Partially holomorphic functions in several variables.
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Moment conditions in complex analysis

1) Morera type conditions. $\int_{C_\alpha} f dz = 0$ for a collection of curves C_α. Classical, many results. Question: what if the Morera condition holds on a family of ellipses instead of circles?

2) 1-dimensional extension property. Given a function f in one or several variables, what happens if for some collection of curves C_α, each bounding an analytic disc D_α, $f|_{C_\alpha}$extends holomorphically to D_α

EXAMPLES:

i) Let $C_r = \{z: |z| = r\}, D_r = \{z: |z| < r\}$. A function $f(z)$ has the 1-dimensional extension property for this family if it can be written as $g(r, z)$, holomorphic in z. Basic building block.

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ii) $D \subseteq \boldsymbol{C}^2$ is a smooth strictly convex domain containing 0. Suppose for every complex line through 0, a function $f \in C(\partial D)$ extends holomorphically on that line.

iii) $D \subseteq \boldsymbol{C}^2$ is a smooth strictly convex domain. Suppose that a function $f \in C(\partial D)$ has a holomorphic extension on every slice $z = c$.

Old question: Find conditions of 1-dimensional extension type which are sufficient and efficient for determining that a function is holomorphic, or has holomorphic extension from a boundary to a domain.

EXAMPLES:

i) (Stout) Let $D \subseteq \boldsymbol{C}^n$ be any domain with C^1 boundary, and let $f \in C(\partial D)$. If f has holomorphic extensions on every affine slice of D, then f extends holomorphically to D.

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ii) Let $D \subseteq \boldsymbol{C}^2$ be a suitable perturbation of $\rho_\epsilon < 0$, where $\rho_\epsilon = |z|^2 + |w|^2 - 1 + \epsilon(zw + \overline{zw})$. If $f \in L^1(\partial D)$ and if for almost all $z = c$ and $w = k$, f extends holomorphically on the slices, then f extends holomorphically to D.

iii) (Tumanov, with improvements by Lawrence). Set $S = \{z = x + iy: |y| < 1\}$. Let $C_t = \{z: |z - t| = 1\}, D_t = \{z: |z - t| < 1\}$. If $f \in C(\overline{S})$ and if for all t, $f|_{C_t}$ extends holomorphically to D_t, then f is holomorphic on S.

New (?) Question: Can moment conditions force a function to be real analytic without being holomorphic? Question due to [L]?

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Theorem: Let $X \subseteq \boldsymbol{C}$ be a discrete set. Let $f \in C(\boldsymbol{C})$ and suppose that for every $a \in X$ and every $r > 0, f|_{|z-a|=r}$ extends holomorphically to $|z-a| < r$. If X is sufficiently spread out at infinity (mild condition), then f is real analytic with infinite radius of convergence for the power series.

Example: $X = \boldsymbol{Z} \cup \omega\boldsymbol{Z} \cup \omega^2\boldsymbol{Z}, \omega^3 = 1, \omega \neq 1$ works . Every such f can be written as $f(z) = \sum_{m,n\geq 0} a_{mn} z^m \left(\overline{z} g(z)\right)^n$, where $g(z) = \frac{\sin z \sin \omega z \sin \omega^2 z}{z^2}$.

In other words, this is a class of non-holomorphic power series which is closed under uniform convergence on compacta.

Bergman spaces: The above class of functions can made into an $L^2(\boldsymbol{C}, e^{-|z|^2} dA)$ closed space with bounded point evaluation---like Bergman spaces, but not using holomorphic functions.

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The slicing problem for the ball.

Consider $\boldsymbol{B} = \{(z,w): |z|^2 + |w|^2 < 1\}, S = \partial\boldsymbol{B}$. The function $f \in C(S)$ given by $f(z) = |z|^2 = 1 - |w|^2$ has holomorphic extensions from all vertical and horizontal slices (because it is constant on those slices).

Theorem: Let $CRH = \{f \in L^2(S): f|_{(z=c)\cap S}\ extends\ holomorphically\ to\ \boldsymbol{B} \cap (z = c) for\ a.e.\ c, and\ f|_{w=k} extends\ holomorphically\ to\ \boldsymbol{B} \cap (w = k) for\ a.e. k\}$. Then $CRH = \overline{P[z,w,|z|^2]}$.

Consider a function $p(z,w,|z|^2)$ for a polynomial p. The w-holomorphic extension to **B** is $p_1 = p(z,w,|z|^2)$; the z-holomorphic extension is $p_2 = p(z,w,1-|w|^2)$.

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Inside the ball, on every w-slice, p_1 is holomorphic in w, and has holomorphic extensions from every circle $|z| = r$; on every z-slice of the ball, p_2 is holomorphic in z and has holomorphic extensions from every circle $|w| = r$.

Definition: Given a domain $D \subseteq \boldsymbol{C}^{\boldsymbol{n}}, n > 1$, a function f on D is said to be partially holomorphic if it is holomorphic on slices in one affine direction, and satisfies moment conditions for holomorphicity on all slices in one or more other affine directions.

Examples: Assume domains are convex, in $\boldsymbol{C}^{\boldsymbol{2}}$.

1) Let D be $\rho < 0$ for a function $\rho(z, w)$ with one critical point 0 in D. Consider a function $f(z, w)$ on D which is holomorphic in z, and

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such that for each slice $z = c$, has holomorphic extensions from $(\rho = t) \cap (z = c)$ for $t < 0$.

2) Specifically, let $\rho = |z|^2 + |w|^2 - 1 + \epsilon(zw + \overline{zw})$. Then on each $\rho = t, t < 0$, f extends holomorphically on z slices and on w slices, so has a holomorphic extension to $\rho < t$ which must be f. Thus f is holomorphic.

Functions which extend holomorphically from the sphere in many coordinate directions.

Let $(z, w), (z_1, w_1) \dots, (z_n, w_n)$ be distinct orthogonal coordinate systems in $\boldsymbol{C^2}$.

Lemma: $f(z, w) = |z|^2 \pi_{i=1}^{n} z_i w_i$ extends holomorphically from S on every slice $z_k = c$, or $w_k = c$.

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Conclusion: For the sphere, no finite number of directions is enough to guarantee holomorphic extension of a function.

Observation: Consider the function $f(z,w) = |z|^2 \pi_{i=1}^{n} z_i w_i$ on $\boldsymbol{B}$. For every slice $w = c, z_i = c, w_i = c,\ f$ extends holomorphically from each circle $|z| = r, |w_i| = r, |z_i| = r$. Thus f is partially holomorphic (holomorphic in w).

Theorem: Let $f(z,w)$ be a continuous function on S which extends holomorphically from every slice $z = c, w = c, z_i = c, w_i = c, 1 \leq i \leq n$. Let $F(z,w)$ be the z-holomorphic extension to $\boldsymbol{B}$**.** Then for every slice of $\boldsymbol{B}$ by $z = c, z_i = c, w_i = c,$ F has holomorphic extensions from each $|w| = r, |z_i| = r, |w_i| = r$.

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QUESTION: Does F have any regularity better than continuity, and being holomorphic in z?

The proof of the theorem depends on residue calculations, using

i) $dV = \frac{dz_i}{z_i} \wedge dw_i \wedge d\overline{w_i}$ which holds in all orthonormal coordinate systems.

ii) $\int_{\partial\boldsymbol{B}} \frac{f(z,w)}{w-a} |w|^{2n} z_i^{n+1} z^k dz \wedge dw \wedge d\overline{w} = 0$

iii) Rewrite iii) as $\int_{\partial\boldsymbol{B}} \frac{f(z,w)}{w-a} |w|^{2n} z_i^{n} z^{k+1} dz_i \wedge dw_i \wedge d\overline{w_i} = 0$

Take residues in iii), and after some calculation and induction, get the theorem.

Remark: A more direct proof would use polynomial approximation by sums of monomials as above. Should be possible.

The attempt to prove real analyticity:

I attempt to copy the strategy that worked for proving real analyticity for an algebra of functions on the plane.

Tools:

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i) If $g(z, w)$ defined on $|z| = s, |w| = t$ has holomorphic extensions on w-slices and z slices, then it has a holomorphic extension to the bidisc (easy Fourier series argument). Consider $F(z, w)$. On tori $|z_i| = s, |w_i| = t, s^2 + t^2 < 1$, F extends holomorphic to the bidisc (value depending on s and t).

ii) Easy CR wedge extension case. In $\boldsymbol{C}^2$, if $M_1 = \{(z, w): Im(z) = 0, Im(w) \geq 0, M_2 = \{(z, w): Im(z) \geq 0, Im(w) = 0\}$ a continuous function g on $M_1 \cup M_2$ which is holomorphic on the complex lines, extends holomorphically to $Im(z) > 0, Im(w) > 0$. Local version is true also. In $\boldsymbol{C}^4$ an analogous result is true for 6 dimension M_1, M_2, each of which is foliated by 2 dimensional affine complex spaces.

Complexification.

In a coordinate system (z_i, w_i), we have that $F(z, w)$ extends holomorphically from each $|z_i| = s, |w_i| = t, s^2 + t^2 < 1$.

Denote the holomorphic extension by $F_{s,t}$

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Complexify by defining new variables ζ_i, τ_i. $\boldsymbol{C^2}$is complexified by setting

$$\zeta_i = \overline{z_i}, \tau_i = \overline{w_i}$$

$|z_i|^2 = s^2, |w_i|^2 = t^2$ become $z_i\zeta_i = s^2$, $w_i\tau_i = t^2$. F_{st} is lifted by using

$\zeta_i = \frac{s^2}{z_i}, \tau_i = \frac{t^2}{w_i}$.

For all s, t, these analytic varieties lie in

$Im(z_i\zeta_i) = 0, Im(w_i\tau_i) = 0$, a 6 dimensional manifold foliated by complex two dimensional leaves. We get a half-space in this 6 dimensional manifold. Call this manifold M_i. We get a lift of F to this manifold which is holomorphic on the complex leaves.

For $i \neq j,\ M_i \cup M_j \subseteq \{(z, w, \zeta, \tau: \zeta = \overline{z}, \tau = \overline{w}.\}$

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Because of CR wedge extension, the lifts of F to $M_i \cup M_j$ extend to an open wedge coming out of $\{(z, w, \zeta, \tau : \zeta = \overline{z}, \tau = \overline{w}.\}$

QUESTION: How could you prove real analyticity? If by using enough coordinate systems you can get holomorphic extension to a neighborhood of $\{(z, w, \zeta, \tau : \zeta = \overline{z}, \tau = \overline{w}.\} \subseteq \boldsymbol{C}^4$

Problem: All such analytic continuation occurs in $Im(z\zeta + w\tau) > 0$. Not enough.

Still, there is room for more study of this class of functions.

1) What about non-tangential boundary values?

2) How can you describe how these function spaces approximate the holomorphic functions as the number of slices increases?

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Related questions:

For $X = \{0,1\} \subseteq \boldsymbol{C}$ consider the functions with holomorphic extensions from circles centered at these points. Are there examples which are not real analytic? Prediction: no such functions. Need better proof techniques.