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Seminar on Analysis, Differential Equations
and Mathematical Physics

Geometry of Convex Sets in Spaces of Homogeneous Polynomials

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Convex Sets and Convex Hull. Definitions

Let X be a real vector space.

- **Definition.** A nonempty subset C of X is said to be convex, whenever for any arbitrary vectors x and y and $0 < \lambda < 1$ one has $\lambda x + (1-\lambda)y \in C$.
- **Definition.** A convex hull (denoted by $\mathrm{co}(A)$) of $A \subset X$ is the smallest (with respect to inclusion) convex set that includes A:

$$\mathrm{co}(A) := \bigcap \{C \subset X : \; A \subset C, \; C \text{ is convex}\}.$$

- **Proposition.** The following representation holds:

$$\mathrm{co}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}.$$

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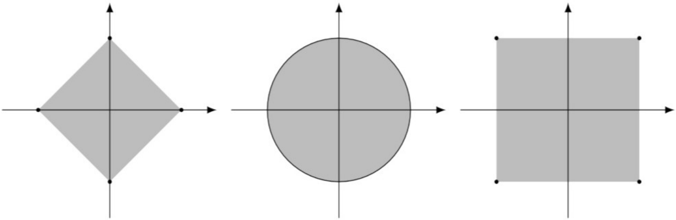
Extreme Points. Definitions

Let X be a vector space, and $C \subset X$ a convex subset in it.

- Definition. An arbitrary vector $x \in C$ is called extreme point, if x is not the inner point of any line segment in C.
- We denote by a symbol $\text{ext}(C)$ the set of all extreme points of a convex set C.



$\|x\|_1 \leq 1$ $\|x\|_2 \leq 1$ $\|x\|_\infty \leq 1$

Kreĭn–Mil'man Theorem (1940). A compact convex subset of a Hausdorff locally convex topological vector space coincides with the closed convex hull of its extreme points:

$$C \text{ is convex and compact} \Longrightarrow C = \overline{\mathrm{co}}(\mathrm{ext}(C)).$$

- By symbol $\overline{\mathrm{co}}(\cdot)$ we denote the topological closure of the convex hull $\mathrm{co}(\cdot)$: $\overline{\mathrm{co}}(A) := \overline{\mathrm{co}(A)}$.
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Classical Kreĭn–Mil'man Theorem

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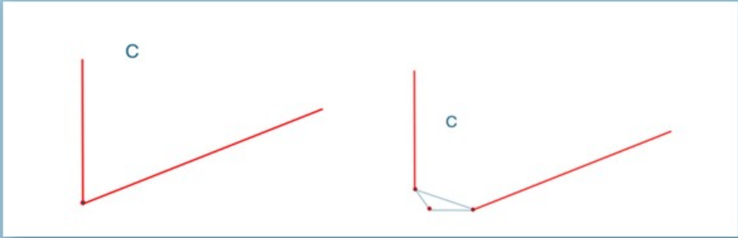
Let C be a subset of a linear vector space X.

- Definition. An extreme ray of C is an open half-line $\rho \subset C$ such that any line segment $[x, y] \subset C$ lies entirely on ρ provided that $\lambda x + (1-\lambda)y \in \rho$ for some $0 < \lambda < 1$.
- The set of extreme rays of C is denoted by $\operatorname{extr}(C)$.
- Illustration.



Krein-Milman Type Theorem

Theorem (Klee, 1958). Let C be a closed convex locally compact subset of a Hausdorff locally convex topological vector space. Suppose C contains no straight lines, then

$$C = \overline{\mathrm{co}(\mathrm{ext}(C) \cup \mathrm{extr}(C))}.$$

- Klee V.L. Extremal structure of convex sets. II // Mathematische Zeitschrift. 1958. V. 69. P. 90–104.
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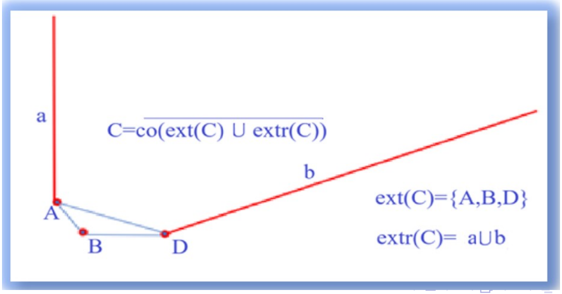

Problem

Starting from 1960s an increasing interest is paid to the following question:

What can be said if the convex set C consists of linear operators, that is $C \subset \mathcal{L}(X, Y)$ with X and Y Banach spaces?

More precisely:

- Under what additional conditions a convex set of operators $A \subset \mathcal{L}(X, Y)$:
  - The Existence and Description Problem: has extreme points? and what is their analytical description?
  - The Recovery Problem: can be recovered from it's extreme points and rays?

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Motivation

Interest in Kreĭn–Mil'man type terms for convex sets of linear operators was initiated by a series of papers published by Phelps and his co-authors in the 1960s:

- Phelps R. R. Extreme positive operators and homomorphisms // Trans. Amer. Math. Soc. 1963.V. 108. P. 265-274.
- Blumenthal R. M., Lindenstrauss J. and Phelps R. R. Extreme operators into $C(K)$ // Pacific J. Math. 1965. V. 15. P. 747-756.
- Phelps R R. Theorems of Krein—Milman type for certain convex sets of functions operators // Annales de institut Fourier. 1970. V. 20, N 2. P. 45-54.
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Theorem (Morris, Phelps, 1970). If K is totally disconnected then the unit ball B of $\mathcal{L}(X, C(K))$ is the pointwise norm closure of the operator convex hull of its extreme points:

$$B = \overline{\mathrm{co}}_{\Lambda}(\mathrm{ext}(B)).$$

- Here X is a Banach space, $C(K)$ the space of continuous real functions on the compact Hausdorff space K, and $\mathcal{L}(X, C(K))$ denotes the space of linear bounded operators from X to $C(K)$.

A Krein–Mil'man Type Result

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- Oates D. K. A non-compact Krein — Milman theorem // Pacific J. Math. 1971. V. 36, N 3. P. 781-788.
- Rubinov A.M. Sublinear operators and operator-convex sets // Sib. Math. J. 1976. V. 17, N 2. P. 289-296.
- Kutateladze S.S. The Krein-Mil'man theorem and its inverse // Sib. Math. J. 1980. V. 21, N 1. P. 97-103.
- Kutateladze S.S. Caps and faces of sets of operators // Soviet Math. Dokl. 1985. V. 31, N 1. P.66-68.

Vector Lattices. Definition

Definition. A real vector space E equipped with an order relation $\geq$, which satisfies the following two axioms:

✓ If $x \geq y$, then $x + z \geq y + z$ holds for all $z \in E$

✓ If $x \geq y$, then $\alpha x \geq \alpha y$ holds for all $\alpha \geq 0$

is a vector lattice if and only if for any pair $x, y \in E$ there exist:

✓ $x \vee y := \sup\{x, y\}$, the supremum,

✓ $x \wedge y := \inf\{x, y\}$, the infimum.

Examples. Classical Banach spaces are vector lattices:

$$C(Q), L^p, l^p (1 \leq p \leq \infty), c_o, c$$

- An order relation is introduced point-wise in function spaces and coordinate-wise in sequences spaces.

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Main Example. Consider σ-finite measure space (Q, Σ, μ). Denote

$$L^0(\mu) := L^0(Q, \Sigma, \mu)$$

the space of all (equivalence classes of) measurable almost everywhere finite real-valued functions on Q.

Here are some necessary definitions, which we need in the sequel:

- Recall that the absolute value (modulus) in a vector lattice is defined as $|x| := x \vee (-x)$.
- The positive cone E_+ of a vector lattice E is defined as $E_+ := \{x \in E : x \geq 0\}$.
- We say that a set B in a vector lattice E is order bounded if $B \subset [a, b] := \{x \in E : a \leq x \leq b\}$ for some $a, b \in E$.

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Homogeneous Polynomials: Definition

- Definition. Let X and Y be vector spaces, $1 \le n \in \mathbb{N}$. A mapping $P : X \to Y$ is said to be an n-homogeneous polynomial if for some multilinear (n-linear) operator $\varphi : X^n = X \overbrace{\times \cdots \times}^{n \text{ times}} X \to Y$ the representation holds:

$$P(x) = \varphi(x, \ldots, x) \quad (x \in X).$$

- The multilinear operator φ is called generating.
- There is exactly one symmetric multilinear operator, generating P; it is called associated and denoted by $\check{P}$.

- Definition. A polynomial P is said to be positive, if

$$0 \le x_1, \ldots, x_n \in E \Rightarrow \check{P}(x_1, \ldots, x_n) \ge 0.$$

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Homogeneous Polynomials: Example and Notation

- Notation. $\mathcal{P}(^nX, F)$ denotes the vector space of all n-homogeneous polynomials from X to F; $\mathcal{P}(^nX, F)_+$ is the part of $\mathcal{P}(^nX, F)$ consisting of positive polynomials.
- Definition. $\mathcal{P}(^nX, F)$ is ordered by $\mathcal{P}(^nX, F)_+$ as

$$P \geq Q \overset{\text{df}}{\Longleftrightarrow} P - Q \in \mathcal{P}(^nX, F)_+.$$

- Example ($F = L^0$). Let $T : X \to L^0$ be a linear operator and define a mapping $P : X \to L^0$ as

$$P(x) = T(x)^n \quad (x \in X).$$

Then P is an n-homogeneous polynomial from X to L^0, whose associated multilinear operator $\check{P} \in \mathcal{L}(X^n, L^0)$ is:

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- Definition. Vector lattice F is said to be Dedekind complete, if given $\varnothing \neq A \subset F$ with $A \subset [a, b]$ for some $a, b \in F$ ($\equiv A$ is order bounded), there exist $\sup(A) \in F$ and $\inf(A) \in F$.
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- It can be easily seen, that $\mathrm{co}_\Lambda(\Omega)$ consists of all Λ-convex combinations, i.e. the following holds:

$$\mathrm{co}_\Lambda(\Omega) = \Big\{ \sum_{i=1}^{m} \lambda_i P_i : \ P_1, \dots, P_k \in \Omega, \ \lambda_1, \dots, \lambda_k \in \Lambda_+, \sum_{i=1}^{k} \lambda_i = I_Y, \ k \in \mathbb{N} \Big\}.$$

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Operator Convex Hulls

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$$\mathrm{co}_\Lambda(\Omega) = \Big\{ \sum_{i=1}^{m} \lambda_i P_i : \ P_1, \dots, P_k \in \Omega, \ \lambda_1, \dots, \lambda_k \in \Lambda_+, \\ \sum_{i=1}^{k} \lambda_i = I_Y, \ k \in \mathbb{N} \Big\}.$$

- where $\Lambda := \mathrm{Orth}(F) := \{\varphi \mapsto \lambda\varphi \quad (\varphi \in F)\}$.

Operator Convex Hulls

- Definition. The subset Ω of $\mathcal{P}(^nX, L^0)$ is called operator convex (or Λ-convex, $\Lambda := L^0$), if $\lambda_1 P_1 + \lambda_2 P_2 \in \Omega$ for all $P_1, P_2 \in \Omega$ and $\lambda_1, \lambda_2 \in \Lambda_+$ with $\lambda_1 + \lambda_2 = \mathbf{1}$.
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Theorem 1 (Krein-Milman's Type Theorem for Homogeneous Polynomials). Let X be an arbitrary vector space, F a Dedekind complete vector lattice (for ex. $F = L^0$) and $\Lambda := \mathrm{Orth}(F)$. If $\Omega \subset \mathcal{P}({}^n X, F)$ is point-wise order bounded, operator convex and point-wise order closed set, then the following representation holds:

$$\Omega = o\text{-}\,\mathrm{cl}(\mathrm{co}_\Lambda(\mathrm{ext}(\Omega))).$$

- [Kusraeva Z.A. Extremal structure of cones of positive homogeneous polynomials // Journal of Mathematical Analysis and Applications. 2024.V. 532, N 1.]
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The First Main Result

Theorem 1 (Krein-Milman's Type Theorem for Homogeneous Polynomials). Let X be an arbitrary vector space, F a Dedekind complete vector lattice (for ex. $F = L^0$) and $\Lambda := \mathrm{Orth}(F)$. If $\Omega \subset \mathcal{P}(^n X, F)$ is point-wise order bounded, operator convex and point-wise order closed set, then the following representation holds:

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Let E and F be vector lattices.

- Definition. A polynomial $P : E \to F$ is called orthogonally additive if P is additive on any pair of disjoint elements, i.e.

$$|x| \wedge |y| = 0 \;\Rightarrow\; P(x+y) = P(x) + P(y).$$

- Denote by $\mathcal{P}_o({}^nE, F)$ the space of all n-homogeneous orthogonally additive polynomials acting between E and F.
- Definition. $\mathcal{P}_o({}^nX, F)$ is ordered by $\mathcal{P}_o({}^nX, F)_+$ as

$$P \geq Q \stackrel{\text{df}}{\Longleftrightarrow} P - Q \in \mathcal{P}_o({}^nX, F)_+.$$

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Definition. A linear operator $T : E \to F$ is said to be lattice homomorphism, if $T(|x|) := |T(x)|$ for $x \in E$.

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$$(Tx)(t) = w(t)x(\varphi(t)) \quad (x \in E),$$

  where $E \subset L^0(\Omega_1, \Sigma_1, \mu_1)$, $\quad 0 \leq w \in F \subset L^0(\Omega_2, \Sigma_2, \mu_2)$, and $\varphi : \Omega_2 \to \Omega_1$ is a measurable mapping.

- Denote by:
  - $\mathcal{H}(E, F)$ the set of lattice homomorphisms from E to F;
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Notation. If $E \subset L^0$, then $E_{(p)} := \{f^p \in L^0 : \ f \in E\}$.

Theorem 2. For a vector lattice E with total $E^\sim$, the following conditions are equivalent:

- $\big(E_{(n)}\big)^\sim_+$ is the $\sigma\big(E^\sim_{(n)}, E_{(n)}\big)$-closed convex hull of $\operatorname{extr}\big((E_{(n)})^\sim_+\big)$.
- $\mathcal{P}_o({}^nE)_+$ is the point-wise closed convex hull of the set $\{\varphi^n : \ \varphi \in \operatorname{extr}(E^\sim_+)\}$.
- For each Dedekind complete vector lattice F (for ex. $F = L^0$), the cone $\mathcal{P}_o({}^nE, F)_+$ is the point-wise order closed convex hull of the set $\{T^n : \ T \in \mathcal{H}(E, F_{(1/n)})\}$, symbolically:

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In linear case, when $n = 1$, we have

- Corollary. For a vector lattice E with total $E^{\sim}$, the following conditions are equivalent:
  - $(E)^{\sim}_{+}$ is the $\sigma(E^{\sim}, E)$-closed convex hull of $\operatorname{extr}((E)^{\sim}_{+})$.
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For the rest fix $n, \nu, k_1, \dots, k_\nu \in \mathbb{N}$ with $n = k_1 + \cdots + k_\nu \geq 2$.

- Definition. For linear $T_1, \dots, T_\nu : E \to F (= L^0)$ define

$$Q := T_1^{k_1} \cdot \ldots \cdot T_\nu^{k_\nu} : E \ni x \mapsto T_1(x)^{k_1} \cdot \ldots \cdot T_\nu(x)^{k_\nu} \in F,$$

where $n, \nu, k_1, \dots, k_\nu \in \mathbb{N}$ and $k_1 + \ldots + k_\nu = n$.

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$$\check{Q}(x_1, \dots, x_n) =$$
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- Lemma. The polynomial Q is orthogonally additive if and only if there exist a lattice homomorphism $T : E \to F$ and a multiplication operator γ such that

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Product of Powers of Linear Operators

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Third Main Result

The third main result deals with necessary and sufficient conditions under which the following holds:

$$\mathcal{P}(^n E, F)_+ =$$

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The above presented 3 theorems are the generalization of Klee's and Krein-Milman's theorems for homogeneous polynomials.

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- Theorem (Choquet,1962). If a closed convex cone in a locally convex topological vector space is well-capped, then it is the closed convex hull of its extremal rays.
- [Choquet G. Les cônes convexes faiblement complets dans l'analyse // Proc. Internat. Congress Math. Stockholm, 1962. P. 317-330].
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Further Motivation

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- And the nearest problem to be solved is to obtain the analogues of Azimov's theorem for homogeneous polynomials, which will generalizes Krein-Milman's Type Theorem, Klee's Type and Choquet's Type Theorems for Homogeneous Polynomials.

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THANK YOU FOR ATTENTION!

Dear OTHA people, dear Friends!
All the best wishes for you and your families this Christmas!
Be happy, healthy and blessed not just on Christmas day,
but thought the whole year!
Wish you a Merry Xmas and a Happy New 2025 Year!