Spectral Geometry of Graphs

Pavel Kurasov

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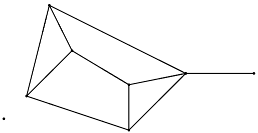
Analysis, Differential Equations and Mathematical Physics

Quantum graphs = Schrödinger operators on metric graphs

- **Metric graph Γ**



- **Differential expression** on the edges

$$\ell_{q,a} = \left(i\frac{d}{dx} + a(x) \right)^2 + q(x)$$

- **Matching conditions** at every vertex: only standard conditions today

$$\begin{cases} \psi(x_i) = \psi(x_j),\ x_i, x_j \in V_m & \text{– continuity condition;} \\ \displaystyle\sum_{x_j \in V_m} \partial\psi(x_j) = 0 & \text{– Kirchhoff condition.} \end{cases}$$

Graph's spectrum = spectrum of the standard Laplacian on Γ
Schrödinger spectrum = spectrum of the Schrödinger operator on Γ

Secular equation

$$L_{q,a}\psi(x) = \lambda\psi(x), \quad \lambda = k_n^2, n = 1, 2, \ldots, \; x \in \Gamma$$

- scattering approach - most suitable for Laplacians

$$\boxed{p_\Gamma(k) = \det\left(\mathbf{S_e}(k) - \mathbf{S_v}\right) = 0}$$

where $\mathbf{S_e}$ and $\mathbf{S_v}$ are the edge and vertex scattering matrices.

$$\mathbf{S_e} \sim \begin{pmatrix} 0 & e^{ik\ell_n} \\ e^{ik\ell_n} & 0 \end{pmatrix}, \quad \mathbf{S_v} \sim -\mathbf{I} + \frac{2}{d}\mathbf{J}, \;\; \mathbf{J} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

- M-function approach - can be used for Schrödinger

An edge $E_1 = [x_1, x_2] \Rightarrow M^1(\lambda) : \begin{pmatrix} \psi(x_1) \\ \psi(x_2) \end{pmatrix} \mapsto \begin{pmatrix} \partial\psi(x_1) \\ \partial\psi(x_2) \end{pmatrix}$

$$\Rightarrow \mathbf{M} = \begin{pmatrix} M_{11}^1 + M_{11}^2 & M_{12}^1 + M_{12}^2 & 0 \\ M_{21}^1 + M_{21}^2 & M_{22}^1 + M_{22}^2 + M_{11}^3 & M_{12}^3 \\ 0 & M_{21}^3 & M_{22}^3 \end{pmatrix}$$

$$\boxed{\det \mathbf{M}(\lambda) = 0}$$

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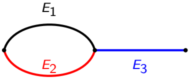

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Secular polynomials

$$e^{ik\ell_n} = z_n \Rightarrow P_\Gamma(\mathbf{z}) = \det \left(\begin{pmatrix} 0 & z_1 & 0 & 0 & \dots \\ z_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & z_2 & \dots \\ 0 & 0 & z_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \mathbf{S_v} \right)$$

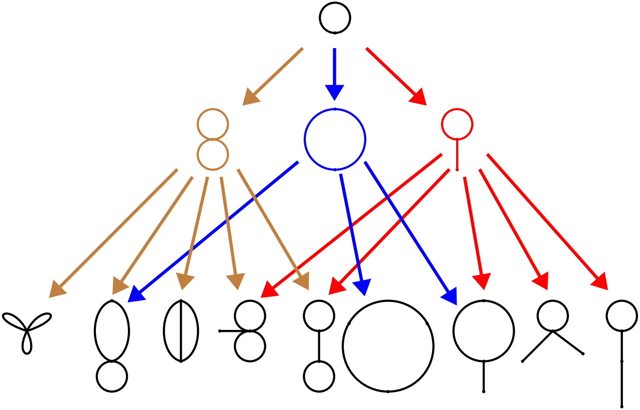
- second degree in each variable
- connected to stable polynomials $\Rightarrow$ the spectrum of Γ is real
- hierarchy with respect to the number of edges

$$\Gamma' = \Gamma/_{E_N} \Rightarrow P_{\Gamma'}(z_, \dots, z_{N-1}) = P_\Gamma(z_1, \dots, z_{N-1}, 1)$$

- hierarchy with respect to edge lengths

$$\ell_N = 2\ell_1 \Rightarrow P_\Gamma(z_1, \dots, z_N) = P_\Gamma(z_1, \dots, z_1^2)$$

Hierarchy with respect to the number of edges



Colin de Verdière's conjecture

Colin de Verdière's conjecture
The secular polynomial is irreducible iff the graph has no symmetries for rationally independent edge lengths $\Leftrightarrow$ the graph has no loops and is not a watermelon graph.

- E_j is a loop
$$\Rightarrow P_\Gamma = (z_j - 1) P_\Gamma^*(\mathbf{z})$$
- Γ - watermelon
$$\Rightarrow P_\Gamma(\mathbf{z}) = P^a(\mathbf{z}) P^s(\mathbf{z})$$

  $P^{a,s}(\mathbf{z})$ - degree one in each variable.

Secular polynomials and the spectrum

Substitute back $\mathbf{z} = e^{ik\ell}$ - the spectrum is obtained as intersection between the line $k\ell$ and the zero set of $P_\Gamma(\mathbf{z})$ (Barra-Gaspard).

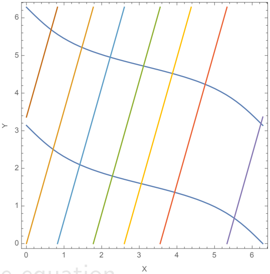
$P_\Gamma(\mathbf{z}) = \mathbf{z^n} P_\Gamma(1/\mathbf{z})$ - stable pair $\Rightarrow$ the spectrum is real (with Sarnak)

Lasso graph

$$P(z_1, z_2) = 1 - \frac{1}{3} z_1 + \frac{1}{3} z_2^2 - z_1 z_2^2$$

$$z_1 z_2^2 P(1/z_1, 1/z_2) = P(z_1, z_2)$$

The zero set on the torus: $z_1 = e^{ix}, z_2 = e^{iy}$, $(x, y) \in [0, 2\pi] \times [0, 2\pi]$



The spectrum is given by the equation

$$3 \sin\left((\frac{\ell_1}{2} + \ell_2) k \right) + \sin\left((\frac{\ell_1}{2} - \ell_2) k \right) = 0 \Rightarrow \{k_j\}$$

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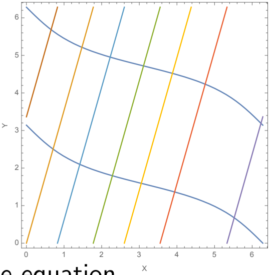

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M-function and contact set

Contact set $\partial\Gamma$ - any subset of vertices that include all degree one vertices.
Associated M-function

$$M_\Gamma : \psi|_{\partial\Gamma} \mapsto \partial\psi|_{\partial\Gamma}$$

where $\psi(x,\lambda)$ satisfies

- the eigenfunction equation on the edges,
- standard conditions at internal vertices $V \notin \partial\Gamma$
- just continuity condition at contact vertices $V \in \partial\Gamma$.

M-function = energy dependent Dirichlet-to-Neumann map

$$\mathbf{M}(\lambda) : \vec{\psi}(\cdot,\lambda)|_{\partial\Gamma} \mapsto \partial\vec{\psi}(\cdot,\lambda)|_{\partial\Gamma}$$

– matrix-valued Herglotz-Nevanlinna function
Inverse problems:
I. Gelfand - B. Levitan, V. Marchenko, ..., $\underbrace{\text{B. Simon, F. Gesztesy, A. Ramm,}}_{\text{A new approach to inverse spectral theory}}$ ...

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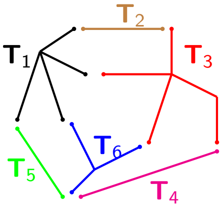
M-function and contact set

- hierarchy with respect to gluing graphs

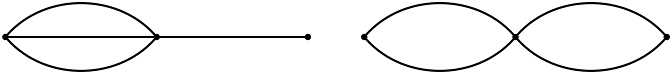
$$\mathbf{M}(\lambda) = \mathbf{M}^1(\lambda) + \mathbf{M}^2(\lambda) + \cdots + \mathbf{M}^N(\lambda)$$



$$\mathbf{M}(\lambda)=\left(\begin{array}{c|c|c|c|c|c|c|c}
M^1+M^5 & M^1 & M^1 & M^1 & & & & M^5 \\ \hline
M^1 & M^1+M^6 & M^1 & M^1 & & & M^6 & M^6 \\ \hline
M^1 & M^1 & M^1 & M^1+M^3 & M^3 & M^3 & M^3 & \\ \hline
M^1 & M^1 & M^1 & M^1+M^2 & M^2 & & & \\ \hline
 & & & M^2+M^3 & M^2+M^3 & M^3 & M^3 & \\ \hline
 & & & M^3 & M^3 & M^3+M^4 & M^3 & M^4 \\ \hline
 & M^6 & & M^3 & M^3 & M^3 & M^3+M^6 & M^6 \\ \hline
M^5 & M^6 & & & & M^4 & M^6 & M^4+M^5+M^6
\end{array}\right)$$

M-function

- **Explanation of isospectrality** (with J. Muller)
  The simplest (equilateral) isospectral pair



The explanation is based on the studies of Steklov subspaces.

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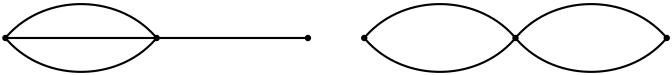

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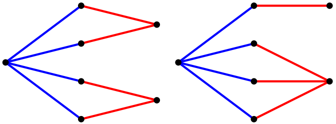

M-functions

- **Explicit formulas**
  Consider two operators on Γ:
  - L^{st} - standard conditions everywhere, λ_n^{st} and ψ_n^{st}
  - L^{D}- Dirichlet conditions on $\partial\Gamma$ and standard elsewhere, λ_n^D and ψ_n^D

$$\mathbf{M}_\Gamma(\lambda) = -\left(\sum_{n=1}^{\infty} \frac{\langle \psi_n^{\mathrm{st}}|_{\partial\Gamma}, \cdot \rangle_{\ell_2(\partial\Gamma)} \psi_n^{\mathrm{st}}|_{\partial\Gamma}}{\lambda_n^{\mathrm{st}} - \lambda} \right)^{-1},$$

$$\mathbf{M}_\Gamma(\lambda) - \mathbf{M}_\Gamma(\lambda') = \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^D - \lambda)(\lambda_n^D - \lambda')} \langle \partial\psi_n^D|_{\partial\Gamma}, \cdot \rangle_{\mathbb{C}^B} \partial\psi_n^D|_{\partial\Gamma},$$

- These formulas determine where zeroes and singularities of M-functions may be situated.
- Possible existence of invisible eigenfunctions.

Spectral estimates

- Rayleigh quotient
- Symmetrisation technique (Friedlander)
- Eulerian paths (P.K., Naboko)
- Transformation of graphs (P.K., Naboko, Kennedy, Malenova, Mugnolo, Berkolaiko, Serio)
- Scaling arguments, Hadamar-type formula (Band, Levy)

$$\frac{d\lambda}{d\ell_n} = -\left(\psi'(x)^2 + \lambda\psi(x)^2\right)|_{x \in E_n}$$

Typical results:

- Estimation of the spectral gap:
  $k_2 \geq \frac{\pi}{\mathcal{L}}$
  (Nicaise, Friedlander, P.K., Naboko)
- Estimation of higher eigenvalues
  $\frac{\pi}{\mathcal{L}}\left(n + 1 - \frac{|N| + \beta_1}{2}\right) \leq k_n \leq \frac{\pi}{\mathcal{L}}\left(n - 1 + \beta_1 + |D| + \frac{|N| + \beta_1}{2}\right)$
  (Berkolaiko, Kennedy, Kurasov, Mugnolo)

Spectral gap and M-function

Theorem (P.K., Naboko). Γ_1, Γ_2 *are glued together in* Γ. *The spectral gap does not decrease* $\lambda_2(\Gamma) \geq \min_j\{\lambda_2(\Gamma_j)\}$, *iff either or*

1. $\min_j\{\lambda_2(\Gamma_j)\} \leq \min_j\{\lambda_1^{\mathrm{D}}(\Gamma_j)\}$ *and*

$$\lim_{\epsilon \searrow 0} \# \left\{ \textit{positive eigenvalues of } \mathbf{M}_\Gamma(\min_j\{\lambda_2(\Gamma_j)\} - \epsilon) \right\} = 1;$$

2. $\min_j\{\lambda_1^{\mathrm{D}}(\Gamma_j)\} < \min_j\{\lambda_2(\Gamma_j)\} < \max_j\{\lambda_1^{\mathrm{D}}(\Gamma_j)\}$ *and*

$$\lim_{\epsilon \searrow 0} \# \left\{ \textit{positive eigenvalues of } \mathbf{M}_\Gamma(\min_j\{\lambda_2(\Gamma_j)\} - \epsilon) \right\} = 0;$$

3. $\lambda_1^{\mathrm{D}}(\Gamma_1) = \lambda_1^{\mathrm{D}}(\Gamma_2) = \min_j\{\lambda_2(\Gamma_j)\}$ *and*

$$\lim_{\epsilon \searrow 0} \# \left\{ \textit{positive eigenvalues of } \mathbf{M}_\Gamma(\min_j\{\lambda_2(\Gamma_j)\} - \epsilon) \right\} = 1.$$

Trace formula

J.-P. Roth, B. Gutkin, T. Kottos, U. Smilansky, P.K., M. Nowaczyk

$$(1+\beta_1)\delta(k)+\sum_{k_n\neq 0}\left(\delta(k-k_n)+\delta(k+k_n)\right)=\frac{\mathcal{L}}{\pi}+\frac{1}{\pi}\sum_{p\in\mathcal{P}} l(\mathrm{prim}\,(p))S_{\mathrm{v}}(p)\cos k\ell(p)$$

β_1 - the number of cycles
$\mathcal{L}$ - the total length of the graph
$\mathcal{P}$ - the set of periodic orbits
$S_{\mathbf{v}}(p)$ - the product of scattering coefficients along the orbit p

- The Euler characteristic is determined by the spectrum

$$\chi \equiv 1-\beta_1 = 2m_s(0)+2\lim_{t\to\infty}\sum_{k_n\neq 0}\cos k_n/t\left(\frac{\sin k_n/2t}{k_n/2t}\right)^2$$

The asymptotics of the spectrum is crucial.

- One may prove trace formula using stability of secular polynomials.

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Equilateral graphs

The spectrum k_n of the Laplacian on an equilateral graph is related to the spectrum μ_j of the normalised discrete Laplacian as (von Below)

$$1 - \cos k_n = \mu_j$$

$$\boxed{(1+\beta_1)\delta(k) + \sum_{k_n \neq 0} \left(\delta(k-k_n) + \delta(k+k_n)\right) = \frac{\mathcal{L}}{\pi} + \frac{1}{\pi} \sum_{p \in \mathcal{P}} l(\mathrm{prim}\,(p)) S_{\mathrm{v}}(p) \cos k\ell(p)}$$

- The trace formula formula is a finite combination of Poisson summation formulas
$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(2\pi m)$$
- The coefficient $1 + \beta_1$ repairs wrong multiplicity of the ground state λ_0.

Non-equilateral graphs

- Spectral concentration = the spectrum is not uniformly discrete

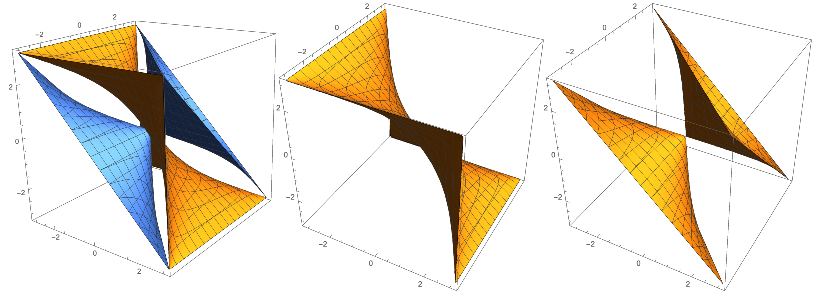


Figure: Zero sets for the watermelon graph on three edges.

Crystalline measures

Definition 1 (Y.Meyer). *A purely atomic measure μ is a crystalline measure iff*

1. *the support of μ is a locally finite set,*
2. *μ is a tempered distribution,*
3. *the distributional Fourier transform $\hat{\mu}$ is also purely atomic measure that is supported by a locally finite set.*

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \qquad \hat{\mu} = \sum_{s \in S} b_s \delta_s$$

Λ, S - discrete sets
$|\lambda_i - \lambda_j| \geq d > 0$ - uniformly discrete

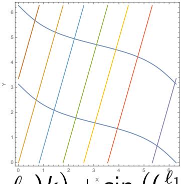
Λ - support, S - spectrum
(opposite to what we have for metric graphs).
Y. Meyer, Kolountzakis, Lev, Olevskii, ...

Crystalline measures from metric graphs

Metric graphs may give non-trivial examples of crystalline measures.
Non-trivial = not a finite combination of Poisson summation formulas.

Lemma A measure on $\mathbb{R}$ given by a finite linear combination of Dirac combs is uniformly discrete iff it is periodic.

Consider the lasso graph.



The secular function $3\sin\left((\frac{\ell_1}{2}+\ell_2)k\right)+\sin\left((\frac{\ell_1}{2}-\ell_2)k\right)$, is not periodic hence the trace formula **is not** a finite combination of Dirac combs, provided ℓ_1 and ℓ_2 are rationally independ.

The first explicit non-trivial example if a crystalline measure.
(with Sarnak)
Recent developments: characterisation of all idempotent crystalline measures (Olevskii, Ulanovskii), several alternative approaches to obtain crystalline measures (Meyer)

Liardet's theorem

Theorem 2. Lang's conjecture *(P. Sarnak, P. Liardet, W. Schmidt, M. Laurent, J-H. Evertse ...)*

- $V \subset (\mathbb{C}^*)^N$ *- an algebraic subvariety given by the zero set of Laurent polynomial;*
- Γ *- finitely generated subgroup of rank* r *of the torus* $T \subset (\mathbb{C}^*)^N$ *considered as a group under coordinatwise product;*
- $\overline{\Gamma}$ *- the division group of* Γ

$$\overline{\Gamma} = \left\{ z \in T : z^m \in \Gamma \text{ for some } m \geq 1 \right\}.$$

Then there exist finitely many translates of (may be low dimensional) subtori $T_1, T_2, \dots, T_\nu$ *contained in* V *such that*

$$\overline{\Gamma} \cap V = \overline{\Gamma} \cap (T_1 \cup T_2 \cup \dots \cup T_\nu)$$

and

$$\nu \leq \Big(C(V) \Big)^r,$$

where $C(V)$ *is an effectively computable constant.*

Arithmetic structure

How to apply this to metric graphs?

We want to prove that: $\boxed{\dim_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}}\{k_n\} = \infty}$

Assume the opposite $\dim_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}}\{k_n\} < \infty$, *i.e.*

$$k_n = \alpha_1^n k_1 + \alpha_2^n k_2 + \cdots + \alpha_{n_0}^n k_{n_0}, \quad \alpha_j^n \in \mathbb{Q}$$

for a certain n_0 and arbitrary n.

It follows that

$$e^{ik_n \ell_j} = \left(e^{ik_1 \ell_j}\right)^{\alpha_1^n} \left(e^{ik_2 \ell_j}\right)^{\alpha_2^n} \ldots \left(e^{ik_{n_0} \ell_j}\right)^{\alpha_{n_0}^n}$$

in other words, all $(e^{ik_n \ell_1}, \ldots, e^{ik_n \ell_N})$ belong to the division group for the multiplicative group Γ generated by

$$(e^{ik_i \ell_1}, e^{ik_i \ell_2}, \ldots, e^{ik_i \ell_N}), \quad i = 1, 2, \ldots, n_0.$$

Multiplication is carried coordinatwise.

It feels like Liardet's theorem was proven especially to serve our purposes!

Arithmetic structure

Theorem (P.K., Sarnak)**.**

- $E_1, E_2, \dots, E_\nu$, $\nu \le N$ – *loops.*
- ℓ_j, $j = 1, 2, \dots, N$, *are rationally independent*
- Γ *is neither the segment graph, nor the cycle graph, nor the figure eight graph.*

Then the spectrum of the standard Laplacian is a union of multisets

$$\mathrm{Spec}\,(\Gamma) = L_1(\Gamma) \cup L_2(\Gamma) \cup \dots \cup L_\nu(\Gamma) \cup \mathrm{Spec}^*(\Gamma), \tag{1}$$

where $L_j(\Gamma) = \left\{ \frac{2\pi}{\ell_j} n,\ n \in \mathbb{Z} \right\}$ *are full size arithmetic sequences and* $\mathrm{Spec}^*(\Gamma)$ *contains no full size arithmetic progression*

$$\#\Big(\mathrm{Spec}^*(\Gamma) \cap [-T, T]\Big) = \alpha T + \mathcal{O}(1), \quad \text{as } T \to \infty, \tag{2}$$

$$\alpha = \frac{1}{2\pi} \left(\mathcal{L} - \sum_{j=1}^{\nu} \ell_j \right) = \frac{1}{2\pi} \left(\ell_1 + \dots + \ell_\nu + 2(\ell_{\nu+1} + \dots \ell_N) \right). \tag{3}$$

Rigidity of the spectrum

Graph's spectrum is given by the zeroes of the trigonometric polynomial

$$p_\Gamma(k) = \sum_{j \in J} a_j e^{i\omega_j k}$$

– almost periodic function.

Theorem. *Two graphs are asymptotically isospectral*

$$k_n - k_n' \to 0, \quad n \to \infty$$

if and only if they are isospectral.

Generalised by S. Favorov to arbitrary almost periodic functions.

Theorem. *Two Schrödinger operators are asymptotically isospectral*

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Ambartsumian-type theorems

Theorem (Davies). *The standard Schrödinger operator is isospectral to the standard Laplacian on* Γ

$$\lambda_n(L_q^{\mathrm{st}}(\Gamma)) = \lambda_n(L^{\mathrm{st}}(\Gamma)),$$

if and only if $q(x) \equiv 0$ *almost everywhere.*

Theorem (K., Naboko). *Assume that the first non-trivial eigenvalue of* $L^{\mathrm{st}}(\Gamma)$ *coincides with the first nontrivial eigenvalue of the standard Laplacian* $L^{\mathrm{st}}(I)$ *on the interval* I *of the same length* $\mathcal{L}(I) = \mathcal{L}(\Gamma)$

$$\lambda_2(L^{\mathrm{st}}(\Gamma)) = \lambda_2(L^{\mathrm{st}}(I)) \equiv \left(\frac{\pi}{\mathcal{L}}\right)^2,$$

then the graph Γ *coincides with the interval* I.

Inverse problem

Task: reconstruct:

- the metric graph
- the potential(s) $q(x)$

Magnetic potential can be eliminated on each edge $\Rightarrow$ only magnetic fluxes through cycles $\Phi_j = \int_{C_j} a(x)dx$ are relevant

$\Rightarrow$ DRIVING IDEA: use magnetic fluxes to solve the inverse problem

Contact set $\partial\Gamma$ **- non-empty** set of vertices containing **all** degree one vertices.

Spectral data: two equivalent sets = two different languages

- response operator = dynamical Dirichlet-to-Neumann map $\mathbf{R}^T$;
- Titchmarsh-Weyl M-function $\mathbf{M}(\lambda)$.

$$\boxed{\widehat{\left(\mathbf{R}\vec{f}\right)}(s) = \mathbf{M}(-s^2)\hat{\vec{f}}(s)}$$

where $\hat{\cdot}$ denotes the Laplace transform.

Response operator

Consider solution to the wave equation subject to **Boundary Control** on the contact set $\partial\Gamma$

$$\begin{cases} \left(i\frac{d}{dx}+a(x)\right)^2 u(x,t)+q(x)u(x,t)=-\frac{\partial^2}{\partial t^2}u(x,t) \\ \text{standard conditions at } V_m \notin \partial\Gamma \\ \text{continuity condition at } V_m \in \partial\Gamma \\ u(x,0)=u_t(x,0)=0 \text{ -- zero initial data} \\ u(\cdot,t)|_{\partial\Gamma}=\vec{f}(t) \text{ -- boundary control} \end{cases}$$

Response operator = dynamical Dirichlet-to-Neumann map

$$\mathbf{R}^T: \quad \underbrace{\vec{f}(t)}_{\left\{u(V_m,t)\right\}_{V_m\in\partial\Gamma}} \quad \mapsto \quad \underbrace{\partial\vec{u}(\cdot,t)|_{\partial\Gamma}}_{\left\{\sum_{x_j\in V_m}\partial u(x_j,t)\right\}_{V_m\in\partial\Gamma}}$$

Inverse problems in $\mathbb{R}$ and $\mathbb{R}^n$ – Boundary Control method = BC-method: A.S. Blagoweschenskii, M. Belishev, Ya. Kurylev, S. Avdonin, ...

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Inverse problems for trees: peeling procedure

1. BC-method determines the potential on all pending edges
   Theorem (Boundary control). *The response operator* $\mathbf{R}^T$ *for the Schrödinger operator on* $[0,\infty)$ *determines the unique potential on the interval* $[0, T/2]$.
2. Knowing potential on a bunch of edges allows one to peel off this bunch and thus reduce the problem to a smaller tree.

Theorem (P.K., Avdonin). *The response operator or the Titchmarsh-Weyl M-function uniquely determines the metric tree and potential on it.*

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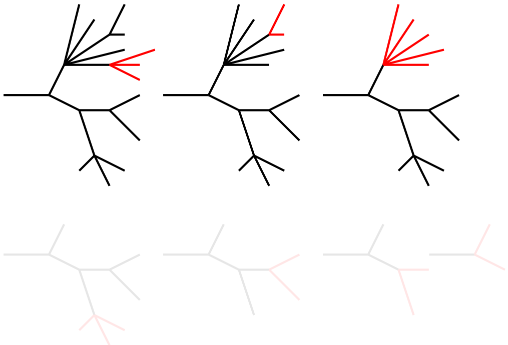

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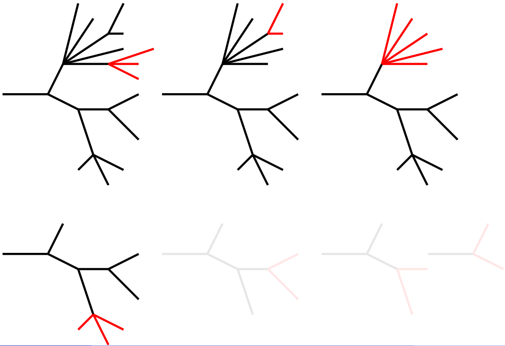

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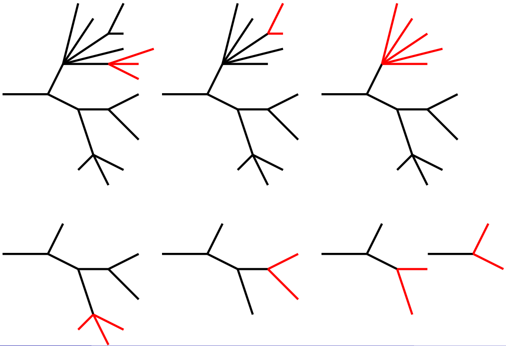


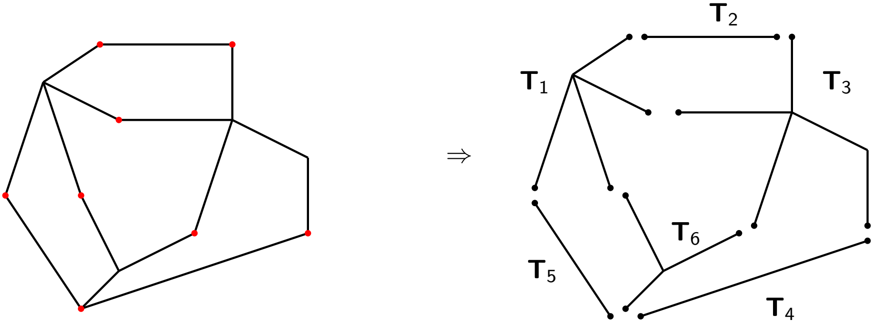
Inverse problems for graphs with cycles

- Magnetic potential on each cycle can be removed up to a phase $\Rightarrow$ magnetic fluxes Φ_j
- The response operator and M-matrices are considered as functions of the set of fluxes Φ_j, fixed each time.
  Magnetic Boundary Control - MBC-method

$$\mathbf{R}(\Phi_j) \mapsto q(x) \quad \text{or} \quad \mathbf{M}(\lambda, \Phi_j) \mapsto q(x)$$

- **Driving idea**: reduction to trees.
  - Dismantling graphs
  - Dissolving vertices using dependence on the magnetic fluxes
- The inverse problems will be solved **generically –** under mild generically satisfied conditions.

Dismantling graphs



Graph Γ Dismantled graph

Figure: Dismantling graphs

Lemma. **T** *a metric tree, then it holds:*

1. *every Dirichlet eigenfunction on* **T** *is visible, i.e. the corresponding eigenvalue* λ_n^{D} *is a singularity of the M-function;*
2. *every Dirichlet eigenfunction* ψ_n^{D} *on* **T** *has non-zero derivatives at (at least) two pendent vertices,*
3. *for every Dirichlet eigenvalue* λ_n^{D} *at least two diagonal elements of* $\mathbf{M}(\lambda)$ *are singular;*
4. *the following asymptotic representation holds*

$$\mathbf{M}(-s^2) = -s\mathbf{I} + o(1), \quad s \to \infty.$$

Definition. *The set of subtrees* $\{\mathbf{T}_j\}$ *of a metric graph* Γ *is called* **independent** *if any pair has at most one common vertex.*

Theorem.
$\partial\Gamma$ *dismantles the metric graph* Γ *into a set of subtrees* $\{\mathbf{T}_j\}$*, such that*

1. *no subtree* $\mathbf{T}_j$ *has two pendent vertices coming from the same vertex in* Γ*;*
2. *the subtrees* $\mathbf{T}_j$ *are independent.*

$\Rightarrow$ *M-function generically determines* Γ *and* q*, i.e. provided*

a) *the Schrödinger operators* $L_q^{\mathrm{st,D}}(\mathbf{T}_j)$, $j = 1, 2, \dots$ *have disjoint spectra*

$$\lambda_n^{\mathrm{D}}(\mathbf{T}_j) \neq \lambda_m^{\mathrm{D}}(\mathbf{T}_i), \quad j \neq i.$$

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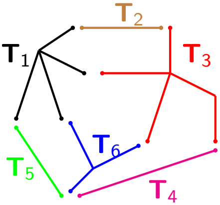
Idea of the proof:

Reconstruct the M-functions for the subtrees from the M-function for the original graph

$$\mathbf{M}(\lambda) = \mathbf{M}^1(\lambda) + \mathbf{M}^2(\lambda) + \cdots + \mathbf{M}^N(\lambda)$$



$$\mathbf{M}(\lambda)=\left(\begin{array}{c|c|c|c|c|c|c|c}
M^1+M^5 & M^1 & M^1 & M^1 & & & & M^5 \\ \hline
M^1 & M^1+M^6 & M^1 & M^1 & & & & M^6 \\ \hline
M^1 & M^1 & M^1 & M^1 + M^3 & M^3 & M^3 & M^3 & \\ \hline
M^1 & M^1 & M^1 & M^1+M^2 & M^2 & & & \\ \hline
 & & & M^2+M^3 & M^2+M^3 & M^3 & M^3 & \\ \hline
 & & & M^3 & M^3 & M^3+M^4 & M^3 & M^4 \\ \hline
 & M^6 & & M^3 & M^3 & M^3 & M^3+M^6 & M^6 \\ \hline
M^5 & M^6 & & & & M^4 & & M^4+M^5+M^6
\end{array}\right)$$

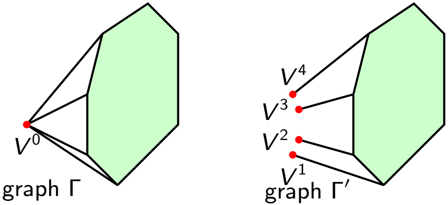
Step 1 For every subtree we have explicit formula through the traces of the Dirichlet eigenfunctions

$$\mathbf{M}_\Gamma(\lambda) - \mathbf{M}_\Gamma(\lambda') = \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^D - \lambda)(\lambda_n^D - \lambda')} \langle \partial \psi_n^D|_{\partial\Gamma}, \cdot \rangle_{\mathbb{C}^B} \partial \psi_n^D|_{\partial\Gamma},$$

Step 2 Using condition *a)* we can sort the singular terms into classes associated with the subtrees (without knowing the subtree a priori)
Step 3 The regular term $\mathbf{M}_j(\lambda)$ is determined from the asymptotics

$$\mathbf{M}(-s^2) = -s\mathbf{I} + o(1), \quad s \to \infty.$$

Dissolving vertices



The M-functions for Γ and Γ' are related as:

$$\underbrace{\mathbb{M}(\lambda, \vec{\Phi}'')}_{= \mathbf{M}_{00}(\lambda, \vec{\Phi}'')} = \sum_{i,j=1}^{d_0} e^{i(\Phi_i - \Phi_j)} \underbrace{\mathbb{M}'_{ij}(\lambda, \vec{0})}_{= \mathbf{M}'_{ij}(\lambda, \vec{0})}$$

In particular for the singularities

$$\mathbf{M}_{00}(\lambda, \vec{\Phi}'') \underset{\lambda \to \lambda_n^{\mathrm{D}}}{\sim} \frac{1}{\lambda_n^{\mathrm{D}} - \lambda} \sum_{i,j=1}^{d_0} e^{i(\Phi_i - \Phi_j)} \partial\psi_n^{\mathrm{D}}(V_i) \overline{\partial\psi_n^{\mathrm{D}}(V_j)}$$

$$= \frac{1}{\lambda_n^{\mathrm{D}} - \lambda} \Big(\sum_{i=1}^{d_0} |\partial\psi_n^{\mathrm{D}}(V_i)|^2 + \sum_{i,j=1,}^{d_0} 2\cos(\Phi_i - \Phi_j) \partial\psi_n^{\mathrm{D}}(V_i) \overline{\partial\psi_n^{\mathrm{D}}(V_j)} \Big).$$

Theorem. *Assume that*

1. *the graph* Γ' *and hence* Γ *is connected;*
2. *the degree* d_0 *of the contact vertex* V_0 *is at least three:* $d_0 \geq 3$.

Then the diagonal entry $\mathbf{M}_{00}(\vec{\Phi}')$ **generically** *determines the diagonal* $d_0 \times d_0$ *block of* $\mathbf{M}'$ *associated with the new pendent vertices, i.e. provided*

a) *the spectrum of the Dirichlet operator on* Γ' *is simple;*

b) *every Dirichlet eigenfunction* ψ_n^{D} *on* Γ' *has at least three non-zero normal derivatives at the pendent vertices.*

Why $d_0 \geq 3$?
It is simple:
$a^2 + b^2$ and ab do not determine a and b.
$a^2 + b^2 + c^2$, ab, bc and ac do determine a, b, c (up to multiplication by -1).

Reconstruction by dissolving vertices

- The whole graph is reconstructed

- Only a part of the graph is reconstructed ⇒ infiltration domain

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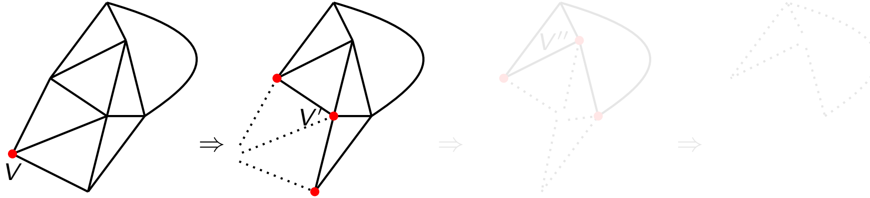

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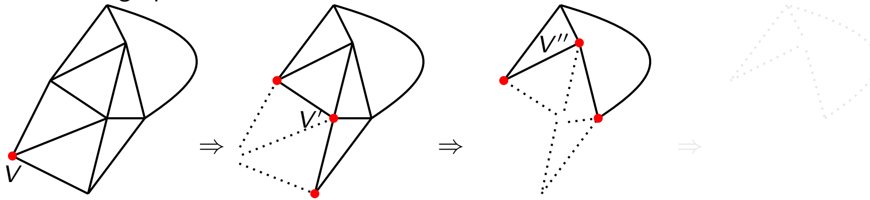

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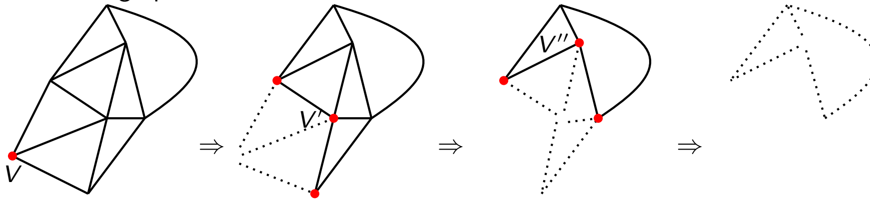

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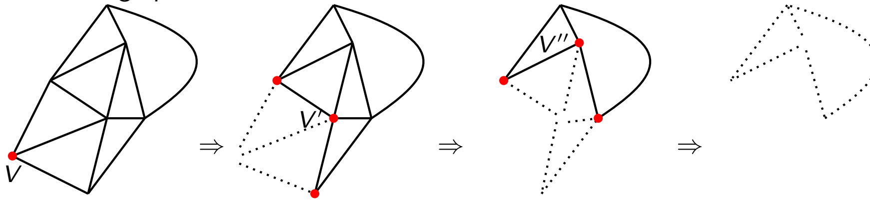

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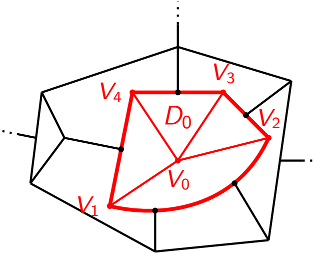

Magnetic Boundary Control

Theorem. *Assume that*

1. *the infiltration domains corresponding to $V_j \in \partial\Gamma$ cover the original graph Γ*

$$\bigcup_{V_j \in \partial\Gamma} D_j = \Gamma.$$

Then the $M(\Phi_j)$, $\Phi_i = 0, \pi$, generically determines the graph Γ and potential q, i.e. provided that

a) *the Dirichlet eigenfunctions on proper subgraphs of Γ do not vanish identically on any edge.*

Skeleton

$$\mathbb{S} := \Gamma \setminus \bigcup_{V_j \in \partial\Gamma} D_j$$

The skeleton is empty in this case.

Magnetic Boundary Control

Idea of the proof.

1. Determine distances between the contact points.
2. Recover infiltration domains (keeping in mind possible inclusion of other contact points)
3. Connect different infiltration domains together.

Note that:

- no a priori knowledge of the metric graph;
- procedure can start from one contact point adding new points if necessary if it is seen that obtained data are not sufficient (degree two vertices, the distances between contact points do not match reconstructed graph);
- procedure is local allowing to determine a part of an infinite graph, or even some infinite graphs.

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