Toeplitz operators and Bergman projections on weighted spaces of holomorphic functions.

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1 Introduction

We consider the area measure $dm = dxdy$ on $\mathbb{C}$ and an open subset $O \subset \mathbb{C}$,
(in the following mostly $O = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$).
A *weight* v on O is a continuous map $O \to]0, \infty[$.
For measurable f on O put

$$||f||_{v,p} = \left(\int_O |f(z)|^p v(z) dm(z) \right)^{1/p} \quad \text{if } 1 \leq p < \infty,$$

$$||f||_{v,\infty} = \operatorname*{ess\,sup}_{z \in O} |f(z)| v(z) \quad \text{and}$$

$$L^p_v = \{f : O \to \mathbb{C} \text{ measurable} : ||f||_{v,p} < \infty\},$$

$$H^p_v = \{h \in L^p_v : h \text{ holomorphic }\}.$$

For each $z_0 \in O$ there is a constant $c(z_0) > 0$ such that every $h \in H^2_v$ satisfies $|h(z_0)| \leq c(z_0)||h||_{v,2}$.
Hence there is a unique $K_{z_0} \in H^2_v$ with

$$h(z_0) = \langle h, K_{z_0} \rangle_v := \int_O h(z) \overline{K_{z_0}(z)} v(z) dm(z),$$

called the *reproducing kernel*.

Let $P_v : L^2_v \to H^2_v$ be the orthogonal projection, called *Bergman projection*, i.e.

$$P_v(g)(z_0) = \int_O g(z)\overline{K_{z_0}(z)}v(z)dm(z), \quad z_0 \in O, \; g \in L^2_v.$$

For measurable f and holomorphic h let

$$T_f(h)(z_0) = P_v(fh)(z_0) = \int_O f(z)h(z)\overline{K_{z_0}(z)}v(z)dm(z)$$

be the *Toeplitz operator* whenever the preceding integral exists.

Questions:

- For which f and v is T_f a bounded operator $H^\infty_v \to H^\infty_v$?
- For which weights $v, \tilde{v}$ and $1 \leq p \leq \infty$ is $P_{\tilde{v}}$ a bounded operator $L^p_v \to H^p_v$?

2 Toeplitz operators

2.1.Definition. A weight v on $\mathbb{D}$ is a *standard weight* if $v(z) = v(|z|)$, $z \in \mathbb{D}$, $v(s) \leq v(r)$ whenever $0 \leq r \leq s < 1$ and $\lim_{r\to 1} v(r) = 0$.

Examples for $v(r)$, $0 \leq r < 1$: $(1 - \log(1-r))^{-1}$, $(1-r)^\gamma$ for some $\gamma > 0$, $\exp(-\alpha(1-r)^{-\beta})$ for some $\alpha, \beta > 0$.

For a standard weight v we obtain

$$K_{z_0}(z) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{z^j \bar{z}_0^j}{\int_0^1 r^{2j+1} v(r) dr}.$$

In particular K_{z_0} is bounded and δ_{z_0} with $\delta_{z_0}(h) = h(z_0)$ is a bounded linear functional on H_v^∞,
i.e. $\delta_{z_0} \in (H_v^\infty)^*$.

2.2.Theorem. *Let v be a standard weight on $\mathbb{D}$ and let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Then*

T_f is a bounded operator $H_v^\infty \to H_v^\infty \Leftrightarrow f$ is bounded.

Proof. " $\Leftarrow$ ": Here, for $h \in H_v^\infty$, $fh \in H_v^\infty$ and $||P_v(fh)||_{v,\infty} = ||fh||_{v,\infty} \leq \sup_z |f(z)| \cdot ||h||_{v,\infty}$.
" $\Rightarrow$ ": Here we have $T_f^* \delta_z = f(z)\delta_z$, i.e.

$$|f(z)| = ||T_f^*(\delta_z/||\delta_z||)|| \leq ||T_f^*|| = ||T_f|| < \infty, z \in \mathbb{D}.$$

□

2.3.Theorem. *There is a bounded harmonic function f on $\mathbb{D}$ such that for NO standard weight v on $\mathbb{D}$ the Toeplitz operator T_f is a bounded operator $H_v^\infty \to H_v^\infty$.*

Sketch of proof. Let

$$f(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}(z^{2j+1} + \bar{z}^{2j+1}),$$

$$i.e. \quad f(e^{i\varphi}) = \begin{cases} \frac{\pi}{2}, & -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \\ \\ -\frac{\pi}{2}, & \text{else} \end{cases}$$

Put $\quad h_m(z) = \dfrac{z^{4m}}{||z^{4m}||_{v,\infty}} = \dfrac{z^{4m}}{r_{4m}^{4m} v(r_{4m})}, \quad m = 1, 2, \ldots,$

if r_m is a maximum point of $r \mapsto r^m v(r)$.

We have $||h_m||_{v,\infty} = 1$ for all m. Moreover we obtain $\lim_{m\to\infty} r_m = 1$.

A computation shows there are $\beta_{m,j} \geq 0$ with

$$T_f(h_m)(z) = \sum_{0\leq 2j+1 \leq 4m} (-1)^j \beta_{m,j} z^{2j+1} +$$

$$+ \sum_{4m+1\leq 2j+1} \left(\frac{(-1)^j}{2j+1-4m} \right) \left(\frac{z^{2j+1}}{r_{4m}^{4m} v(r_{4m})} \right).$$

Cancelling the summands with negative coefficients yields

$$\frac{1}{2}\left(T_f(h_m)(z) - iT_f(h_m)(iz)\right) = \sum_{0\leq 4j+1\leq 4m} \beta_{m,j} z^{4j+1} +$$

$$+ \sum_{4m+1\leq 4j+1} \left(\frac{1}{4(j-m)+1} \right) \left(\frac{z^{4j+1}}{r_{4m}^{4m} v(r_{4m})} \right).$$

For $z = r_{4m}$ this yields

$$\frac{r_{4m}}{5} \log\left(\frac{1}{1-r_{4m}^4} \right) = \frac{r_{4m}}{5} \sum_{k=1}^{\infty} \frac{r_{4m}^{4k}}{k} \leq \sum_{k=0}^{\infty} \frac{r_{4m}^{4k+1}}{4k+1}$$

$$\stackrel{(k=j-m)}{=} \sum_{4m+1\leq 4j+1} \left(\frac{1}{4(j-m)+1} \right) \left(\frac{r_{4m}^{4j+1} v(r_{4m})}{r_{4m}^{4m} v(r_{4m})} \right)$$

$$\leq \frac{1}{2}(T_f(h_m)(r_{4m}) - iT_f(h_m)(ir_{4m}))v(r_{4m})$$

$$\leq ||T_f(h_m)||_{v,\infty} \to \infty \quad \text{while} \quad ||h_m||_{v,\infty} = 1.$$

□

Theorem 2.3. is no longer true on other open subsets of $\mathbb{C}$. Let

$$\mathbb{G} = \{w \in \mathbb{C} : Im\ w > 0\} \quad \text{(upper half plane)}\ .$$

2.4.Theorem. *Let* $v(w) = (Im\ w)^\gamma$, $w \in \mathbb{G}$, *for some* $\gamma > 0$. *If* $f : \mathbb{G} \to \mathbb{C}$ *is measurable and bounded then* T_f *is a bounded operator* $H_v^\infty \to H_v^\infty$.

2.5.Definition. A standard weight v on $\mathbb{D}$ is *normal* if

$$\sup_{n\in\mathbb{N}} \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty \quad \text{and}$$

$$\exists k: \quad \limsup_{n\to\infty} \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1.$$

("The decay of $v(r)$ for $r \to 1$ is not too slow and not too fast".)

Examples. $1/(1-\log(1-r))$ is not normal (too slow). $(1-r)^\gamma$ for $\gamma > 0$ is normal, $\exp(-\alpha/(1-r)^\beta)$ for $\alpha, \beta > 0$ is not normal (too fast).

Remark. One can show that normal weights and exponential weights satisfy the following **condition** (B) ("inner regularity condition"):

Let r_m be point of absolute maximum of $r \mapsto r^m v(r)$, $0 \le r < 1$, for $m > 0$. Then

$\forall b_1 > 1\, \exists b_2 > 1\, \exists c > 0\, \forall m, n > 0 :$

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \quad \text{and} \quad m, n, |n-m| \ge c$$

$$\Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \le b_2$$

2.6.Theorem (L. 2006). *Let v be a standard weight on $\mathbb{D}$. Then there are only two isomorphism classes of H_v^∞ as Banach spaces, namely*

$H_v^\infty \sim l^\infty$ *if and only if v satisfies* (B)

and

$H_v^\infty \sim H^\infty = \{h : \mathbb{D} \to \mathbb{C} : h \text{ holomorphic, bounded}\}$ *if and only if v does not satisfy* (B).

Remark. Normal weights and exponential weights satisfy (B). $(1 - \log(1 - r))^{-1}$ does not satisfy (B).

In contrast to Theorem 2.3. we have

2.7.Theorem *Let v be normal. If f is a trigonometric polynomial on $\mathbb{D}$ then T_f is a bounded operator $H_v^\infty \to H_v^\infty$.*

In the following we consider Toeplitz operators with radial symbols f:

2.8.Theorem. *Let f be radial, i.e. $f(z) = f(|z|)$ for $z \in \mathbb{D}$, and satisfy $\int_0^1 |f(r)| r dr < \infty$.*

Then T_f is a bounded operator $H_v^\infty \to H_v^\infty$ provided that one of the following conditions is satisfied:
(i) v is a normal weight on $\mathbb{D}$ and

$$\limsup_{r \to 1} |f(r)| \cdot |\log(1-r)| < \infty,$$

(ii) v is a normal weight on $\mathbb{D}$ and $f(r)$ is well-defined on $r \in [0,1]$ and is differentiable in 1,

(iii) $v(r) = \exp(-\alpha/(1-r)^\beta)$, $0 \leq r < 1$,

$$\limsup_{r \to 1} |f(r)| \cdot |1-r|^{-1/2-\beta/4} < \infty.$$

T_f is compact on H_v^∞ if

- *v is normal and f satisfies*

$$\lim_{r \to 1} |f(r)| \cdot |\log(1-r)| = 0$$

or

- *$v(r) = \exp(-\alpha/(1-r)^\beta)$ and f satisfies*

$$\lim_{r \to 1} |f(r)| \cdot |1-r|^{-1/2-\beta/4} = 0.$$

Main ingredient of the proof.

Let $h(z) = \sum_{k=0}^{\infty} b_k z^k$ for some $b_k \in \mathbb{C}$. Since f is radial we obtain

$$(T_f(h))(z) = \sum_{k=0}^{\infty} \gamma_k b_k z^k \text{ with } \gamma_k = \frac{\int_0^1 f(r) r^{2k+1} v(r) dr}{\int_0^1 r^{2k+1} v(r) dr},$$

i.e. T_f is a "multiplier".

(ii): We have

$$T_f = T_{f-f(1)} + T_{f(1)} = T_{f-f(1)} + f(1) id_{H_v^\infty}$$

and $\limsup_{r \to 1} |f(r) - f(1)| \cdot |\log(1-r)| < \infty$ (due to differentiability). Hence (ii) follows from (i). $\square$

3 Bergman projections for exponential weights.

Let $v, \tilde{v}$ be standard weights on $\mathbb{D}$. Recall, the Bergman projection $P_{\tilde{v}}$ is the orthogonal projection $L^2_{\tilde{v}} \to H^2_{\tilde{v}}$.

Question. Let $1 \leq p \leq \infty$. Under which condition is $P_{\tilde{v}}$ a bounded projection $L^p_v \to H^p_v$?

Dostanic: Let $v(r) = \tilde{v}(r) = \exp(-\alpha/(1-r)^\beta)$ for some $\alpha, \beta > 0$. Then P_v is bounded on L^p_v if and only if $p = 2$.

Consequence of Theorem 2.3.: For any standard weight v on $\mathbb{D}$ the Bergman projection P_v is unbounded on L^∞_v.

In the following we consider

$$v(r) = \exp\left(-\frac{\alpha}{(1-r)^{\beta}}\right) \text{ and } \tilde{v}(r) = \exp\left(-\frac{\tilde{\alpha}}{(1-r)^{\tilde{\beta}}}\right)$$

for $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} > 0$.

3.1.Theorem (Arroussi, Constantin, Pelaez). *Let $\beta = \tilde{\beta}$.*

(i) Assume that $1 \leq p < \infty$. If $\tilde{\alpha} = 2\alpha/p$ then $P_{\tilde{v}}$ is a bounded projection $L^p_v \to H^p_v$.

(ii) If $\tilde{\alpha} = 2\alpha$ then $P_{\tilde{v}}$ is a bounded projection $L^\infty_v \to H^\infty_v$.

3.2.Theorem (BLT).

(i) If $\tilde{\beta} = \beta$ and $\tilde{\alpha} \neq 2\alpha/p$ then $P_{\tilde{v}}$ is unbounded on L^p_v for all $1 \leq p < \infty$.

(ii) If $\tilde{\beta} = \beta$ and $\tilde{\alpha} \neq 2\alpha$ then $P_{\tilde{v}}$ is unbounded on L^∞_v.

(iii) If $\tilde{\beta} \neq \beta$ then $P_{\tilde{v}}$ is unbounded on L^p_v for all p and all $\alpha, \tilde{\alpha} > 0$.

Sketch of proof of (ii) if $\tilde{\alpha} > \alpha$.

Put $g_k(re^{i\varphi}) = e^{ik\varphi}/v(r)$ for $k \in \mathbb{N}$.
Then $g_k \in L^\infty_v$ and $||g_k||_{v,\infty} = 1$.
A calculation shows

$$||P_{\tilde{v}}(g_k)||_{v,\infty} \geq \int_0^1 r^{k+1} \exp\left(\frac{-\alpha}{(1-r)^\beta}\right) dr \cdot$$

$$\cdot \frac{\int_0^1 r^{k+1} \exp(-(\tilde{\alpha}-\alpha)(1-r)^{-\beta}) dr}{\int_0^1 r^{2k+1} \exp(-\tilde{\alpha}(1-r)^{-\beta}) dr}$$

$$\geq c_1 \exp\Bigg(c_2 k^{\beta/(\beta+1)} \cdot$$

$$\cdot \underbrace{(2^{\beta/(\beta+1)}\tilde{\alpha}^{1/(\beta+1)} - \alpha^{1/(\beta+1)} - (\tilde{\alpha}-\alpha)^{1/(\beta+1)})}_{=:f(\tilde{\alpha})}\Bigg)$$

for constants $c_1, c_2 > 0$. We have $f(\tilde{\alpha}) > 0 \Leftrightarrow \tilde{\alpha} \neq 2\alpha$. So, if $\tilde{\alpha} \neq 2\alpha$ then

$$||P_{\tilde{v}}(g_k)||_{v,\infty} \geq c_1 \exp\left(c_2 f(\tilde{\alpha}) k^{\beta/(\beta+1)}\right) \to \infty \text{ as } k \to \infty.$$

Hence $P_{\tilde{v}}$ is unbounded.

If $\tilde{\alpha} = 2\alpha$ then $f(\tilde{\alpha}) = 0$. Hence $||P_{\tilde{v}}(g_k)||_{v,\infty} \geq c_1$??? But here Theorem 3.1. says that $P_{\tilde{v}}$ is bounded.

Summing up:

- *If* $1 \leq p < \infty$ *and* $\beta = \tilde{\beta}$*:*

  $P_{\tilde{v}}$ *is a bounded projection* $L^p_v \to H^p_v$

  $$\Leftrightarrow \quad \tilde{\alpha} = 2\alpha/p.$$

- *If* $p = \infty$ *and* $\beta = \tilde{\beta}$*:*

  $P_{\tilde{v}}$ *is a bounded projection* $L^\infty_v \to H^\infty_v$

  $$\Leftrightarrow \quad \tilde{\alpha} = 2\alpha.$$

- *If* $\beta \neq \tilde{\beta}$ *then* $P_{\tilde{v}}$ *is never bounded on* L^p_v*.*

Final Remark. The preceding assertions remain true if we replace H^p_v by

$$h^p_v := \{g \in L^p_v : g \text{ harmonic } \}.$$

THANK YOU !