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Minimal commutant and double commutant property for analytic Toeplitz operators

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Contents

Basics

... *some algebras*

Some algebras associated to an operator.

Let T be a Hilbert space operator.

1. Set $\mathrm{Alg}(T)$ the algebra generated by T and the identity,
2. $\{T\}' = \{A : AT = TA\}$ the commutant of T.
3. $\{T\}'' = \{B : BA = AB \ : A \in \{T\}'\}$ the double commutant of T.

$$\overline{\mathrm{Alg}(T)}^{WOT} \subset \{T\}'' \subset \{T\}'$$

Basics

... *Von Neumann's theorem*

Von Neumann

Let T be a Hilbert space selftadjoint operator then

$$\overline{\mathrm{Alg}(T)}^{WOT} = \{T\}''$$

Basics

... *double commutant property*

Von Neumann

Let T be a Hilbert space operator. We say that T has the double commutant property if

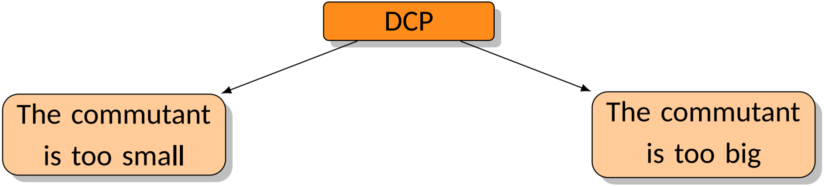
$$\overline{\mathrm{Alg}(T)}^{WOT} = \{T\}''$$

Basics

... *main question*

1970s

Let T be a Hilbert space operator. When T has the double commutant property?



Basics

... *the size of the commutant*

Minimal commutant property (MCP)

Let T be a Hilbert space operator. We say that T has the minimal commutant property if

$$\overline{\mathrm{Alg}(T)}^{WOT} = \{T\}'$$

Basics

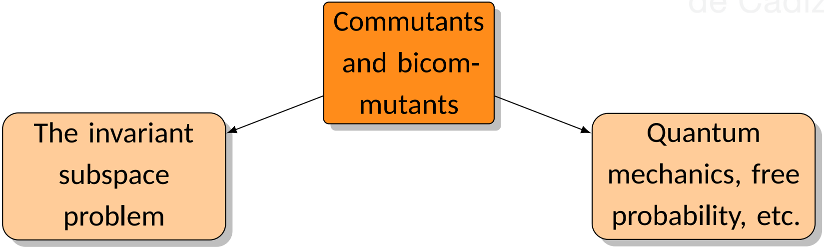
... *the size of the commutant*

Fact

1. If T has the MCP then $\overline{\mathrm{Alg}(T)}^{WOT} = \{T\}'' = \{T\}'$.
2. If $\{T\}'$ "increases" then $\{T\}''$ decreases. For big commutants we have also the DCP.

Basics

... *double commutant property, origin*



Basics

... *double commutant property, origin*

Victor Lomonosov Theorem (1973)

Assume that T is a Banach space operator that commute with an operator A, and such that A also commutes with a non-trivial compact operator. Then T has a non-trivial invariant subspace.

Question

Is there a non-trivial operator which does not satisfy the Lomonosov hypothesis?

Basics

...analytic Toeplitz operators

1. $H^2(\mathbb{D}) = \left\{f(z) = \sum_n a_n z^n \in H(\mathbb{D}) \,:\, \sum_n |a_n|^2 < \infty\right\}$
2. $M_z f(z) = zf(z)$.
3. $\{M_z\}' = \{M_\phi \,:\, \phi \in H^\infty\}$; $M_\phi f(z)(z) = \phi(z)f(z)$
4. Cowen: M_z satisfies Lomonosov Hypothesis.

Basics

....Hadwin, Nordgren, Radjavi, Rosenthal.

JOURNAL OF FUNCTIONAL ANALYSIS 38, 410–415 (1980)

An Operator Not Satisfying Lomonosov's Hypothesis ([1])

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An example is presented of a Hilbert space operator such that no non-scalar operator that commutes with it commutes with a non-zero compact operator.

Questions

...*analytic Toeplitz operators*

1. Given $\varphi \in H^{\infty}(\mathbb{D})$ describe $\{M_\varphi\}'$.
2. (Turner 1971) For which maps φ; $\{M_\varphi\}''$ is minimal?
3. For which maps φ, $\{M_\varphi\}'$ is minimal?

Selected background

...analytic Toeplitz operators

1. Cowen-1980. M_z satisfies Lomonosov hypothesis.
2. Turner-1972 If T is algebraic then T has the DCP.
3. Turner-1971 A normal operator has the DCP if and only if each invariant subspace of T is invariant under $T^\star$.
4. Abrahamse-Cowen-Deddens-Turner-Thomson'70s They characterized the commutant of several analytic Toeplitz operators.

Analytic Toeplitz operators

...with a minimal commutant

Shields and Wallen (1970/71)

If $\varphi \in H^\infty(\mathbb{D})$ is univalent then $\{M_\varphi\}' = \{M_z\}'$. That is,

$$\{M_\varphi\}' = \{M_\phi : \phi \in H^\infty(\mathbb{D})\}$$

M.J.González-L.

If φ is not univalent in $H^\infty(\mathbb{D})$ then M_φ don't have a minimal commutant.

Analytic Toeplitz operators

Toeplitz minimal commutant's theorem

M.J. González- L.

Set $\varphi \in H^{\infty}(\mathbb{D})$. The following conditions are equivalents:

1. M_{φ} has a minimal commutant.
2. The polynomials on φ are dense in $H^2(\mathbb{D})$.

Analytic Toeplitz operators

Toeplitz minimal commutant's theorem

Facts

If the polynomials on φ are dense in $H^2(\mathbb{D})$ then

1. φ must be univalent.
2. Then φ must be univalent almost every where on $\partial\mathbb{D}$, that is, there exists a set $S \subset \partial\mathbb{D}$ of zero Lebesgue measure such that φ is univalent on $\partial\mathbb{D} \setminus S$.

Analytic Toeplitz operators

Toeplitz minimal commutant's theorem

Example

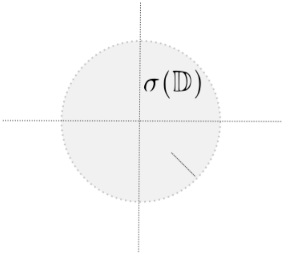

Figure: σ maps univalently $\mathbb{D}$ onto a slit disk.

Analytic Toeplitz operators

Toeplitz minimal commutant's theorem

Facts

For instance, if φ maps U univalently onto G, and G is a simply connected domain whose boundary is a Jordan curve, Walsh's Theorem asserts that the polynomials in φ are dense in $H^2(\mathbb{D})$.

Analytic Toeplitz operators

Toeplitz minimal commutant's theorem

Example

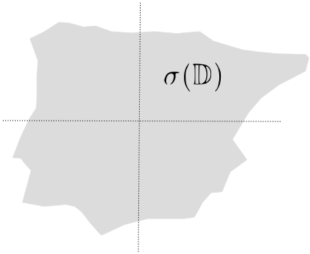

Figure: σ is univalent and the boundary of $G = \sigma(\mathbb{D})$ is a Jordan curve.

Analytic Toeplitz operators

Toeplitz minimal commutant's theorem

Remarks

1. $M_\varphi = M_{\varphi-\varphi(0)} + \varphi(0)I$, that is, the commutant of M_φ is invariant under translation. Also it is invariant under scalar multiplications.
2. $\phi = \frac{\varphi - c}{R}$, $\phi(\mathbb{D}) \subset \mathbb{D}$.
3. M_φ has a minimal commutant if and only if for some c and R, C_ϕ is cyclic.

Double commutant property

...univalent case

Univalent Case

Assume that $\varphi \in H^\infty(\mathbb{D})$ is univalent. Then M_φ has the double commutant property if and only if the polynomials in φ are dense in $H^2(\mathbb{D})$.

Question

Is there a non-univalent map $\varphi \in H^\infty(\mathbb{D})$ such that M_φ has the double commutant property?

Double commutant property

... non-univalent case

Deddens-Wong (1973)

Assume that $\varphi \in H^\infty(\mathbb{D})$ single cover a non-empty neighbourhood W of $\varphi(\mathbb{D})$, then $\{M_\varphi\}' = \{M_z\}'$.

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If $\varphi \in H^\infty(\mathbb{D})$ is not univalent but φ single cover a non-empty neighbourhood W of $\varphi(\mathbb{D})$, then M_φ don't have the double commutant property.

Double commutant property

... non-univalent case

Example

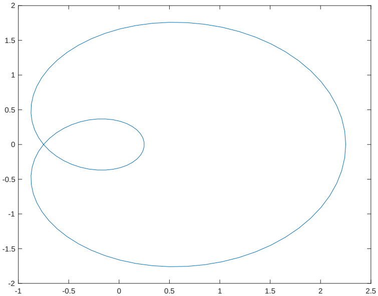

Figure: The cardioid map $\varphi(z) = (z + 1/2)^2$. M_φ not DCP.

Double commutant property

... non-univalent case

By considering $\varphi(z) = z^n$, $n \geq 2$, the commutant of M_{z^n} is very big.

Čučković (1994)

Čučković described the elements in the algebra of all Toeplitz operators that commutes with M_{z^n}.

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M_{z^n} has the double commutant property.

Double commutant property

... non-univalent case

$f \in H^\infty(\mathbb{D})$ have radial limits $f^\star(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ almost every where on $\partial\mathbb{D}$.

Inner function

$f \in H^\infty(\mathbb{D})$ is called an inner function if $|f^\star(e^{i\theta})| = 1$ almost everywhere.

Turner (1971)

Turner proved that non-unitary isometries have the double commutant property. In particular, if ϕ is inner, then M_φ has the double commutant property.

Double commutant property

... multiplication by entire functions

γ the unit circle $\{e^{it} \ : \ 0 \le t \le 2\pi\}$. Set φ analytic on $\overline{\mathbb{D}}$, and $a \notin f(\gamma)$. Denote $n(\varphi(\gamma), a)$ the winding number of φ about γ and we set

$$k(\varphi) = \inf\{n(\varphi(\gamma, a) \ : \ n(\varphi(\gamma), a) \neq 0\}.$$

Baker-Deddens-Ullman (1974)

If φ is a non-constant entire function and $k = k(\varphi)$ then there exists an entire function h such that $\varphi(z) = h(z^k)$ and $k(h) = 1$.

1. $\{M_\varphi\}' = \{M_{z^k}\}'$.
2. If $\varphi(z) = \sum a_n z^n$ then $k(\varphi) = g.c.d\{n \ : \ a_n \neq 0\}$.

Double commutant property

... multiplication by entire functions

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Assume that φ is an entire function with $k(\varphi) = k$. If the image of a point under φ has $p > k = k(\varphi)$ preimages then M_φ does not have the double commutant property.

M.J. González-L.

Assume that φ is an entire function with $k(\varphi) = k$. If there exists of a point $c \in \varphi(\mathbb{D})$ such that $n(\varphi(\gamma), c) = p$ and $p > k = k(\varphi)$ then M_φ does not have the double commutant property.

Double commutant property

... multiplication by entire functions

For $\theta \in H^{\infty}(\mathbb{D})$, let us denote $H_{\theta} = \bigvee\{1, \theta, \theta^2, \cdots\}$.

M.J. González-L.

Assume that φ is an entire function with $k(\varphi) = k$. The following conditions are equivalents:

1. M_{φ} has the double commutant property.
2. The polynomials on φ are dense in H_{z^k}

Double commutant property

... multiplication by functions in the Thomson-Cowen's class

Thomson-Cowen's class: $TC(\mathbb{D})$.

$\mathrm{TC}(\mathbb{D})$) is the class of bounded analytic functions φ for which there exists a point $\lambda \in \mathbb{D}$ such that the inner part of $\varphi - \varphi(\lambda)$ is a finite Blaschke product.

$\mathrm{TC}(\mathbb{D})$ contains all non-constant functions in $H^\infty(\overline{\mathbb{D}})$

Thomson (1976)-Cowen(1978)

Assume $\varphi \in \mathrm{TC}(\mathbb{D})$. Then there exists a finite Blaschke product B and a function $h \in H^\infty(\mathbb{D})$ such that $\phi = h(B)$ and $\{M_\varphi\}' = \{M_B\}'$.

Double commutant property

... multiplication by functions in the Thomson-Cowen's class

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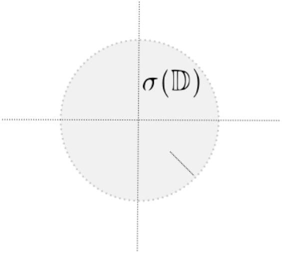
Assume that $\phi \in \mathrm{TC}(\mathbb{D})$, $\varphi = h(B)$ for some $h \in H^\infty(\mathbb{D})$ and B is a finite Blascke product. Then the following conditions are equivalents:

1. M_φ has the double commutant property.
2. The polynomials on ϕ are dense in H_B.

Double commutant property

... multiplication by functions in the Thomson-Cowen's class

Example: σ maps univalently $\mathbb{D}$ onto a slit disk.



If B is a finite Blascke product, then $M_{\sigma(B)}$ don't have the double commutant property.

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3. M. Lacruz, F León-Saavedra, S Petrovic, L Rodríguez-Piazza, The double commutant property for composition operators, Collectanea mathematatica 70 (3), (2019) 501-532

Thank you very much for your attention