A survey on current results in Theory of Lieb-Thirring inequalities

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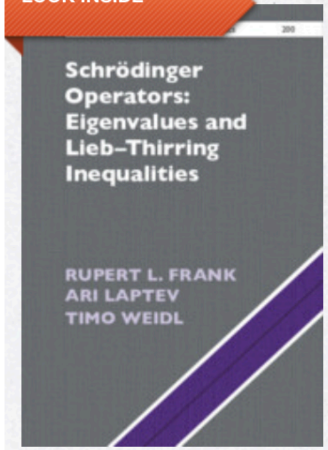

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Schrödinger Operators: Eigenvalues and Lieb–Thirring Inequalities

Part of Cambridge Studies in Advanced Mathematics

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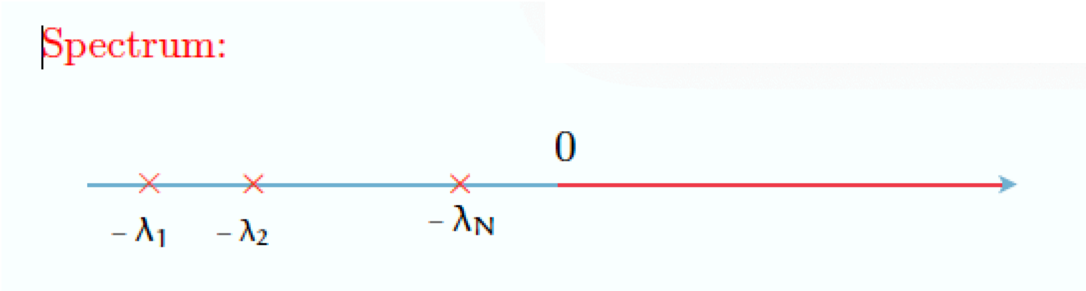
- CLR and Lieb-Thirring inequalities

Consider a Schrödinger operator

$$\mathcal{H} = -\Delta - V(x), \qquad \text{in } L^2(\mathbb{R}^d),$$

where $V \to 0$ as $|x| \to \infty$ and $V \geq 0$ and let $\{-\lambda_k\}$ be negative eigenvalues of $\mathcal{H}$.



$$\sum_j \lambda_j^\gamma = \sum_j \lambda_j^\gamma(V) \leq C_{d,\gamma} \int\int \Big(|\xi|^2 - V(x)\Big)_-^\gamma \, dx d\xi = L_{d,\gamma} \int V(x)_+^{\gamma+d/2} \, dx.$$

This inequality holds true for $d = 1$, $\gamma \geq 1/2$; $d = 2$, $\gamma > 0$; $d \geq 3$, $\gamma \geq 0$.

Compare it with Weyl's asymptotic formula:

$$\sum_j \lambda_j(\alpha V)^\gamma \sim_{\alpha\to\infty} L_{d,\gamma}^{cl} \int (\alpha V_+)^{\gamma+d/2} \, dx = (2\pi)^{-d} \iint (|\xi|^2 - \alpha V)_-^\gamma \, d\xi dx,$$

which implies $L_{d,\gamma}^{cl} \leq L_{d,\gamma}$.

Applications.

- Weyl's asymptotics.
- Stability of matter.
- Study of properties of continuous spectrum of Schrödinger operators.
- Estimate of dimensions of attractors in theory of Navier-Stokes equations.
- Bounds on the maximum ionization of atoms.

$$\sum_j \lambda_j(V)^\gamma \leq C_{d,\gamma} \int\int \Big(|\xi|^2 - V(x)\Big)_-^\gamma \, dx d\xi.$$

Remark.

If $\Omega \in \mathbb{R}^d$ is a set of finite measure and $\mathcal{H} = -\Delta + V$,

$$V(x) = \begin{cases} -\lambda, & x \in \Omega, \\ +\infty, & x \notin \Omega, \end{cases}$$

then the spectrum of $\mathcal{H}$ coincides with the spectrum of the Dirichlet Laplacian in Ω.

In this case L-Th inequalities look as

$$\sum_k (\lambda - \lambda_k)_+^\gamma \leq C_{d,\gamma} \, |\Omega| \, \lambda^{d/2+\gamma}.$$

The sharp constants $C_{d,\gamma}$ are known for all $\gamma \geq 1$ and $C_{d,\gamma} = L^{cl}_{d,\gamma}$ (Berezin-Li-Yau).
If $\gamma = 0$ then we have Pólya's inequalities that could be considered as a special case of the L-Th inequalities.

• CLR and Lieb-Thirring inequalities.

Theorem. Let $\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d \geq 2$, $\gamma \geq 0$ if $d \geq 3$ and let $0 \leq V \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then the negative eigenvalues $\{-\lambda_k\}$ of the operator $-\Delta - V$ satisfy

$$\sum_k \lambda_k^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V^{\gamma+d/2}\, dx.$$

M.S.Biman, M.Z.Solomyak, G.Rosenblum, M.Cwikel, J.G. Conlon, E.H.Lieb, W.Thirring, M.Aizenmann, D.Hundertmark, L.Thomas, AL & T.Weidl, R.Frank, M. Jex, P. T. Nam, R.Benguria & M.Loss, J.Dolbeault, M.Rumin.

Sharp constant were obtained in the following cases:

Theorem. It is known that $L_{1,1/2} = 1/2$ ($L^{cl}_{1,1/2} = 1/4$) and $L_{d,\gamma} = L^{cl}_{d,\gamma}$ if $\gamma \geq 3/2$, $d \geq 1$.

In other cases the sharp constants are unknown.

• L-Th inequality $\gamma = 3/2$, $d = 1$ by using the Darboux transform (Benguria & Loss).

Let $(-\lambda_1, \psi_1)$ be the lowest eigenvalue and its respective eigenfunction

$$\mathcal{H}\psi_1 = -\frac{d^2}{dx^2}\psi_1 - V(x)\,\psi_1 = -\lambda_1\,\psi_1.$$

It is known that $\psi_1 \neq 0$ and we can choose $\psi_1 > 0$.

Denote

$$f_1 = \frac{\psi_1'}{\psi_1}, \qquad f_1' = \frac{\psi_1''}{\psi_1} - \left(\frac{\psi_1'}{\psi_1}\right)^2.$$

Therefore

$$f_1' + f_1^2 = \frac{\psi_1''}{\psi_1} = \lambda_1 - V.$$

Let us introduce

$$Q_1 = \frac{d}{dx} - f_1 \qquad \& \qquad Q_1^* = -\frac{d}{dx} - f_1.$$

Then

$$Q_1^* Q_1 = \left(-\frac{d}{dx} - f_1\right)\left(\frac{d}{dx} - f_1\right) = -\frac{d^2}{dx^2} + f_1' + f_1^2$$

$$= -\frac{d^2}{dx^2} - V + \lambda_1 = \mathcal{H} + \lambda_1.$$

The discrete spectrum $\sigma_d(Q_1^* Q_1)$ of the operator $Q_1^* Q_1$ coincides with

$$\sigma_d(Q_1^* Q_1) = \{0, -\lambda_2 + \lambda_1,\ -\lambda_3 + \lambda_1, \dots\}.$$

In particular,

$$Q_1^* Q_1 \psi_1 = 0,$$

where

$$\psi_1(x) \sim \begin{cases} e^{-\sqrt{\lambda_1} x}, & x \to +\infty, \\ e^{\sqrt{\lambda_1} x}, & x \to -\infty. \end{cases}$$

and also

$$f_1(x) = \frac{\psi_1'(x)}{\psi_1(x)} \sim \begin{cases} -\sqrt{\lambda_1}, & x \to +\infty, \\ \sqrt{\lambda_1}, & x \to -\infty. \end{cases}$$

Commuting Q_1^* and Q_1 we obtain

$$Q_1 Q_1^* = \left(\frac{d}{dx} - f_1\right)\left(-\frac{d}{dx} - f_1\right) = -\frac{d^2}{dx^2} - f_1' + f_1^2$$
$$= -\frac{d^2}{dx^2} - 2f_1' - V + \lambda_1 = \mathcal{H} - 2f_1' + \lambda_1.$$

The operators $Q_1^* Q_1$ and $Q_1 Q_1^*$ have the same non-zero spectrum.

Moreover, $0 \notin \sigma(Q_1 Q_1^*)$, indeed, assume that there is $\psi \in L^2(\mathbb{R})$ s.t.

$$Q_1 Q_1^* \psi = 0 \quad \Longrightarrow \quad \|Q_1^* \psi\| = 0 \quad \Longrightarrow$$
$$-\psi' - f_1 \psi = 0 \quad \Longrightarrow \quad (f_1 \sim -\sqrt{\lambda_1},\, x \to +\infty) \quad \Longrightarrow$$
$$\psi \sim e^{\sqrt{\lambda_1}\, x}, \quad x \to +\infty.$$

Therefore $\psi \notin L^2(\mathbb{R})$.

Conclusion:

$$\sigma_d(\mathcal{H}) = \{-\lambda_1, -\lambda_2, -\lambda_3, \dots\}$$

and

$$\sigma_d(\mathcal{H} - 2f_1') = \{-\lambda_2, -\lambda_3, \dots\}.$$

Denote now $V_1 = V + 2f_1'$, $\mathcal{H}_1 = \mathcal{H} - 2f_1'$.

Considering the class of potentials with the finite number of eigenvalues and repeating this process, we obtain a non-negative Schrödinger operator with the potential

$$-V_n = -V - 2f_1' - 2f_2' - \dots - 2f_n',$$

where

$$f_n' + f_n^2 = \lambda_n - V_{n-1}$$

and

$$\sigma_d(\mathcal{H} - 2f_1' - 2f_2' - \dots - 2f_n') = \emptyset.$$

Finally we have

$$
\begin{aligned}
0 \le \int_{\mathbb{R}} (V_{n-1} + 2f_n')^2 \, dx &= \int_{\mathbb{R}} \left[V_{n-1}^2 + 4f_n'(V_{n-1} + f_n') \right] \, dx \\
= \int_{\mathbb{R}} \left[V_{n-1}^2 + 4f_n'(\lambda_n - f_n^2) \right] \, dx &= \int_{\mathbb{R}} V_{n-1}^2 \, dx + 4\lambda_n f_n \Big|_{-\infty}^{\infty} - \frac{4}{3} \, f_n^3 \Big|_{-\infty}^{\infty} \\
= \int_{\mathbb{R}} V_{n-1}^2 \, dx - 8\lambda_n^{3/2} + \frac{8}{3} \, \lambda_n^{3/2} &= \int_{\mathbb{R}} V_{n-1}^2 \, dx - \frac{16}{3} \, \lambda_n^{3/2} \\
&= \cdots = \int_{\mathbb{R}} V^2 \, dx - \frac{16}{3} \sum_{j=1}^{n} \lambda_j^{3/2}.
\end{aligned}
$$

Theorem.

Let $\mathcal{H} = -\frac{d^2}{dx^2} - V$, in $L^2(\mathbb{R})$, where $V \in L^2(\mathbb{R})$, $V \ge 0$. Then for the negative eigenvalues $\{\lambda_j\}$ of the operator $\mathcal{H}$ we have

$$
\sum_j \lambda^{3/2} \le \frac{3}{16} \int_{\mathbb{R}} V^2 \, dx = L_{1,3/2}^{cl} \int_{\mathbb{R}} V^2 \, dx.
$$

Theorem. (Aisenman-Lieb) The inequality

$$\sum_j \lambda^{3/2} \leq L^{cl}_{1,3/2} \int_{\mathbb{R}} V^2\, dx$$

implies

$$\sum_j \lambda^{\gamma} \leq L^{cl}_{1,\gamma} \int_{\mathbb{R}} V^{3/2+\gamma}\, dx, \qquad \gamma > 3/2.$$

Proof. Denote by $B(p,q)$ the classical Beta function

$$\mathcal{B}(p,q) = \int_0^1 (1-t)^{q-1} t^{p-1}\, dt.$$

Then

$$\begin{aligned}
\sum_n |\lambda_n(V)|^{\gamma} &= \frac{1}{\mathcal{B}(\gamma-3/2,5/2)} \sum_n \int_0^{\infty} (|\lambda_n(V)|-t)_+^{3/2}\, t^{\gamma-3/2-1}\, dt \\
&= \frac{1}{\mathcal{B}(\gamma-3/2,5/2)} \sum_n \int_0^{\infty} (|\lambda_n(V+t)|)^{3/2}\, t^{\gamma-3/2-1}\, dt \\
&\leq \frac{1}{\mathcal{B}(\gamma-3/2,5/2)} L^{cl}_{1,3/2} \int_{-\infty}^{\infty} \int_0^{\infty} ((V(x)+t)_-)^2 t^{\gamma-3/2-1}\, dt dx \\
&= \frac{\mathcal{B}(\gamma-3/2,3)}{\mathcal{B}(\gamma-3/2,5/2)} L^{cl}_{1,3/2} \int_{-\infty}^{\infty} (V_-(x))^{\gamma+1/2}\, dx = L^{cl}_{1,\gamma} \int_{-\infty}^{\infty} (V_-(x))^{\gamma+1/2}\, dx.
\end{aligned}$$

- Sharp multidimensional Lieb-Thirring inequalities, $\gamma \geq 3/2$.

The main argument is based on a Lieb-Thirring inequality for Schrödinger operators with matrix-valued potentials.

Theorem. (AL & T.Weidl)
Let $M \geq 0$ be a Hermitian $m \times m$ matrix-function and let $\mathcal{H} = -d^2/dx^2 - M$ in $L^2(\mathbb{R})$. Then

$$\sum_n \lambda_n^{3/2}(\mathcal{H}) \leq \frac{3}{16} \int \operatorname{Tr} M^2(x)\, dx.$$

Our proof was based on the matrix valued version of the so-called second trace (BFZ) formula related to the integral of motion for the KdV equation

$$\sum_k \lambda_k^{3/2} + \frac{3}{2\pi} \int_{\mathbb{R}} k^2 \log |a(k)|\, dk = \frac{3}{16} \int V^2\, dx$$

and the fact that $|a(k)| \geq 1$.

Theorem. (AL & T.Weidl)

Let $V \geq 0$ with $\gamma \geq 3/2$. Then for the negative eigenvalues $\{-\lambda_k\}$ of the Schrödinger operator $\mathcal{H} = -\Delta - V$ in $L^2(\mathbb{R}^d)$ we have

$$\sum \lambda_n^\gamma \leq L_{d,\gamma}^{cl} \int_{\mathbb{R}^d} V^{\gamma+d/2}(x)\, dx.$$

For the proof we use the "lifting argument" with respect to dimension.

Let for simplicity $d = 2$, $V \in C_0^\infty(\mathbb{R}^2)$, $V \geq 0$, $x = (x_1, x_2)$ and $\gamma = 3/2$. Then

$$H = -\Delta - V = -\partial^2_{x_1 x_1} - \underbrace{(\partial^2_{x_2 x_2} + V)}_{\tilde{H}(x_1)}.$$

Spectrum $\sigma(\tilde{H})$ of $\tilde{H}(x_1)$ has a finite number of positive eigenvalues $\mu_l(x_1)$. Thus $\tilde{H}_+(x_1)$ has a finite rank. We find

$$\sum_j \lambda_j^{3/2}(H) \leq \sum_j \lambda_j^{3/2}(-\partial^2_{x_1 x_1} - \tilde{H}_+)$$

$$\leq \frac{3}{16} \int \operatorname{Tr} \tilde{H}_+^2(x_1)\, dx_1 \leq \underbrace{\frac{3}{16} L_{1,2}}_{L^{cl}_{2,3/2}} \iint V^{3/2+1}(x)\, dx.$$

- The sum of the square roots $d = 1$, $\gamma = 1/2$.

Theorem. (D.Hundertmark, E.Lieb and L.Thomas)
Let $V \geq 0$, $d = 1$ and $\gamma = 1/2$. Then the negative eigenvalues $\{-\lambda_k\}$ of the operator $\mathcal{H} = -\frac{d^2}{dx^2} - V$, satisfy

$$\sum_k \sqrt{\lambda_k} \leq \frac{1}{2} \int_{\mathbb{R}} V \, dx = 2L^{cl}_{1,1/2} \int_{\mathbb{R}} V \, dx.$$

- Properties of self-adjoint compact operators.

Denote by $\mu_n = \mu_n(A)$ eigenvalues of a compact, self-adjoint operator A.

Proposition. Let A be a compact, self-adjoint operator. Then for any $N \in \mathbb{N}$ the sum of its highest eigenvalues

$$|||A|||_N = \sum_{n=1}^{N} |\mu_n(A)|$$

is a norm. In particular, if A and B are compact, self-adjoint operators, then for any N we have

$$|||A + B|||_N \leq |||A|||_N + |||B|||_N.$$

• Birman–Schwinger principle.

Let $-\lambda$ be the eigenvalue of $-\Delta - V$, $V \geq 0$. Then there is ψ s.t.

$$-\Delta\psi - V\psi = -\lambda\psi \Rightarrow -\Delta\psi + \lambda\psi = V\psi \Rightarrow \psi = (-\Delta + \lambda)^{-1}V\psi.$$

• A modified Birman–Schwinger operator.

Let $0 \leq W \in L^2(\mathbb{R})$. For $\varepsilon > 0$ we consider the operator

$$\mathcal{L}_\varepsilon := 2\varepsilon\, W \left(-\frac{d^2}{dx^2} + \varepsilon^2\right)^{-1} W, \qquad \text{in} \quad L^2(\mathbb{R}).$$

Note that $\mathcal{L}_\varepsilon \geq 0$ and

$$2\varepsilon \left(-\frac{d^2}{dx^2} + \varepsilon^2\right)^{-1} = 2\pi\, \mathcal{F}^* g_\varepsilon \mathcal{F},$$

where $\mathcal{F}$ denote the Fourier transform and where

$$g_\varepsilon(\xi) = \frac{\varepsilon}{\pi(\xi^2 + \varepsilon^2)}.$$

The function g_ε is a probability density, that is,

$$\int_{\mathbb{R}} g_\varepsilon(\xi)\, d\xi = 1 \qquad \forall \varepsilon > 0.$$

Lemma. The operator $\mathcal{L}_\varepsilon \geq 0$ and for any $\varepsilon > 0$

$$\operatorname{Tr} \mathcal{L}_\varepsilon = \int_{\mathbb{R}} W^2 \, dx.$$

- A property of the operator $\mathcal{L}_\varepsilon$.

Let us denote by $U(\xi)$ the unitary in $L^2(\mathbb{R})$ operator of multiplication by the function $e^{-i\xi x}$.

Proposition. Let $0 < \varepsilon' \leq \varepsilon$. Then

$$\mathcal{L}_\varepsilon = \int_{\mathbb{R}} U^*(\xi) \mathcal{L}_{\varepsilon'} \, U(\xi) \, g_{\varepsilon - \varepsilon'}(\xi) \, d\xi.$$

Proof. The operator $\mathcal{L}_\varepsilon = 2\pi W\mathcal{F}^* g_\varepsilon\,\mathcal{F}\,W$ is an integral operator with integral kernel

$$\mathcal{L}_\varepsilon(x,y) = \int_{\mathbb{R}} W(x)e^{i\xi(x-y)}W(y)\,g_\varepsilon(\xi)\,d\xi.$$

Note that

$$\widehat{g_\varepsilon} = (\mathcal{F}g_\varepsilon)(x) = e^{-\varepsilon|x|}.$$

Therefore for $0<\varepsilon'<\varepsilon$ we have

$$g_\varepsilon(\xi) = \mathcal{F}^{-1}\widehat{g_\varepsilon}(\xi) = \mathcal{F}^{-1}\left(\widehat{g_{\varepsilon'}}\,\widehat{g_{\varepsilon-\varepsilon'}}\right)(\xi) = \int_{\mathbb{R}} g_{\varepsilon'}(\xi-\eta)g_{\varepsilon-\varepsilon'}(\eta)\,d\eta.$$

This implies that the kernel of $\mathcal{L}_\varepsilon$ can be written as

$$\begin{aligned}
\mathcal{L}_\varepsilon(x,y) &= \int_{\mathbb{R}} W(x)e^{i\xi(x-y)}W(y)\,g_\varepsilon(\xi)\,d\xi \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\eta(x-y)}\,W(x)\,e^{i(\xi-\eta)(x-y)}\,W(y)\,g_{\varepsilon'}(\xi-\eta)g_{\varepsilon-\varepsilon'}(\eta)\,d\eta\,d\xi \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\eta(x-y)}\,W(x)\,e^{i\rho(x-y)}\,W(y)\,g_{\varepsilon'}(\rho)g_{\varepsilon-\varepsilon'}(\eta)\,d\rho d\eta \\
&= \int_{\mathbb{R}} e^{i\eta(x-y)}\,\mathcal{L}_{\varepsilon'}(x,y)\,g_{\varepsilon-\varepsilon'}(\eta)\,d\eta.
\end{aligned}$$

• Monotonicity lemma.

Let $\mu_n = \mu_n(\mathcal{L}_\varepsilon)$ be eigenvalues of $\mathcal{L}_\varepsilon$ arranged in non-increasing order and repeated according to multiplicities.

Lemma. Let $0 < \varepsilon' \leq \varepsilon$. Then for any $N \in \mathbb{N}$ we have

$$|||\mathcal{L}_\varepsilon|||_N \leq |||\mathcal{L}_{\varepsilon'}|||_N,$$

that is

$$\sum_{n=1}^{N} \mu_n(\mathcal{L}_\varepsilon) \leq \sum_{n=1}^{N} \mu_n(\mathcal{L}_{\varepsilon'}).$$

Proof. Using the fact that $|||\cdot|||$ is a norm we find

$$|||\mathcal{L}_\varepsilon|||_N = \left|\left|\left| \int_{\mathbb{R}} U^*(\xi)\mathcal{L}_{\varepsilon'}\, U(\xi)\, g_{\varepsilon-\varepsilon'}\, d\xi \right|\right|\right|_N$$

$$\leq \int_{\mathbb{R}} |||U^*(\xi)\mathcal{L}_{\varepsilon'}\, U(\xi)|||_N\, g_{\varepsilon-\varepsilon'}(\xi)\, d\xi = |||\mathcal{L}_{\varepsilon'}|||_N.$$

- Proof of $\sum_n \sqrt{\lambda_n} \leq \frac{1}{2} \int_{\mathbb{R}} V\,dx$.

We let $W = \sqrt{V} \in L^2(\mathbb{R})$ and consider the Birman–Schwinger operator

$$\frac{1}{2\varepsilon}\,\mathcal{L}_\varepsilon = W\left(-\frac{d^2}{dx^2} + \varepsilon^2\right)^{-1} W, \quad \varepsilon > 0.$$

Let $\mu_n(\mathcal{L}_\varepsilon)$ be the n's eigenvalue of $\mathcal{L}_\varepsilon$ and let $-\lambda_n = -\lambda_n(\mathcal{H})$ be the n's negative eigenvalue of $\mathcal{H} = -d^2/dx^2 - V$.

According to the Birman–Schwinger principle

$$1 = \frac{1}{2\sqrt{\lambda_n}}\,\mu_n\left(\mathcal{L}_{\sqrt{\lambda_n}}\right), \quad \forall n. \qquad (*)$$

We now show that

$$2\sum_{n=1}^{N} \sqrt{\lambda_n} \leq \sum_{n=1}^{N} \mu_n\left(\mathcal{L}_{\sqrt{\lambda_N}}\right), \quad \forall N.$$

If $N = 1$ this follows from $(*)$. Let $N = 2$, then again from $(*)$ and also by applying Corollary we have

$$2(\sqrt{\lambda_1} + \sqrt{\lambda_2}) = \mu_1\left(\mathcal{L}_{\sqrt{\lambda_1}}\right) + \mu_2\left(\mathcal{L}_{\sqrt{\lambda_2}}\right) \le \mu_1\left(\mathcal{L}_{\sqrt{\lambda_2}}\right) + \mu_2\left(\mathcal{L}_{\sqrt{\lambda_2}}\right).$$

Let $N = 3$. Then

$$\begin{aligned}2(\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}) &= \mu_1\left(\mathcal{L}_{\sqrt{\lambda_1}}\right) + \mu_2\left(\mathcal{L}_{\sqrt{\lambda_2}}\right) + \mu_3\left(\mathcal{L}_{\sqrt{\lambda_3}}\right)\\ &\le \mu_1\left(\mathcal{L}_{\sqrt{\lambda_2}}\right) + \mu_2\left(\mathcal{L}_{\sqrt{\lambda_2}}\right) + \mu_3\left(\mathcal{L}_{\sqrt{\lambda_3}}\right)\\ &\le \mu_1\left(\mathcal{L}_{\sqrt{\lambda_3}}\right) + \mu_2\left(\mathcal{L}_{\sqrt{\lambda_3}}\right) + \mu_3\left(\mathcal{L}_{\sqrt{\lambda_3}}\right).\end{aligned}$$

Repeating this N-times we obtain the claimed inequality

$$2\sum_{n=1}^{N} \sqrt{\lambda_n} \le \sum_{n=1}^{N} \mu_n\left(\mathcal{L}_{\sqrt{\lambda_N}}\right), \quad \forall N.$$

We conclude the proof of the theorem by computing the trace of $\mathcal{L}_{\sqrt{\lambda_N}}$

$$\begin{aligned}2\sum_{n=1}^{N} \sqrt{\lambda_k} &\le \operatorname{Tr} \mathcal{L}_{\sqrt{\lambda_N}} = \int \mathcal{L}_{\sqrt{\lambda_N}}(x,y)\Big|_{x=y} dx\\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} W(x) e^{i\xi(x-y)} W(y)\, g_\varepsilon(\xi)\, d\xi\Big|_{x=y} dx = \int_{\mathbb{R}} W(x)^2 dx = \int_{\mathbb{R}} V(x) dx.\end{aligned}$$

The proof is complete.

- L-Th inequalities for $\gamma \geq 1/2$, $d \in \mathbb{N}$.

The latter theorem can be proven for 1D Schrödinger operators with matrix-valued potentials and using the technique that was presented before we obtain

Corollary. (D.Hundertmark, AL, T.Weidl)

Let $V \geq 0$, $d \in \mathbb{N}$ and $1/2 \leq \gamma < 3/2$. Then the negative eigenvalues $\{-\lambda_k\}$ of the operator $\mathcal{H} = -\Delta - V$, satisfy

$$\sum_k \lambda_k^\gamma \leq 2L_{d,\gamma}^{cl} \int_{\mathbb{R}} V^{d/2+\gamma}\, dx.$$

- L-Th inequality for $\gamma = 1$.

Theorem. (R.L.Frank, D.Hndertmark, M.Jex, P.T,Nam (2021)) Let $V \geq 0$, $d = 1$, $\gamma = 1$. Then for the negative eigenvalues $\{-\lambda_k\}$ of the operator $\mathcal{H} = -d^2/dx^2 - V$ we have

$$\sum_k \lambda_k \leq R_{1,1} L^{cl}_{1,1} \int V^{3/2} dx,$$

where $R_{1,1} = 1.456...$

Remark.
Some ideas of the proof are related to the ideas suggested by M.Rumin.
This constant $R_{1,1}$ is an improvement of the previously obtained constant 1.813 by J.Dolbeault, AL and M.Loss (2008).

- Two big open problems.

1. Find sharp L-Th constants for $d \in \mathbb{N}$ and $1/2 < \gamma < 3/2$. In particular, it is conjectured that

$$L_{d,1} = L_{d,1}^{cl}, \qquad d \geq 3$$

This inequality cannot be true in dimensions $d = 1, 2$.

Remark. The sharp constants in the CLR inequalities are also unknown. The best known constants for $d = 3, 4$ are due to E.Lieb. Recently D.Hundertmark, P.Kunstmann, T.Ried and S.Vugalter obtained new constants for $d \geq 5$ by using Cwikel's approach.

2. Prove Pólya's conjecture saying that for the eigenvalues $\{\lambda_k\}$ of the Dirichlet Laplacian $-\Delta^{\mathcal{D}}$ in $L^2(\Omega)$, where $\Omega \in \mathbb{R}^d$ is a domain of finite measure,

$$\#\{k : \lambda_k < \lambda\} \leq (2\pi)^{-d} |\Omega| |B_d| \lambda^{d/2}.$$

Remark. This was proved by Pólya for tiling domains and by AL for some product type domains. Last year M.Levitin, I.Polterovich and D.A.Sher proved the Pólya conjecture for the disc and also N.Filonov obtained it for balls and circular sectors.

Thank you