method for single barrier options

Numerical experiments in Kudryavtsev and Luzhetskaya (2020) for the case of single barrier options show that the approach similar to the SWHF-method can be a competitor for the efficient algorithm developed in Kudryavtsev and Levendorskiǐ (2009) that uses more accurate construction of an approximate Wiener-Hopf factorization.

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Simple Wiener-Hopf factorization method for double barrier options in Lévy models with jumps of bounded variation

Numerical experiments in Kudryavtsev (2021) for the case of double barrier options in Lévy models of bounded variation show that the approach similar to the SWHF-method can be a competitor for the efficient algorithm developed in Boyarchenko M. and Levendorskii S.(2011) that uses more involved construction of an approximate Wiener-Hopf factorization and an iterative procedure.

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Numerical examples. Double barrier options

Knock-and-out put option

As a basic example, we consider the knock-and-out put option with the strike K, the lower barrier D, the upper barrier U and time to expiry T.

Model parameters

We take the KoBoL (TSL) model of order $\nu \in (0, 1)$, with the parameters
$\sigma = 0$, $\nu = 0.5$, $\lambda_+ = 9$, $\lambda_- = -8$, $c = 1$
($C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ in CGMY parametrization).

Option parameters

the instantaneous interest rate: $r = 0.03$,
time to expiry: $T = 0.1$ year,
the strike price: $K = 3500$,
the barriers: $D = 2800$ and $U = 4200$.
In this case, the drift parameter μ is approximately -0.0423.

Numerical examples

We check the performance of the SWHF-method against prices obtained by a Monte Carlo method (MC-method) and the approximate Wiener-Hopf factorization method (AWHF-method) developed in Boyarchenko M. and Levendorskii S.(2011).

The computations of the option prices by the SWHF-method performed in 5 points $x_k = \ln(S/K)(= 0.81, 0.90, 1.00, 1.10, 1.19)$, where S – initial spot price.

We use the prices calculated by the MC-method with 500, 000 sample paths simulations and 2000 time steps along each trajectory as the benchmark.

Bibliography

Boyarchenko M., Levendorskii S. “Valuation of continuously monitored double barrier options and related securities”, *Math Financ.*, 2011.

Performance of the SWHF method

Prices: knock-and-out put in KoBoL (CGMY) model

	MC[a]	AWHF[b]	SWHF[c]	SWHF	SWHF
M	$5 \cdot 10^5$	10^{12}	10^{11}	10^{12}	10^{12}
N	2000	80	80	80	320
$S/K = 0.81$	222.545	222.5256	222.3233	220.3986	221.5624
$S/K = 0.90$	303.337	302.2708	302.3494	302.2064	302.8453
$S/K = 1.00$	78.672	78.19000	79.0976	79.1141	78.8553
$S/K = 1.10$	14.578	14.5985	14.8533	14.8482	14.6945
$S/K = 1.19$	3.541	3.5222	3.6042	3.5885	3.5624

[a] The MC-method: M – the number of sample paths, N – the number of time steps.
[b] The AWHF-method: M – the number of space points, N – the number of time steps.
[c] The SWHF-method: M – the number of space points, N – the number of time steps.

Performance of the SWHF method

Relative errors w.r.t. MC: knock-and-out put in KoBoL (CGMY) model

	AWHF[a]	SWHF[b]	SWHF	SWHF
M	10^{12}	10^{11}	10^{12}	10^{12}
N	80	80	80	320
$S/K = 0.81$	-0.01%	-0.1%	-1.0%	-0.4%
$S/K = 0.90$	-0.35%	-0.3%	-0.4%	-0.2%
$S/K = 1.00$	-0.61%	0.5%	0.6%	0.2%
$S/K = 1.10$	0.14%	1.9%	1.9%	0.8%
$S/K = 1.19$	-0.52%	1.8%	1.4%	0.6%

[a] The AWHF-method: M – the number of space points, N – the number of time steps.

[b] The SWHF-method: M – the number of space points, N – the number of time steps.

Conclusion

- We suggest a new approach for pricing path-dependent options with a payoff depending on the infimum and supremum of Lévy processes at expiry
- The SFWH-prices converge sufficiently fast, and the relative errors reported show that increasing the number of time steps improves the accuracy of the method.
- The same algorithm's parameters for the AWHF and SWHF-methods lead to similar results. However, the SWHF-method is rather simpler to implement into program.
- The calculating knock-and-out put prices takes a fraction of a second.
- The method suggested makes it easy to implement such a sophisticated tool as the matrix Wiener-Hopf factorization for general Lévy models with jumps of finite variation.