Koopman semigroups and integral operators on Lebesgue spaces

Monatshefte für Mathematik, **206**, (2025).
arXiv: 2511.13868, (2025)

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Departamento de Matemáticas Universidad Zaragoza

Seminar on Analysis, differential equations and mathematica physics, March 26th, 2026

OTHA Research Network in Operator Theory and Harmonic Analysis

Ernesto Cesàro (1859 Naples - 1906 Torre Annunziata)

Cesàro's main contributions are in the field of differential geometry. Lessons of intrinsic geometry, written in 1894, explains in particular the construction of a fractal curve. After that, Cesàro also studied the "snowflake curve" of von Koch, continuous but not differentiable in any of its points.

Among his other works are *Introduction to the mathematical theory of infinitesimal calculus* (1893), *Algebraic analysis* (1894), *Elements of infinitesimal calculus* (1897). He proposed a possible definition of a limit of divergent sequence, known today as "Cesàro's sum," given by the limit of the mean of the sequence partial terms' sum.

Cesàro died in tragic circumstances. His seventeen year old son went swimming in the sea near Torre Annunziata and got into difficulties in rough water. Cesàro went to rescue his son but sustained injuries which led to his death.

TARJETA POSTAL.

A Monsieur Ernest Cesàro
professeur à l'Université
Torre-Annunziata
(Italia) Napoli

En este lado debe escribirse solamente la dirección.

Muy Sr mio y distinguido colega:

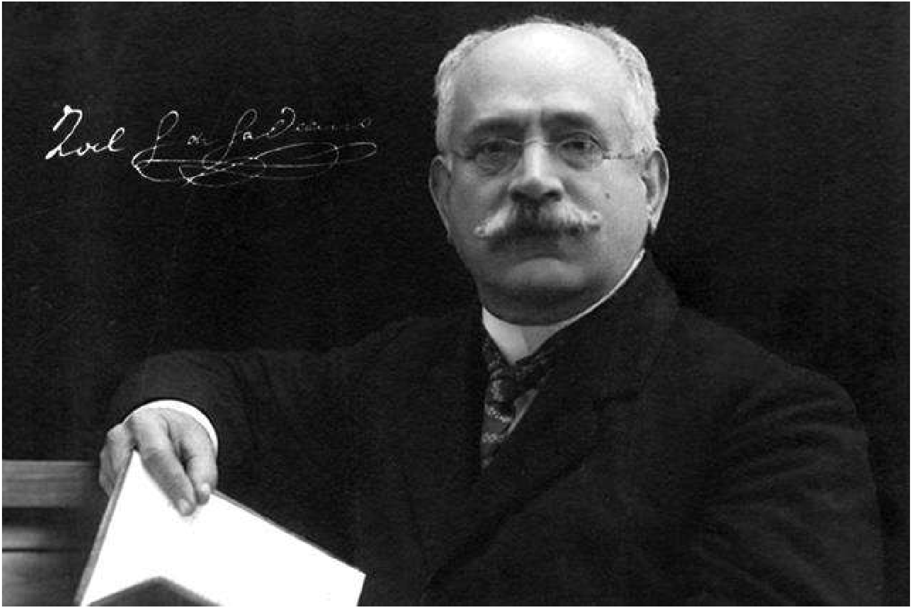
la satisfaccion de recibir su importante

Memoria contestacion de Vd. trat. Asunt. que le agradezco mucho.

A fin de mes le remitiré el nº 97 del Prog. mat.º y si no ha recibido V algun numero por haberse extraviado tendré mucho gusto en remitirle los que le fallen.

Soy de V con la mayor consideracion af.mo servidor y colega Z. G. de Galdeano

Zaragoza 19 Enero 1894



Zoel García de Galdeano (1846, Pamplona - 1924, Zaragoza)

Año III | Zaragoza 15 de Enero de 1893 | Núm. 25

EL PROGRESO MATEMÁTICO

PERIÓDICO DE MATEMÁTICAS PURAS Y APLICADAS

DIRECTOR

Don Zoel G. de Galdeano

CATEDRÁTICO EXCEDENTE

DE LA

UNIVERSIDAD DE ZARAGOZA

PRECIOS DE SUSCRIPCION

En toda España, un año, 10 ptas. Países de la union postal, 11 fr.

Si, pour n croissant à l'infini, a_n et b_n tendent respectivement vers a et b, on a

$$\lim \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1) = ab. \tag{3}$$

On sait d'ailleurs, par un théorème de Cauchy, cas particulier de celui que nous sommes en train de démontrer, que

$$\lim \frac{1}{n}\sum_1^n a_i = a, \qquad \lim \frac{1}{n}\sum_1^n |a_i| = |a|.$$

Conséquemment,

$$\lim \frac{1}{n}[a_1(b_n - b) + a_2(b_{n-1} - b) + \ldots + a_\nu(b_{n-\nu+1} - b)] = 0,$$

d'où

$$\lim \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \ldots + a_\nu b_{n-\nu+1}) = \tfrac{1}{2} ab.$$

E. Cesàro, *Sur la multiplication des series*. Bull. Sci. Math. 14, 114-120 (1890).

The Cesàro operators

$$(f(n))_{n\geq 0} \mapsto C(f)(n) = \frac{1}{n+1}\sum_{k=0}^{n} f(k),$$

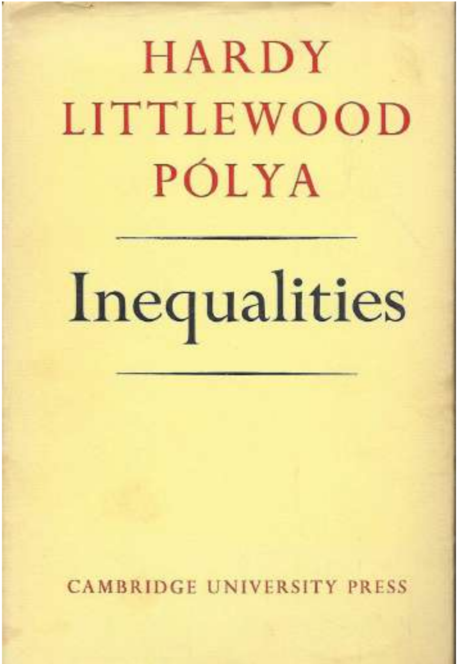
$$(f(z))_{z\in\mathbb{D}} \mapsto C(f)(z) = \frac{1}{z}\int_0^z \frac{f(w)}{1-w}dw, \qquad z\in\mathbb{D},$$

$$(f(t))_{t>0} \mapsto C(f)(t) = \frac{1}{t}\int_0^t f(s)ds, \qquad t>0,$$

A large numbers of mathematicians.... a nice survey W.T. Ross.

A fruitful tool: composition semigroups and Cesàro operators.

Introduction



Introduction

[HLP]

Introduction

[HLP]

327.[a] *If* $p > 1$, $f(x) \geqq 0$, *and* $F(x) = \int_0^x f(t)\,dt$, *then*

(9.8.2) $$\int_0^\infty \left(\frac{F}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p\,dx,$$

unless $f \equiv 0$. *The constant is the best possible.*

Introduction

For $1 < p < \infty$, $f \in L^p(\mathbb{R}^+)$,

$$\left(\int_0^\infty \left| \frac{1}{t} \int_0^t f(s) ds \right|^p dt \right)^{1/p} \leq \frac{p}{p-1} ||f||_p.$$

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The so-called Cesàro transformation $\mathcal{C}$, defined by

$$\mathcal{C}(f)(t) = \frac{1}{t} \int_0^t f(s) ds, \quad t > 0,$$

is a bounded operator on $L^p(\mathbb{R}^+)$ with $||\mathcal{C}|| = \frac{p}{p-1}$ for $1 < p < \infty$.

Introduction

[HLP]

Introduction

[HLP]

329. *If* $p>1$, $r>0$, *and*

(9.9.4) $$f_r(x)=\frac{1}{\Gamma(r)}\int_0^x (x-t)^{r-1}f(t)\,dt,$$

then

(9.9.5) $$\int_0^\infty \left(\frac{f_r}{x^r}\right)^p dx<\left\{\frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(r+1-\frac{1}{p}\right)}\right\}^p\int_0^\infty f^p\,dx,$$

unless $f\equiv 0$. *If*

(9.9.6) $$f_r(x)=\frac{1}{\Gamma(r)}\int_x^\infty (t-x)^{r-1}f(t)\,dt,$$

then

(9.9.7) $$\int_0^\infty f_r^p\,dx<\left\{\frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(r+\frac{1}{p}\right)}\right\}^p\int_0^\infty (x^r f)^p\,dx,$$

unless $f\equiv 0$. *In each case the constant is the best possible.*

Introduction

For $\beta > 0$, $1 < p < \infty$, $f \in L^p(\mathbb{R}^+)$,

$$\left(\int_0^\infty \left|\frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1} f(s)ds\right|^p dt\right)^{1/p} \leq \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p})}{\Gamma(\beta+1-\frac{1}{p})}\|f\|_p,$$

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$$\left(\int_0^\infty \left|\beta\int_x^\infty \frac{(t-x)^{\beta-1}}{t^\beta} f(t)dt\right|^p dx\right)^{\frac{1}{p}} \leq \frac{\Gamma(\beta+1)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\beta+\frac{1}{p}\right)}\|f\|_p.$$

Introduction

The operators $\mathcal{C}_\beta$, $\mathcal{C}_\beta^*$, defined by

$$\mathcal{C}_\beta f(t) := \frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1} f(s)ds, \quad \mathcal{C}_\beta^* f(s) := \beta \int_s^\infty \frac{(t-s)^{\beta-1}}{t^\beta} f(t)dt,$$

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By Fubini theorem, the dual operator of $\mathcal{C}_\beta$ on $L^p(\mathbb{R}^+)$ is $\mathcal{C}_\beta^*$ on $L^{p'}(\mathbb{R}^+)$,

$$\int_0^\infty \mathcal{C}_\beta f(t) g(t)dt = \int_0^\infty f(s)\mathcal{C}_\beta^* g(s)ds, \qquad f \in L^p(\mathbb{R}^+), \quad g \in L^{p'}(\mathbb{R}^+),$$

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where $1 < p, p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Other properties in [Br-Ha-Sh], [Bo], [Mo].

Introduction

J. Math. Anal. Appl. 419 (2014) 373–394

Contents lists available at ScienceDirect

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On the boundedness of generalized Cesàro operators on Sobolev spaces

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[b] *Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain*

[c] *Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile*

Introduction

A B S T R A C T

For $\beta > 0$ and $p \geq 1$, the generalized Cesàro operator

$$\mathscr{C}_\beta f(t) := \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1} f(s) ds$$

and its companion operator $\mathscr{C}_\beta^*$ defined on Sobolev spaces $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ and $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$ (where $\alpha \geq 0$ is the fractional order of derivation and are embedded in $L^p(\mathbb{R}^+)$ and $L^p(\mathbb{R})$ respectively) are studied. We prove that if $p > 1$, then $\mathscr{C}_\beta$ and $\mathscr{C}_\beta^*$ are bounded operators and commute on $\mathscr{T}_p^{(\alpha)}(t^\alpha)$ and $\mathscr{T}_p^{(\alpha)}(|t|^\alpha)$. We calculate explicitly their spectra $\sigma(\mathscr{C}_\beta)$ and $\sigma(\mathscr{C}_\beta^*)$ and their operator norms (which depend on p). For $1 < p \leq 2$, we prove that $\widehat{\mathscr{C}_\beta(f)} = \mathscr{C}_\beta^*(\widehat{f})$ and $\widehat{\mathscr{C}_\beta^*(f)} = \mathscr{C}_\beta(\widehat{f})$ where $\widehat{f}$ denotes the Fourier transform of a function $f \in L^p(\mathbb{R})$.

Introduction

X Banach space, a C_0-semigroups,, $(T(t))_{t\geq 0} \subset \mathcal{B}(X)$, $u(t) = T(t)x = e^{tA}x$, are solutions of the Cauchy problem

$$(T(t))_{t>0} \mapsto \begin{array}{rcll} u'(t) & = & Au(t), & t > 0, \\ u(0) & = & x. & \end{array}$$

Theorem

For $1 \leq p$ the family of operators $(T_{t,p})_{t\in\mathbb{R}}$ defined by

$$T_{t,p}f(s) := e^{-\frac{t}{p}}f(e^{-t}s), \qquad f \in L^p(\mathbb{R}^+),$$

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$$(\Lambda f)(s) := -sf'(s) - \frac{1}{p} f(s),$$

with domain $D(\Lambda) := \{f \in L^p(\mathbb{R}^+) \ : \ tf' \in L^p(\mathbb{R}^+)\}$.

Introduction

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The operator $\mathcal{C}_\beta$ is a bounded operator on $L^p(\mathbb{R}^+)$ and

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$$B(u,v) = \int_0^1 r^{u-1}(1-r)^{v-1} dr = \int_0^\infty (1-e^{-s})^{u-1} e^{-vs} ds$$

Introduction

Theorem

Let $1 < p < \infty$, and $\mathcal{C}_\beta : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ the generalized Cesàro operator. Then

$$\sigma(\mathcal{C}_\beta) = \Gamma(\beta+1)\overline{\left\{ \frac{\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)} \ : \ t \in \mathbb{R} \right\}}$$

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$p > 1$ and $\beta > 0$.
If $f \in L^p(\mathbb{R}^+)$, then

$$\mathcal{C}_\beta^* f(t) = \beta \int_{-\infty}^{0} (e^{-r}-1)^{\beta-1} e^{-r(1-1/p-\beta)} T_{r,p} f(t) dr, \quad t \geq 0.$$

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The dual operator of $\mathcal{C}_\beta$ on $L^p(\mathbb{R}^+)$ is $\mathcal{C}_\beta^*$ on $L^p(\mathbb{R}^+)$, where $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\langle \mathcal{C}_\beta f, g\rangle_\alpha = \langle f, \mathcal{C}_\beta^* g\rangle_\alpha, \qquad f \in L^p(\mathbb{R}^+), \quad g \in L^p(\mathbb{R}^+).$$

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Let $\beta > 0$, $1 \le p < \infty$, and $\mathcal{C}_\beta^* : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ the generalized dual Cesàro operator. Then

$$\sigma(\mathcal{C}_\beta^*) = \Gamma(\beta+1)\overline{\left\{ \frac{\Gamma(\frac{1}{p}+it)}{\Gamma(\beta+\frac{1}{p}+it)} \ : \ t \in \mathbb{R} \right\}}$$

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Composition semigroups on $\mathbb{D}$: A wonderful review [Si]

TABLE 1. Examples of semigroups

	$G(z)$	$h(z)$	$\phi_t(z)$
$b=0$	$-zc,\ \Re c \geq 0$	z	$e^{-ct}z$
	$-z(1-z)$	$\frac{z}{1-z}$	$\frac{e^{-t}z}{(e^{-t}-1)z+1}$
	$-(1-z)\log\frac{1}{1-z}$	$\log\frac{1}{1-z}$	$1-(1-z)^{e^{-t}}$
	$-z(1-z^n)$	$\frac{z}{(1-z^n)^{\frac{1}{n}}}$	$\frac{e^{-t}z}{((e^{-nt}-1)z^n+1)^{\frac{1}{n}}}$
	$-\frac{1-z^2}{2}\log\frac{1+z}{1-z}$	$\log\frac{1+z}{1-z}$	$\frac{(1+z)^{e^{-t}}-(1-z)^{e^{-t}}}{(1+z)^{e^{-t}}+(1-z)^{e^{-t}}}$
$b=1$	$1-z$	$\log\frac{1}{1-z}$	$e^{-t}z+1-e^{-t}$
	$c(1-z)^2,\ \Re c \geq 0$	$\frac{z}{1-z}$	$\frac{(1-ct)z-1+ct}{-ctz+1+ct}$
	$\frac{1}{2}(1-z^2)$	$\frac{1}{2}\log\frac{1+z}{1-z}$	$\frac{(1+e^t)z-1+e^t}{(-1+e^t)z+1+e^t}$
	$\frac{(1-z)^\alpha}{1+\alpha},\ \alpha \in (-1,1]$	$\frac{1}{1+\alpha}\left((1-z)^{-\alpha-1}-1\right)$	$1-\left((1-z)^{-\alpha-1}+t\right)^{-\frac{1}{\alpha+1}}$

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For $t > 0$, we define the maps (semiflows) in the unit disc

$$\begin{aligned} b = 0, \qquad \phi_t(z) : &= e^{-ct} z, \quad (\textit{elliptic}) \\ \phi_t(z) : &= \frac{e^{-t} z}{z(e^{-t} - 1) + 1}, \\ b = 1, \qquad \phi_t(z) : &= e^{-t} z + 1 - e^{-t}, \quad (\textit{parabolic}) \\ \psi_t(z) : &= \frac{(1 + e^t) z - 1 + e^t}{(-1 + e^t) z + 1 + e^t}, \quad (\textit{hyperbolic}). \end{aligned}$$

(In Ph. Thesisś J. Oliva-Maza, [AGMO1], [AGMO2]).

$$\textit{QUESTION}: \qquad z \in \mathbb{D} \mapsto r > 0$$

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References

Monatshefte für Mathematik (2025) 206:629–664
https://doi.org/10.1007/s00605-024-02041-2

Three weight Koopman semigroups on Lebesgue spaces

Pedro J. Miana[1] · **Verónica Poblete**[2]

Received: 28 April 2024 / Accepted: 21 November 2024 / Published online: 7 February 2025

Abstract

In this paper, we consider three different semiflows $(\phi_t)_{t\geq 0}$, $(\psi_t)_{t\geq 0}$ and $(\varphi_t)_{t\geq 0}$ on the real half-line given by

$$\phi_t(r) := e^{-t}r + 1 - e^{-t},\ \psi_t(r) := \frac{e^t r}{1 + r(e^t - 1)},\ \varphi_t(r) := \frac{(1 + e^t)r - 1 + e^t}{(-1 + e^t)r + 1 + e^t},$$

(...) They argue that in spite of the success of holomorphic semiflows, real semi-flows have not considered so for. But this is not true. In the Springer LN edited by R. Nagel: One parameter-semigroups of positive operators, in Part B, a whole chapter is devoted to such semigroup and further literature is given in the notes. Moreover, much has been done under the notion of Koopman operators, another name for composition operators. But this does not matter at all, since the results of the paper under review are centainly new. (...)

1. Semiflows in the real half-line

For $t > 0$, we define the maps in the real half-line

$$\begin{aligned} \phi_t(r) : &= e^{-t} r + 1 - e^{-t}, \\ \psi_t(r) : &= \frac{e^t r}{1 + r(e^t - 1)}, \\ \varphi_t(r) : &= \frac{(1 + e^t) r - 1 + e^t}{(-1 + e^t) r + 1 + e^t}, \qquad r > 0. \end{aligned}$$

Note

$$\lim_{t \to +\infty} \phi_t(r) = \lim_{t \to +\infty} \psi_t(r) = \lim_{t \to +\infty} \varphi_t(r) = 1, \qquad r \in [0, \infty).$$

i.e. the value $\{1\}$ is an attractive fixed point of these mappings and

$$e^{-t} = \phi'_t(1) = \psi'_t(r) = \varphi'_t(1) = 1, \qquad t \in [0, \infty).$$

These semigroups (semiflows) have more fixed points

$$\lim_{r \to +\infty} \phi_t(r) = +\infty, \qquad \psi_t(0) = 0, \qquad \text{and} \qquad \varphi_t(-1) = -1, \ t \geq 0.$$

1. Semiflows in the real half-line

Proposition

Given the inner maps $\phi_t, \psi_t, \varphi_t$, $\mathbb{R}^+ \to \mathbb{R}^+$ for $t > 0$. Then

(i) $\phi_t(\phi_s) = \phi_{t+s}$, $\psi_t(\psi_s) = \psi_{t+s}$, $\varphi_t(\varphi_s) = \varphi_{t+s}$ for $t, s > 0$.

(ii)
$$\phi_t(\frac{1}{r}) = \frac{1}{\psi_t(r)}, \qquad r > 0.$$

(iii) $\phi_t([0,1]) = [1 - e^{-t}, 1]$ and $\phi_t([1,\infty]) = [1,\infty)$.

(iv) $\psi_t(\infty) := \frac{e^t}{e^t - 1}$. Then $\psi_t([0,1]) = [0,1]$ and $\psi_t([1,\infty]) = [1, \psi_t(\infty))$.

(v) $\varphi_t(\infty) := \frac{e^t+1}{e^t-1}$. Then $\varphi_t([0,1]) = [\frac{1}{\varphi_t(\infty)}, 1]$ and $\varphi_t([1,\infty)) = [1, \varphi_t(\infty))$.

We consider the weight $\omega_\xi(r) := \dfrac{1-\xi_(r)}{1-r}$. Note that $\omega_{\phi_t}(r) = e^{-t}$,

$$\omega_{\psi_t}(r) = \frac{1}{1+r(e^t-1)}, \qquad \omega_{\varphi_t}(r) = \frac{2}{e^t+1+r(e^t-1)}.$$

$(\omega_t)_{t>0}$ is a semicocycles if $\omega_0 = 1$ and $\omega_{t+s} = \omega_t(\omega_s \circ \xi_t)$.

Lemma

Take $f \in C^{(n)}(\mathbb{R}^+)$ $n \in \mathbb{N}$ and $m_1 + 2m_2 + \cdots + nm_n = n$ and $k = m_1 + \cdots + m_n$. Then

$$\frac{d^n}{dr^n}(f(\phi_t(r))) = e^{-nt} f^{(n)}(\phi_t)(r),$$

$$\frac{d^n}{dr^n}(f(\psi_t(r))) =$$

$$\omega^n_{\psi_t}(r)(1-e^t)^n \sum \frac{n!}{m_1! \cdots m_n!} f^{(k)}(\psi_t(r))\omega^k_{\psi_t}(r)\left(\frac{e^t}{1-e^t}\right)^k,$$

$$\frac{d^n}{dr^n}(f(\varphi_t(r))) =$$

$$\omega^n_{\varphi_t}(r)\left(\frac{1-e^t}{2}\right)^n \sum \frac{n!}{m_1! \cdots m_n!} f^{(k)}(\varphi_t(r))\,\omega^k_{\varphi_t}(r)\left(\frac{2e^t}{1-e^t}\right)^k.$$

2. Fractional Sobolev spaces defined on $\mathbb{R}^+$

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The *Weyl fractional integral* of order $\alpha > 0$ is defined by

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$$W_+^{\alpha} f(t) := (-1)^n \frac{d^n}{dt^n} W_+^{-(n-\alpha)} f(t), \quad t \in \mathbb{R}^+,$$

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More details in `[Mi-Ro]` and `[Sa-Ki-Ma]`.

Lemma

Take $\alpha \in \mathbb{R}$, $t > 0$ and $f \in \mathcal{S}_+$. Then

(i) *For $\alpha \in \mathbb{R}$ and $t > 0$,*

$$W_+^{\alpha}(f(\phi_t))(r) = e^{-t\alpha}\left(W_+^{\alpha} f\right)(\phi_t(r)), \qquad r > 0.$$

(ii) *For $\alpha > 0$, $\gamma > \alpha$, and $t > 0$,*

$$W_+^{-\alpha}(\omega_t^{\gamma} f(\psi_t))(r) = \frac{e^{-t\alpha}}{(\psi_t(\infty))^{\gamma-\alpha-1}\omega_t^{\alpha-1}(r)} \times$$

$$\int_{\psi_t(r)}^{\psi_t(\infty)} \frac{(u - \psi_t(r))^{\alpha-1}(\psi_t(\infty) - u)^{\gamma-\alpha-1}}{\Gamma(\alpha)} f(u)du$$

Definition

For $\alpha > 0$ let be the Banach space $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ defined as the completion of the Schwartz class $\mathcal{S}_+$ in the norm

$$||f||_{\alpha,p} := \frac{1}{\Gamma(\alpha+1)} \left(\int_0^\infty |W_+^\alpha f(t)|^p t^{\alpha p} dt \right)^{\frac{1}{p}}.$$

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For $\alpha > 0$ in [Ga-Mi].

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- The operator $D_+^\alpha : \mathcal{T}_p^{(\alpha)}(t^\alpha) \to L^p(\mathbb{R}^+)$ is an isometry, where

$$D_+^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)} t^\alpha W_+^\alpha f(t).$$

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$$D_+^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)} t^\alpha W_+^\alpha f(t).$$

- If $p > 1$ and p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then $(\mathcal{T}_p^{(\alpha)}(t^\alpha))' = \mathcal{T}_{p'}^{(\alpha)}(t^\alpha)$,

$$\langle f, g \rangle_\alpha = \frac{1}{\Gamma(\alpha+1)^2} \int_0^\infty W_+^\alpha f(t) W_+^\alpha g(t) t^{2\alpha} dt.$$

In fact, $\qquad \|f\|_{\alpha,p} = \|D_+^\alpha f\|_p, \qquad \langle f, g \rangle_\alpha = \langle D_+^\alpha f, D_+^\alpha g \rangle_0.$

Examples

(i) Functions $\dfrac{(1-t)^{c-1}}{\Gamma(c)}\chi_{(0,1)}(t) \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ if and only if $c > \alpha + 1 - \frac{1}{p}$.

(ii) $t^\beta(1-t)^c\chi_{(0,1)}(t) \in \mathcal{T}_p^{(m)}(t^m)$ for $\Re\beta > \frac{-1}{p}$ and $\Re c > m - \frac{1}{p}$.

(iii) $\dfrac{(t-1)^a}{t^b}\chi_{(1,\infty)}(t) \in \mathcal{T}_p^{(m)}(t^m)$ for $\Re a > m - \frac{1}{p}$ and $\Re b > \Re a + \frac{1}{p}$.

We introduce in the next definition a decomposition of $\mathcal{T}_p^{(\alpha)}(t^\alpha)$.

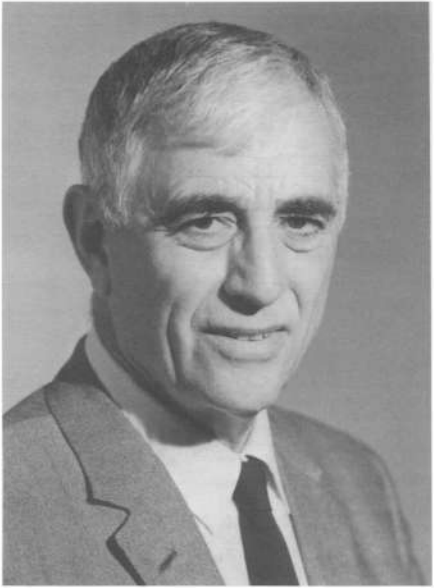
Definition

Given $\alpha \geq 0$, $p \geq$ and the Banach space $\mathcal{T}_p^{(\alpha)}(t^\alpha)$. We introduce the following subspaces

$$\begin{aligned}\mathcal{T}_0 &:= \{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \mid \text{supp}(f) \subset [0,1]\},\\ \mathcal{T}_1 &:= \{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \mid \text{supp}(f) \subset [1,\infty)\}.\end{aligned}$$

The subspaces $\mathcal{T}_0$ and $\mathcal{T}_1$ are closed and $\mathcal{T}_p^{(\alpha)}(t^\alpha) = \mathcal{T}_0 + \mathcal{T}_1$.

Bernard O. Koopman (1900-1981)



Bernard Osgood Koopman

Bernard O. Koopman (1900-1981)

A French-born American mathematician, known for his work in ergodic theory, the foundations of probability, statistical theory and operations research.

(-) Son of Augustus Koopman (painter) and Louise Osgood, cousin of Harvard mathematician William F. Osgood. William F. Osgood. His father died in 1914. Come back to New England.

(-) In 1922 he graduated, summa cum laude, highest honors in mathematics in Harvard with Marshall Stone.
We became very close friends, studying together and enjoying various social activities. Koopman had a special interest in dynamics and mathematical physics. His dissertation was written under George D. Birkhoff and was devoted to dynamics. My interests did not include classical dynamics, but in the fields of differential equations and Fourier analyses we were both interested and took some courses together.

Bernard O. Koopman (1900-1981)

In March of 1931, Koopman published a note in the National Academy Proceedings, transforming the problem into one dealing with one parameter unitary groups in Hilbert space. Since these groups may be represented by self-adjoint transformations and since they were known to have a particularly decent structure, the door was open to rapid extension. Koopman communicated his ideas to von Neumann, who, in a short time, gave a proof of the ergodic theorem in a Hilbert space sense, establishing convergence in the mean but not actual convergence. In a state of considerable excitement Koopman told von Neumann's result to Birkhoff, who worked feverishly and succeeded in proving the theorem, establishing point-wise convergence almost everywhere.

Bernard O. Koopman (1900-1981)

VOL. 17, 1931 *MATHEMATICS: B. O. KOOPMAN* 315

HAMILTONIAN SYSTEMS AND TRANSFORMATIONS IN HILBERT SPACE

BY B. O. KOOPMAN

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

Communicated March 23, 1931

318 *MATHEMATICS: B. O. KOOPMAN* PROC. N. A. S.

$$U_t F(\varphi_1, \varphi_2, \ldots) = F(U_t\varphi_1, U_t\varphi_2, \ldots),$$

an equation which plays an important part in the developments of our theory.

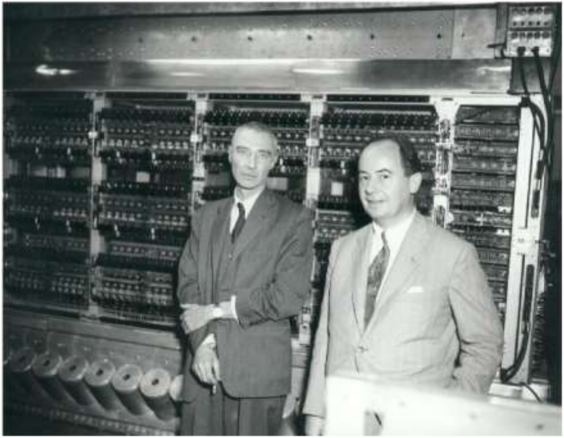
Bernard O. Koopman (1900-1981)

DYNAMICAL SYSTEMS OF CONTINUOUS SPECTRA

By B. O. Koopman and J. v. Neumann

Departments of Mathematics, Columbia University and Princeton University

Communicated January 21, 1932



Koopman operators and semigroups

(i) In [LNM 1184, 1986] a general study about positive semigroups in lattice spaces ($C(K)$ and $C_0(X)$ where K is compact and X locally compact) is developed. A characterization of continuous flows on K is given in terms of semigroups of composition operators and its infinitesimal generators on $C(K)$.

(ii) For a measure space $\mathbf{X} = (X, \Sigma, \mu)$ and $1 \leq p < \infty$, a operator $T : L^p(\mathbf{X}) \to L^p(\mathbf{X})$ is called a Koopman operator if there exists a measurable function $\phi : X \to X$ such that ϕ^{-1} maps null-sets into null-sets and $Tf = f \circ \phi$ for all $f \in L^p(\mathbf{X})$. A nice treatment of Koopman operators (with applications to ergodicity) in $L^p(\mathbf{X})$ may be found in [EFHN, 2015] for measure-preserving system (X, ϕ).

Koopman operators and semigroups

(iii) A Koopman semigroup $(T(t))_{t\geq 0}$ is a C_0-semigroup of operators on $L^p(\mathbf{X})$ such that for each $t \geq 0$ the operator $T(t)$ is a Koopman operator. For a finite measure space, a characterization of Koopman semigroups on $L^p(\mathbf{X})$ is presented in [EGK, 2019] for measure-preserving semiflows. An equivalence result between measure-preserving flows on standard probability spaces and continuous flow on compact Borel probability space in also shown in [EGK].

(iv) The continuous Denjoy-Wolff theorem: if $(\zeta_t)_{t\geq 0}$ is a semigroup (not hyperbolic rotations) of holomorphic self-maps on the unit disc, then there exists a unique point $\tau \in \overline{\mathbb{D}}$ such that

$$\lim_{t\to+\infty} \zeta_t(z) = \tau, \qquad z \in \mathbb{D}.$$

The point τ is called the Denjoy-Wolff point of $(\zeta_t)_{t\geq 0}$. However a similar result of Denjoy-Wolff theorem for arbitrary semiflows in Lebesgue space is unknown.

3. Koopman semigroups in the fractional Sobolev spaces

Definition

Given $p \geq 1$ and a function $f \in \mathcal{S}_+$, we define functions $T_{t,p}f$, $S^{\gamma}_{t,p}f$ and $R^{\gamma}_{t,p}f$ by

$$
\begin{aligned}
T_{t,p}f(r) &:= e^{\frac{-t}{p}} f(\phi_t(r)) = e^{\frac{-t}{p}} f(e^{-t}r + 1 - e^{-t}),\\
S^{\gamma}_{t,p}f(r) &:= e^{\frac{t}{p}} \omega^{\gamma}_{\psi_t}(r) f(\psi_t(r))\\
&= \frac{e^{\frac{t}{p}}}{(1 + r(e^t - 1))^{\gamma}} f\left(\frac{e^t r}{1 + r(e^t - 1)}\right),\\
R^{\gamma}_{t,p}f(r) &:= e^{t(\gamma - \frac{1}{p})} \omega^{\gamma}_{\varphi_t}(r) f(\varphi_t(r))\\
&= \frac{2^{\gamma}\, e^{t(\gamma - \frac{1}{p})}}{(e^t + 1 + r(e^t - 1))^{\gamma}}\, f\left(\frac{(1 + e^t)r + e^t - 1}{e^t + 1 + r(e^t - 1)}\right),
\end{aligned}
$$

and $t \geq 0$. Note that $T_{0,p}f = S^{\gamma}_{0,p}f = R^{\gamma}_{0,p}f = f$.

3.1 The Koopman semigroup $(T_{t,p})_{t>0}$

Theorem

Take $p \geq 1$, $\alpha \geq 0$ and $t \geq 0$. Then

$$\|T_{t,p}f\|_{\alpha,p} \leq \|f\|_{\alpha,p}, \qquad f \in \mathcal{T}_p^{(\alpha)}(t^\alpha).$$

Moreover subspaces $\mathcal{T}_i$ are $(T_{t,p})_{t>0}$-invariant, i.e., $T_{t,p}(\mathcal{T}_i) \subset \mathcal{T}_i$ for $i \in \{0,1\}$ and

$$\|T_{t,p}f\|_{\alpha,p} \leq e^{-t\alpha}\|f\|_{\alpha,p}, \qquad f \in \mathcal{T}_0, \quad t > 0.$$

The restriction of $(T_{t,p})_{t\geq 0}$ on the subspace $\mathcal{T}_1$ are extended for $t \in \mathbb{R}$ and

$$\|T_{t,p}f\|_{\alpha,p} \leq \max\{e^{-t\alpha}, 1\}\|f\|_{\alpha,p}, \qquad f \in \mathcal{T}_1, \quad t \in \mathbb{R}.$$

Theorem

For $p \geq 1$ and $\alpha \geq \mu \geq 0$, the family of operators $(T_{t,p})_{t \geq 0}$ is a contractive C_0-semigroup on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ whose infinitesimal generator A is given by

$$(Af)(s) := (1-s)f'(s) - \frac{1}{p}f(s),$$

with domain $D(A) = \{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \mid ((1-s)f)' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)\}$, and

$$\{f \in \mathcal{T}_p^{(\alpha)}(t^{\alpha+1}) \mid f' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)\} \subset D(A).$$

Theorem

For $1 \leq p < \infty$ *we have*

(i) $\{z \in \mathbb{C} \mid \Re(z) < -\alpha\} \subset \sigma_p(A)$.

(ii) *The set* $\rho(A) \subset \mathbb{C}^+$ *and*

$$R(\lambda, A)f(r) = \frac{1}{|r-1|^{\lambda+\frac{1}{p}}} \int_{\Gamma_{1,r}} |s-1|^{\lambda+\frac{1}{p}-1} f(s)ds, \quad f \in \mathcal{T}_p^{(\alpha)}(t^\alpha),$$

for $\lambda \in \mathbb{C}^+$ *and* $\Gamma_{1,r} := (1, r)$ *when* $r > 1$ *and* $\Gamma_{1,r} := (r, 1)$ *in the case* $0 < r < 1$.

(iii) $\sigma(A) = \{z \in \mathbb{C} \mid \Re(z) \leq 0\}$.

3.2 The Koopman semigroup $(S^{\gamma}_{t,p})_{t>0}$

Theorem

Take $p \geq 1$, $m \in \mathbb{N} \cup \{0\}$, $\gamma \geq \frac{2}{p}$ and $t \geq 0$. Then

$$\|S^{\gamma}_{t,p} f\|_{m,p} \leq C_m \|f\|_{(m),p}, \qquad f \in \mathcal{T}^{(m)}_p(t^m).$$

Moreover subspaces $\mathcal{T}_i$ are $(S^{\gamma}_{t,p})_{t>0}$-invariant, i.e., $S^{\gamma}_{t,p}(\mathcal{T}_i) \subset \mathcal{T}_i$ for $i \in \{0,1\}$. The restriction of $(S^{\gamma}_{t,p})_{t\geq 0}$ on the subspace $\mathcal{T}_0$ is extended for $t \in \mathbb{R}$ and

$$\|S^{\gamma}_{t,p} f\|_{m,p} \leq C_m \max\{e^{-t(m+\gamma-\frac{2}{p})}, 1\} \|f\|_{m,p}, \qquad f \in \mathcal{T}_0, \quad t \in \mathbb{R}.$$

The restriction of $(S^{\gamma}_{t,p})_{t\geq 0}$ on the subspace $\mathcal{T}_1$ holds

$$\|S^{\gamma}_{t,p} f\|_{m,p} \leq C_m e^{-t(\gamma-\frac{2}{p})} \|f\|_{m,p}, \qquad f \in \mathcal{T}_1, \quad t > 0,$$

where the constant C_m is independent on t and f.

Theorem

Take $p \geq 1$, $m \in \mathbb{N} \cup \{0\}$, $p \geq \frac{2}{\gamma}$ and $t \geq 0$. Then the family of operators $(S^{\gamma}_{t,p})_{t \geq 0}$ is a contractive C_0-semigroup on $\mathcal{T}_p^{(m)}(t^m)$ whose infinitesimal generator B is given by

$$(Bf)(r) := r(1-r)f'(r) + (\frac{1}{p} - \gamma r)f(r), \qquad r > 0,$$

with domain
$D(B) = \{f \in \mathcal{T}_p^{(m)}(t^m) \, |(r(1-r)f)' - (\gamma - 2)rf \in \mathcal{T}_p^{(m)}(t^m)\}$, *and*

$$\{f \in \mathcal{T}_p^{(m+1)}(t^{m+1}) \, | rf, r^2 f' \in \mathcal{T}_p^{(m)}(t^m)\} \subset D(B).$$

Theorem

Take $p \geq 1$, $m \in \mathbb{N} \cup \{0\}$, $p \geq \frac{2}{\gamma}$ and the C_0-semigroup $(S^{\gamma}_{t,p})_{t\geq 0}$ whose infinitesimal generator is $(B, D(B))$. Then

(i) $\{z \in \mathbb{C} \,|\Re(z) < -(m + \gamma - \frac{2}{p})\} \subset \sigma_p(B)$.

(ii) *The set* $\rho(B) \subset \mathbb{C}^+$ *and*

$$R(\lambda, B)f(r) = \frac{r^{\lambda-\frac{1}{p}}}{|r-1|^{\lambda+\gamma-\frac{1}{p}}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\lambda+\gamma-\frac{1}{p}-1}}{s^{\lambda-\frac{1}{p}+1}} f(s)ds, \quad r > 0,$$

for $\lambda \in \mathbb{C}^+$ *and* $\Gamma_{1,r} = (1, r)$ *when* $r > 1$ *and* $\Gamma_{1,r} = (r, 1)$ *in the case* $0 < r < 1$.

(iii) $\sigma(B) = \{z \in \mathbb{C} \,|\Re(z) \leq 0\}$.

3.3 The Koopman semigroup $(R^{\gamma}_{t,p})_{t>0}$

Theorem

Take $p \geq 1, m \in \mathbb{N} \cup \{0\}$, $\gamma \geq \frac{2}{p}$ and $t \geq 0$. Then for $f \in \mathcal{T}_p^{(m)}(t^m)$

$$\|R^{\gamma}_{t,p} f\|_{m,p} \leq C_m \|f\|_{m,p}.$$

Theorem

Take $p \geq 1$ and $\gamma \geq \frac{2}{p}$. Then the family of operators $(R^{\gamma}_{t,p})_{t\geq 0}$ is a C_0-semigroup on $\mathcal{T}_p^{(m)}(t^m)$ whose infinitesimal generator C is given by

$$(Cf)(r) := \frac{1}{2}\left(\gamma - \frac{2}{p} - \gamma r\right) f(r) + \frac{1}{2}(1-r^2)f'(r)$$

with domain
$D(C) = \{f \in \mathcal{T}_p^{(m)}(t^m) \,|\, ((1-r^2)f)' - (\gamma-2)rf \in \mathcal{T}_p^{(m)}(t^m)\}$,
and $\{f \in \mathcal{T}_p^{(m+1)}(t^{m+1}) \,|\, f' \in \mathcal{T}_p^{(m)}(t^m))\} \subset D(C)$.

Theorem

For $1 \leq p < \infty$, $m \in \mathbb{N} \cup \{0\}$, $p \geq \frac{2}{\gamma}$ *and the* C_0*-semigroup* $(R^{\gamma}_{t,p})_{t\geq 0}$ *whose infinitesimal generator is* $(C, D(C))$. *Then*

(i) $\{z \in \mathbb{C} \,|\, \Re z < -m\} \subset \sigma_p(C)$.

(ii) $\sigma(C) = \{z \in \mathbb{C} \,|\, \Re(z) \leq 0\}$.

(iii) *The set* $\rho(C) \subset \mathbb{C}^+$ *and* $f \in \mathcal{T}_p^{(m)}(t^m)$,

$$R(\lambda, C)f(r) = 2\frac{|r+1|^{\lambda-\gamma+\frac{1}{p}}}{|r-1|^{\lambda+\frac{1}{p}}}\int_{\Gamma_{1,r}} |s+1|^{-\lambda+\gamma-\frac{1}{p}-1}|s-1|^{\lambda+\frac{1}{p}-1} f(s)ds,$$

for $r > 0$, $\lambda \in \mathbb{C}^+$ *and* $\Gamma_{1,r} = (1, r)$ *if* $r > 1$ *and* $\Gamma_{1,r} = (r, 1)$ *when* $0 < r < 1$.

C_0-semigroups, $u(t) = T(t)x = e^{tA}x$, are solutions of the Cauchy problem

$$\begin{aligned} (T(t))_{t>0} \mapsto u'(t) &= Au(t), \qquad t > 0, \\ u(0) &= x. \end{aligned}$$

Differential equations

$$\begin{aligned} (T_{t,p})_{t>0} \mapsto \frac{\partial u(t,r)}{\partial t} &= (1-r)\frac{\partial u(t,r)}{\partial r} - \frac{1}{p}u(t,r), \\ (S^{\gamma}_{t,p})_{t>0} \mapsto \frac{\partial v(t,r)}{\partial t} &= r(1-r)\frac{\partial v(t,r)}{\partial r} + (\frac{1}{p} - \gamma r)v(t,r), \\ (R^{\gamma}_{t,p})_{t>0} \mapsto 2\frac{\partial w(t,r)}{\partial t} &= (1-r^2)\frac{\partial w(t,r)}{\partial r} + (\gamma - \frac{2}{p} - \gamma r)w(t,r), \end{aligned}$$

4. Cesàro-like operators subordinated to composition semigroups

Given $\mu, \nu \in \mathbb{R}$, we consider the integral operators

$$\mathcal{C}_{\mu,\nu} f(r) := \frac{1}{|r-1|^{\mu+\nu-1}} \int_{\Gamma_{1,r}} |s-1|^{\mu-1} |r-s|^{\nu-1} f(s) ds,$$

$$\mathfrak{C}^{\gamma}_{\mu,\nu} f(r) := \frac{r^{\mu}}{|r-1|^{\mu+\nu+\gamma-1}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\mu+\gamma-1}}{s^{\mu+\nu}} |r-s|^{\nu-1} f(s) ds,$$

$$\mathbf{C}^{\gamma}_{\mu,\nu} f(r) := \quad 2^{\nu} \frac{|r+1|^{\mu-\gamma}}{|r-1|^{\mu+\nu-1}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\mu-1}}{|s+1|^{\mu+\nu-\gamma}} |r-s|^{\nu-1} f(s) ds,$$

for $r > 0$ whenever this integral converges and $\Gamma_{1,r} := (1, r)$ for $r > 1$ and $\Gamma_{1,r} := (r, 1)$ in the case $0 < r < 1$.

4.1 Cesàro-like operators subordinated to $(T_{t,p})_{t>0}$

Theorem

For $p \geq 1$, $\mu, \nu > 0$,

(i) *If* $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, *then*

$$\mathcal{C}_{\mu+\frac{1}{p},\nu} f(r) = \int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1} T_{t,p} f(r)\,dt, \quad r \geq 0.$$

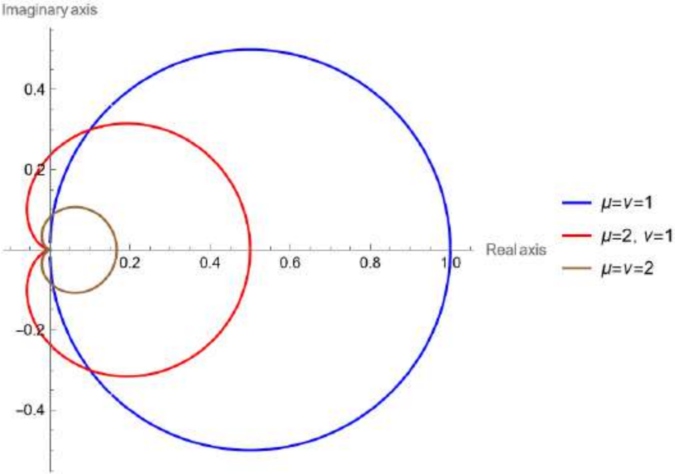
(ii) *the operator* $\mathcal{C}_{\mu+\frac{1}{p},\nu}$ *is a bounded operator on* $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ *for* $\alpha \geq 0$ *and*

$$B(\mu+\alpha,\nu) \leq ||\mathcal{C}_{\mu+\frac{1}{p},\nu}|| \leq B(\mu,\nu).$$

For $\alpha = 0$, *on* $L^p(\mathbb{R}^+)$, *the equality* $||\mathcal{C}_{\mu+\frac{1}{p},\nu}|| = B(\mu,\nu)$ *holds.*

(iii)

$$\sigma(\mathcal{C}_{\mu+\frac{1}{p},\nu}) = \overline{\left\{ B(\mu+z,\nu) \,:\, \Re z \geq 0 \right\}}.$$



Drawings of $\sigma(\mathcal{C}_{1+\frac{1}{p},1})$, $\sigma(\mathcal{C}_{2+\frac{1}{p},1})$ and $\sigma(\mathcal{C}_{2+\frac{1}{p},2})$.

4.2 Cesàro-like operators subordinated to $(S^{\gamma}_{t,p})_{t>0}$

Theorem

Take $p \geq 1$, $\mu, \nu > 0$, $\gamma \geq \frac{2}{p}$ and $f \in \mathcal{T}_p^{(m)}(t^m)$.

(i)

$$\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu} f(r) = \int_0^{\infty} e^{-\mu t}(1-e^{-t})^{\nu-1} S^{\gamma}_{t,p} f(r)dt, \quad r \geq 0,$$

(ii) *The operator $\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}$ is a bounded operator on $\mathcal{T}_p^{(m)}(t^m)$ and*

$$B(\mu + m + \gamma - \frac{2}{p}, \nu) \leq ||\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}|| \leq \sup_{t\geq 0} \|S^{\gamma}_{t,p}\| B(\mu, \nu).$$

On $L^p(\mathbb{R}^+)$, and $\gamma = \frac{2}{p}$, the equality $||\mathfrak{C}^{\frac{2}{p}}_{\mu-\frac{1}{p},\nu}|| = B(\mu, \nu)$ holds.

(iii) $\sigma(\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}) = \overline{\left\{ B(\mu + z, \nu) \,:\, \Re z \geq 0 \right\}}.$

4.3 Cesàro-like operators subordinated to $(R^{\gamma}_{t,p})_{t>0}$

Theorem

For $p \geq 1$, $\mu, \nu > 0$ and $\gamma \geq \frac{2}{p}$. If $f \in \mathcal{T}_p^{(m)}(t^m)$ then

(i)

$$\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu} f(r) = \int_0^{\infty} e^{-\mu t}(1-e^{-t})^{\nu-1} R^{\gamma}_{t,p} f(r)\,dt, \quad r \geq 0,$$

(ii) *The operator $\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu}$ is a bounded operator on $\mathcal{T}_p^{(m)}(t^m)$ and*

$$B(\mu+m,\nu) \leq ||\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu}|| \leq \sup_{t\geq 0} \|R^{\gamma}_{t,p}\| B(\mu,\nu).$$

On $L^p(\mathbb{R}^+)$, and $\gamma = \frac{2}{p}$, the equality $||\mathbf{C}^{\frac{2}{p}}_{\mu+\frac{1}{p},\nu}|| = B(\mu,\nu)$ holds.

(iii) $\sigma(\mathbf{C}^{\gamma}_{\mu,\nu}) = \overline{\left\{ B(\mu+z,\nu) \,:\, \Re z \geq 0 \right\}}.$

TWO KOOPMAN SEMIGROUPS ON DISCRETE LEBESGUE SPACES

PEDRO J. MIANA

To my dear father Francisco Miana García

ABSTRACT. In this paper we are interested to connect Koopman semigroups in Lebesgue funcion spaces $L^p(\mathbb{R}^+)$ and C_0-semigroups in Lebesgue sequence spaces ℓ^p for $1 \leq p < \infty$. To get this we use certain Poisson transformation $\mathcal{P} : L^p(\mathbb{R}^+) \to \ell^p$ and its adjoint $\mathcal{P}^*$ which allows carry semigroup properties from one space to the other one. Two Koopman semigroups on ℓ^p are presented and linked to the standard Koopman semigroup $T_p(t)f(r) := e^{-\frac{t}{p}} f(e^{-t}r)$ and $S_p(t)f(r) := e^{\frac{-t}{p}} f(e^{-t}r + 1 - e^{-t})$ for $t, r > 0$ on $L^p(\mathbb{R}^+)$. In the last section we introduce Cesàro-like operators subordinated to these Koopman semigroups on ℓ^p.

5. The discrete Lebesgue space ℓ^p

For $1 \leq p \leq \infty$, the Banach space $(\ell^p, \| \quad \|_p)$ are formed by $f = (f(n))_{n \geq 0} \subset \mathbb{C}$ such that

$$\begin{aligned} \|f\|_p : &= \left(\sum_{n=0}^{\infty} |f(n)|^p \right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty; \\ \|f\|_\infty : &= \sup_{n \geq 0} |f(n)| < \infty. \end{aligned}$$

$\ell^1 \hookrightarrow \ell^p \hookrightarrow \ell^\infty$, $(\ell^p)' = \ell^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$ and $p = 1$ and $p' = \infty$.

1. Introduction

In the case that $f \in \ell^1$ and $g \in \ell^p$, then $f * g \in \ell^p$ where

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j)g(j), \qquad n \geq 0,$$

and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ for $1 \leq p \leq \infty$. Note that $(\ell^1, *)$ is a commutative Banach algebra with unit (we write $\delta_0 = \chi_{\{0\}}$).

5. The discrete Lebesgue space ℓ^p

We recall that the spectrum of ℓ^1, $\Delta(\ell^1) \simeq \overline{\mathbb{D}}$ and the Güelfand transform $\mathcal{Z} : \ell^1 \to \mathcal{C}(\mathbb{D})$ is the discrete Zeta-transform,

$$\mathcal{Z}(f)(z) := \sum_{n \geq 0} f(n) z^n, \quad z \in \mathbb{D}.$$

Note that

$$\|\mathcal{Z}(f)\|_\infty \leq \|f\|_1, \quad f \in \ell^1.$$

$$\mathcal{Z}(f * g)(z) = \mathcal{Z}(f)(z)\mathcal{Z}(g)(z), \qquad z \in \mathbb{D}, \quad f, g \in \ell^1.$$

We recall that the spectrum of f, denoted as $\sigma_{\ell^1}(f) = \mathbb{C} \backslash \rho_{\ell^1}(f)$, is defined by

$$\rho_{\ell^1}(f) := \{\lambda \in \mathbb{C} \ : \ (\lambda\delta_0 - f)^{-1} \in \ell^1\}.$$

Note that $\sigma_{\ell^1}(f) = \mathcal{Z}(f)(\mathbb{D})$, for $f \in \ell^1$, $\quad \overline{\mathbb{D}} = \{z \in \mathbb{C}\, ; |z| \leq 1\}$.

5. The discrete Lebesgue space ℓ^p

The forward difference operator $\Delta(a)(n) := a(n+1) - a(n)$ is a bounded operator on ℓ^p, $\|\Delta\| = 2$ and

$$(e^{z\Delta}a)(n) = e^{-z}\sum_{j\geq 0}^{\infty} a(j+n)\frac{z^j}{j!}, \qquad a \in \ell^p, \quad 1 \leq p \leq \infty.$$

The backward difference operator $\nabla(a)(n) := a(n-1) - a(n)$ for $n \geq 1$ and $\nabla(a)(0) := -a(0)$ is also a bounded operator on ℓ^p, $\|\nabla\| = 2$ and

$$(e^{z\nabla}a)(n) = e^{-z}\sum_{j\geq 0}^{n} a(j)\frac{z^{n-j}}{(n-j)!}, \qquad a \in \ell^p, \quad 1 \leq p \leq \infty.$$

5. The discrete Lebesgue space ℓ^p

For $1 \leq p < \infty$, the left translation semigroup $(T_{left}(t))_{t>0}$,

$$T_{left}(t)f(s) := f(s+t), \quad s,t > 0, \qquad f \in L^p(\mathbb{R}^+).$$

The infinitesimal generator is the usual derivation $(\frac{d}{ds}, D(\frac{d}{ds}))$,

$$\frac{d}{ds}f(s) := f'(s), \qquad f \in D(\frac{d}{ds}) = \{f \in L^p(\mathbb{R}^+) \ : \ f' \in L^p(\mathbb{R}^+)\}.$$

The right translation C_0-semigroup $(T_{right}(t))_{t>0}$ on $L^p(\mathbb{R}^+)$, $1 \leq p < \infty$, defined by

$$T_{right}(t)f(s) := \begin{cases} f(s-t), & s \geq t, \\ 0, & 0 \leq s < t. \end{cases}$$

Its infinitesimal generator is the derivation $(\frac{d^0}{ds}, D(\frac{d^0}{ds}))$,

$$\frac{d^0}{ds}f(s) := -f'(s), \; f \in D(\frac{d^0}{ds}) = \{f \in L^p(\mathbb{R}^+) : f' \in L^p(\mathbb{R}^+), \; f(0) = 0\}.$$

6. The Poisson transform

Take $j_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. Then

$$(j_\alpha * j_\beta)(t) = \int_0^t \frac{s^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds = j_{\alpha+\beta}(t).$$

Take $a_t(n) = \frac{t^n}{n!}$. Then

$$(a_t * a_s)(n) = \sum_{j=0}^{n} \frac{t^j}{j!} \frac{s^{n-j}}{(n-j)!} = a_{t+s}(n).$$

6. The Poisson transform

Definition

Take $1 \leq p \leq \infty$. We introduce the operator $\mathcal{P} : L^p(\mathbb{R}^+) \to \ell^p$ defined by

$$\mathcal{P}(f)(n) := \int_0^\infty f(t) e^{-t} \frac{t^n}{n!} dt, \qquad n \geq 0, \quad f \in L^p(\mathbb{R}^+). \tag{1}$$

Duality, we consider the operator $\mathcal{P}^* : \ell^p \to L^p(\mathbb{R}^+)$ given by

$$\mathcal{P}^*(a)(t) := e^{-t} \sum_{n=0}^\infty a(n) \frac{t^n}{n!}, \qquad t \in \mathbb{R}^+, \quad a = (a(n))_{n \geq 0} \in \ell^p. \tag{2}$$

6. The Poisson transform

6. The Poisson transform

Examples.(i) We write by $e_\lambda(t) := e^{-\lambda t}$ for $\Re\lambda > 0$. Then

$$\mathcal{P}(e_\lambda)(n) = \int_0^\infty e^{-(\lambda+1)t}\frac{t^n}{n!}dt = \frac{1}{(1+\lambda)^n}, \qquad n \geq 0. \tag{3}$$

Take $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $a_\lambda(n) := \lambda^n$ for $n \geq 0$. Then

$$\mathcal{P}^*(a_\lambda)(t) := e^{-t}\sum_{n=0}^\infty \frac{(\lambda t)^n}{n!} = e^{-(1-\lambda)t}, \qquad t \geq 0.$$

(ii)Take also $j_\alpha(t) := \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha, t > 0$. Then

$$\mathcal{P}(j_\alpha)(n) = \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-t}\frac{t^n}{n!}dt = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!} = k^\alpha(n), \qquad n \geq 0.$$

6. The Poisson transform

Theorem

Take $1 \leq p \leq \infty$ *and operators* $\mathcal{P}$ *and* $\mathcal{P}^*$ *. Then*

(i) *The map* $\mathcal{P} : L^p(\mathbb{R}^+) \to \ell^p$ *is a bounded and* $\|\mathcal{P}\| = 1$*.*

(ii) $\mathcal{P}(f * g) = \mathcal{P}(f) * \mathcal{P}(g)$ *for* $f \in L^p(\mathbb{R}^+)$ *and* $g \in L^1(\mathbb{R}^+)$*.*

(iii) *For* $1 < p \leq \infty$*, the map* $\mathcal{P}^* : \ell^p \to L^p(\mathbb{R}^+)$ *is the adjoint operator of* $\mathcal{P}$ *and* $\|\mathcal{P}^*\| = 1$ *for* $1 \leq p \leq \infty$*.*

(iv) $\mathcal{P}^*(\delta_1 * (a * b)) = \mathcal{P}^*(a) * \mathcal{P}^*(b)$ *for* $a \in \ell^p$ *and* $b \in \ell^1$*.*

6. The Poisson transform

Theorem

Let $(T_{left}(t))_{t>0}$ and $(T_{right}(t))_{t>0}$ be the translations C_0 semigroups defined on $L^p(\mathbb{R}^+)$ and the C_0-semigroups $(e^{t\Delta})_{t\geq 0}$ and $(e^{z\nabla})_{t\geq 0}$ on ℓ^p for $1 \leq p \leq \infty$. The following equalities hold.

(i) $\nabla(\mathcal{P}(f)) = -\mathcal{P}(f')$ *for* $f \in D(\frac{d^0}{ds})$.

(ii) $e^{t\nabla} \circ \mathcal{P} = \mathcal{P} \circ T_{right}(t)$ *for* $t \geq 0$.

(iii) $\mathcal{P}^*(\Delta(a)) = (\mathcal{P}^*(a))'$ *for* $a \in \ell^p$.

(iv) $\mathcal{P}^*(e^{t\Delta}) = \mathcal{P}^*(T_{left}(t))$ *for* $t \geq 0$.

$$\begin{array}{ccc} L^p(\mathbb{R}^+) & \xrightarrow{\mathcal{P}} & \ell^p \\ \big\downarrow -\frac{d}{ds} & & \big\downarrow \nabla \\ L^p(\mathbb{R}^+) & \xrightarrow{\mathcal{P}} & \ell^p \end{array}$$

7. Some Koopman semigroups on $L^p(\mathbb{R}^+)$ and ℓ^p

On $L^p(\mathbb{R}^+)$, we consider the C_0-group of isometries, $(T_p(t))_{t\geq 0}$, defined by

$$T_p(t)f(r) := e^{-\frac{t}{p}} f(e^{-t}r), \qquad r \geq 0, \quad t \in \mathbb{R}. \tag{4}$$

Its infinitesimal generator $(\Lambda_p, D(\Lambda_p))$ is given by

$$\Lambda_p(f)(r) = -rf'(r) - \frac{1}{p} f(r), \qquad r \geq 0,$$

and $D(\Lambda_p) = \{f \in L^p(\mathbb{R}^+) \mid rf'(r) \in L^p(\mathbb{R}^+)\}$.
We denote by $T_p^+(t) := T_p(t)$ and $T_p^-(t) := T_p(-t)$ for $t \geq 0$; Λ_p and $-\Lambda_p$ are its infinitesimal generator, respectively.

7. Some Koopman semigroups on $L^p(\mathbb{R}^+)$ and ℓ^p

Now we consider $(\mathfrak{T}(t))_{t\geq 0}$ and $(\mathfrak{S}(t))_{t\geq 0}$ acting on ℓ^p with $1\leq p\leq\infty$, where

$$\begin{aligned}
\mathfrak{T}_p(t)a(n) &:= e^{-\frac{t}{p}}\sum_{j=0}^{n}\binom{n}{j}e^{-tj}(1-e^{-t})^{n-j}a(j),\\
\mathfrak{S}_p(t)a(n) &:= e^{-t(1-\frac{1}{p})}e^{-tn}\sum_{j=n}^{\infty}\binom{j}{n}(1-e^{-t})^{j-n}a(j), \qquad (5)
\end{aligned}$$

for $n\in\mathbb{N}_0$ and $a\in\ell^p$. Respectively its infinitesimal generators are

$$\begin{aligned}
\mathfrak{A}_p(a)(n) &:= n\nabla(a)(n)-\frac{1}{p}a(n),\ n\geq 1,\\
\mathfrak{A}_p(a)(0) &:= -\frac{1}{p}a(0);\\
\mathfrak{B}_p(a)(n) &:= (n+1)\Delta(a)(n)+\frac{1}{p}a(n),\ n\geq 0,
\end{aligned}$$

for $a\in D(\mathfrak{A}_p)=D(\mathfrak{B}_p)=\{b\in\ell^p\,;(n+1)\Delta(b)\in\ell^p\}$. In fact, $\|\mathfrak{T}_p(t)\|,\|\mathfrak{S}_p(t)\|\leq 1$ and $(\mathfrak{A}_p)^*=\mathfrak{B}_{p'}$ on $\ell^{p'}$ ([AM]).

7. Some Koopman semigroups on $L^p(\mathbb{R}^+)$ and ℓ^p

Theorem

Take the one-parameter families $(T_p(t))_{t\in\mathbb{R}}$ defined on $L^p(\mathbb{R}^+)$ in (4) and $(\mathfrak{T}_p(t))_{t\geq 0}$ and $(\mathfrak{S}_p(t))_{t\geq 0}$ defined on ℓ^p in (6) for $1 \leq p \leq \infty$. Then

(i) $\mathfrak{B}_p(\mathcal{P}(f)) = \mathcal{P}(-\Lambda_p(f))$ *for* $f \in D(\Lambda_p)$.

(ii) $\mathfrak{S}_p(t) \circ \mathcal{P} = \mathcal{P} \circ T_p^-(t)$ *for* $t > 0$.

(iii) $\mathcal{P}^*(\mathfrak{A}_p(a)) = \Lambda_p(\mathcal{P}^*(a))$ *for* $a \in D(\mathfrak{A}_p)$.

(iv) $\mathcal{P}^* \circ \mathfrak{T}_p(t) = T_p^+(t) \circ \mathcal{P}^*$ *for* $t > 0$.

7. Some Koopman semigroups on $L^p(\mathbb{R}^+)$ and ℓ^p

Theorem

Take the one-parameter families $(T_p(t))_{t\in\mathbb{R}}$ *defined on* $L^p(\mathbb{R}^+)$ *in (4) and* $(\mathfrak{T}_p(t))_{t\geq 0}$ *and* $(\mathfrak{S}_p(t))_{t\geq 0}$ *defined on* ℓ^p *in (6) for* $1 \leq p \leq \infty$. *Then*

(i) $\mathfrak{B}_p(\mathcal{P}(f)) = \mathcal{P}(-\Lambda_p(f))$ *for* $f \in D(\Lambda_p)$.

(ii) $\mathfrak{S}_p(t) \circ \mathcal{P} = \mathcal{P} \circ T_p^-(t)$ *for* $t > 0$.

(iii) $\mathcal{P}^*(\mathfrak{A}_p(a)) = \Lambda_p(\mathcal{P}^*(a))$ *for* $a \in D(\mathfrak{A}_p)$.

(iv) $\mathcal{P}^* \circ \mathfrak{T}_p(t) = T_p^+(t) \circ \mathcal{P}^*$ *for* $t > 0$.

Corollary

Take $1 \leq p \leq \infty$. *Then*

(i) $(\nabla + \mathfrak{B}_p)(\mathcal{P}(f)) = \mathcal{P}((-\Lambda_p + \frac{d^0}{ds})(f))$ *for* $f \in D(\Lambda_p) \cap D(\frac{d^0}{ds})$.

(iii) $\mathcal{P}^*((\mathfrak{A}_p + \Delta)(a)) = (\Lambda_p + \frac{d}{ds})(\mathcal{P}^*(a))$ *for* $a \in D(\mathfrak{A}_p)$.

8. Perturbations of Koopman semigroups on ℓ^p

Definition

Take $a \in \ell^p$ for $1 \leq p \leq \infty$ and $t > 0$. We define the operators

$$\mathfrak{T}_{\Delta,p}(t)a(l) := e^{-(\frac{t}{p}+1-e^{-t})} \sum_{j=0}^{l} \binom{l}{j} (1-e^{-t})^{l-j} e^{-tj} \sum_{n=j}^{\infty} \frac{(1-e^{-t})^{n-j}}{(n-j)!} a(n),$$

$$\mathfrak{S}_{\nabla,p}(t)a(l) := e^{-(t(1-\frac{1}{p})+1-e^{-t})} \sum_{j=0}^{l} \frac{(1-e^{-t})^{l-j}}{(l-j)!} e^{-tj} \sum_{n=j}^{\infty} \binom{n}{j} (1-e^{-t})^{n-j} a(n),$$

for $l \geq 0$.

8. Perturbations of Koopman semigroups on ℓ^p

Theorem

Let $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ and $(\mathfrak{S}_{\nabla,p}(t))_{t>0}$ operators given in Definition 21. Then

(i) $\|\mathfrak{T}_{\Delta,p}(t)\| \leq 1$ *for* $t > 0$ *and* $1 \leq p \leq \infty$*.*

(ii) $\|\mathfrak{S}_{\nabla,p}(t)\| \leq 1$ *for* $t > 0$ *and* $1 \leq p \leq \infty$*.*

(iii) $(\mathfrak{T}_{\Delta,p}(t))^* = \mathfrak{S}_{\nabla,p'}(t)$ *and* $(\mathfrak{S}_{\nabla,p}(t))^* = \mathfrak{T}_{\Delta,p'}(t)$ *on* ℓ^p *for* $1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$*.*

8. Perturbations of Koopman semigroups on ℓ^p

([MP]) Given $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^+)$,

$$S_p(t)f(s) := e^{\frac{-t}{p}} f(e^{-t}s + 1 - e^{-t}), \qquad s > 0, \tag{6}$$

and $t \geq 0$, and its infinitesimal generator $(A_p, D(A_p))$ is defined by

$$(A_p f)(s) := (1-s)f'(s) - \frac{1}{p} f(s), \qquad f \in D(A_p), \quad s > 0,$$

$D(A_p) = \{f \in L^p(\mathbb{R}^+) \mid A_p f \in L^p(\mathbb{R}^+)\}$, and $A_p f = (\Lambda_p + \frac{d}{ds})f$.

8. Perturbations of Koopman semigroups on ℓ^p

([MP]) Given $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^+)$,

$$S_p(t)f(s) := e^{\frac{-t}{p}} f(e^{-t}s + 1 - e^{-t}), \qquad s > 0, \qquad (6)$$

and $t \geq 0$, and its infinitesimal generator $(A_p, D(A_p))$ is defined by

$$(A_p f)(s) := (1-s)f'(s) - \frac{1}{p} f(s), \qquad f \in D(A_p), \quad s > 0,$$

$D(A_p) = \{f \in L^p(\mathbb{R}^+) \mid A_p f \in L^p(\mathbb{R}^+)\}$, and $A_p f = (\Lambda_p + \frac{d}{ds})f$.

Similarly, we consider $R_p(t)f$ defined by

$$R_p(t)f(s) := e^{\frac{t}{p}} f(e^t s + 1 - e^t)\chi_{(1-e^{-t},\infty)}(s), \qquad s > 0, \quad (7)$$

and $t \geq 0$. Operators $(R_p(t))_{t>0}$ is a contractive C_0-semigroup and its generator is

$$(B_p f)(s) := (s-1)f'(s) + \frac{1}{p} f(s), \qquad f \in D(A_p), \quad s > 0,$$

$D(B_p) = \{f \in L^p(\mathbb{R}^+) \mid B_p f \in L^p(\mathbb{R}^+)\}$, and $B_p f = (-\Lambda_p + \frac{d}{ds})f$.

8. Perturbations of Koopman semigroups on ℓ^p

Theorem

Take $1 \leq p \leq \infty$, $(S_p(t))_{t>0}$ *and* $(R_p(t))_{t>0}$ *defined on* $L^p(\mathbb{R}^+)$ *and* $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ *and* $(\mathfrak{S}_{\nabla,p}(t))_{t>0}$ *defined on* ℓ^p. *Then*

(i) $\mathfrak{S}_{\nabla,p}(t) \circ \mathcal{P} = \mathcal{P} \circ R_p(t)$ *for* $t > 0$.

(ii) $\mathcal{P}^* \circ \mathfrak{T}_{\Delta,p}(t) = S_p(t) \circ \mathcal{P}^*$ *for* $t > 0$.

8. Perturbations of Koopman semigroups on ℓ^p

Theorem

Take $1 \leq p \leq \infty$, $(S_p(t))_{t>0}$ and $(R_p(t))_{t>0}$ defined on $L^p(\mathbb{R}^+)$ and $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ and $(\mathfrak{S}_{\nabla,p}(t))_{t>0}$ defined on ℓ^p. Then

(i) $\mathfrak{S}_{\nabla,p}(t) \circ \mathcal{P} = \mathcal{P} \circ R_p(t)$ *for* $t > 0$.

(ii) $\mathcal{P}^* \circ \mathfrak{T}_{\Delta,p}(t) = S_p(t) \circ \mathcal{P}^*$ *for* $t > 0$.

Theorem

Take $1 \leq p < \infty$. The one-parameter families $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ and $(\mathfrak{S}_{\nabla,p}(t))_{t>0}$ defined on ℓ^p are contractive C_0-semigroups. Then

(i) *the generator of* $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ *is* $\mathfrak{A}_{\Delta,p} = \mathfrak{A}_p + \Delta$.

(ii) *the generator of* $(\mathfrak{S}_{\nabla,p}(t))_{t>0}$ *is* $\mathfrak{B}_{\nabla,p} = \mathfrak{B}_p + \nabla$.

(iii) *operators* $\mathfrak{A}_{\Delta,p}$ *and* $\mathfrak{B}_{\nabla,p'}$ *and semigroups* $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ *and* $(\mathfrak{S}_{\nabla,p'}(t))_{t>0}$ *are adjoint to each other, i.e.,* $(\mathfrak{A}_{\Delta,p})^* = \mathfrak{B}_{\nabla,p'}$ *and* $(\mathfrak{T}_{\Delta,p}(t))^* = \mathfrak{S}_{\nabla,p'}(t)$ *on* $\ell^{p'}$, $1/p + 1/p' = 1$ *and* $t > 0$.

8. Perturbations of Koopman semigroups on ℓ^p

Corollary

Take $1 \leq p < \infty$, $a \in \ell^p$ and $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ and $(\mathfrak{S}_{\nabla,p}(t))_{t>0}$ the contractive C_0-semigroups defined on ℓ^p in Definition 21.

(i) *For $t > 0$ and $n \geq 0$, we have that*

$$\mathfrak{T}_{\Delta,p}(t)(a)(n) = \mathfrak{T}_p(t)(a)(n) + \int_0^t \mathfrak{T}_p(t-s)\Delta\left(\mathfrak{T}_{\Delta,p}(s)(a))(n)\right) ds.$$

(ii) *For $t > 0$ and $n \geq 1$, we have that*

$$\mathfrak{S}_{\nabla,p}(t)(a)(n) = \mathfrak{S}_p(t)(a)(n) + \int_0^t \mathfrak{S}_p(t-s)\nabla\left(\mathfrak{S}_{\nabla,p}(s)(a)(n)\right) ds,$$

and

$$\mathfrak{S}_{\nabla,p}(t)(a)(0) = \mathfrak{S}_p(t)(a)(0) - \int_0^t \mathfrak{S}_p(t-s)\mathfrak{S}_{\Delta,p}(s)(a)(0) ds.$$

8. Perturbations of Koopman semigroups on ℓ^p

Theorem

Take $1 \leq p < \infty$, $\mathfrak{A}_{\Delta,p}$ and $\mathfrak{B}_{\nabla,p}$ the infinitesimal generators of $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ and $(\mathfrak{S}_{\nabla,p}(t))_{t>0}$ on ℓ^p, $1 \leq p < \infty$ respectively.

(i) *Then $\sigma_{point}(\mathfrak{A}_{\Delta,p}) = \emptyset$ and $\sigma_{point}(\mathfrak{B}_{\nabla,p}) = \mathbb{C}_-$.*

(ii) *The following equalities hold $\sigma(\mathfrak{A}_{\Delta,p}) = \sigma(\mathfrak{B}_{\nabla,p}) = \overline{\mathbb{C}_-}$.*

(iii) *For $\Re\lambda > 0$, and $a = (a(n))_{n\geq 0} \in \ell^p$, we have that*

$$(\lambda - \mathfrak{A}_{\Delta,p})^{-1} a(l) = \sum_{j=0}^{l} \binom{l}{j} \sum_{n=j}^{\infty} \frac{a(n)}{(n-j)!} \mathbb{B}_1(\lambda + \frac{1}{p} + j, n + l - 2j + 1),$$

$$(\lambda - \mathfrak{B}_{\nabla,p})^{-1} a(l) =$$

$$\sum_{j=0}^{l} \frac{1}{(l-j)!} \sum_{n=j}^{\infty} \binom{n}{j} a(n) \mathbb{B}_1(\lambda + 1 - \frac{1}{p} + j, n + l - 2j + 1),$$

where $\mathbb{B}_1(u,v) := \int_0^1 (1-t)^{u-1} t^{v-1} e^{-t} dt$, $u, v \in \mathbb{C}^+$.

Summary

Summary

Table: C_0-semigroups and its infinitesimal generators on $L^p(\mathbb{R}^+)$

$T(t)$	$T(t)f(s)$	A
$T_{left}(t)$	$f(s+t)$	$\frac{d}{ds}$
$T_{right}(t)$	$f(s-t)\chi_{(t,\infty)}(s)$	$\frac{d^0}{ds}$
$T_p(t)$	$e^{-\frac{t}{p}}f(e^{-t}r)$	Λ_p
$S_p(t)$	$e^{\frac{-t}{p}}f(e^{-t}s+1-e^{-t})$	$\Lambda_p+\frac{d}{ds}$
$R_p(t)$	$e^{\frac{t}{p}}f(e^{t}s+1-e^{t})\chi_{(1-e^{-t},\infty)}(s)$	$-\Lambda_p+\frac{d^0}{ds}$

Summary

Summary

Table: C_0-semigroups and its infinitesimal generators on ℓ^p

$T(t)$	$T(t)a(n)$	A
$e^{t\Delta}$	$e^{-t}\sum_{j\geq 0}^{\infty} a(j+n)\frac{t^j}{j!}$	Δ
$e^{t\nabla}$	$e^{-t}\sum_{j\geq 0}^{n} a(j)\frac{t^{n-j}}{(n-j)!}$	∇
$\mathfrak{T}_p(t)$	$e^{-\frac{t}{p}}\sum_{j=0}^{n}\binom{n}{j}e^{-tj}(1-e^{-t})^{n-j}a(j)$	$\mathfrak{A}_p$
$\mathfrak{S}_p(t)$	$e^{-t(1-\frac{1}{p})}e^{-tn}\sum_{j=n}^{\infty}\binom{j}{n}(1-e^{-t})^{j-n}a(j)$	$\mathfrak{B}_p$
$\mathfrak{T}_{\Delta,p}(t)$	$e^{-(\frac{t}{p}+1-e^{-t})}\times$ $\sum_{j=0}^{l}\binom{l}{j}(1-e^{-t})^{l-j}e^{-tj}\sum_{n=j}^{\infty}\frac{(1-e^{-t})^{n-j}}{(n-j)!}a(n)$	$\mathfrak{A}_p+\Delta$
$\mathfrak{S}_{\nabla,p}(t)$	$e^{-(t(1-\frac{1}{p})+1-e^{-t})}\times$ $\sum_{j=0}^{l}\frac{(1-e^{-t})^{l-j}}{(l-j)!}e^{-tj}\sum_{n=j}^{\infty}\binom{n}{j}(1-e^{-t})^{n-j}a(n)$	$\mathfrak{B}_p+\nabla$

9. Cesàro-like operators on ℓ^p

Let $(X, \|\ \ \|)$ be a Banach space and $T = (T(t))_{t>0}$ a uniformly bounded C_0-semigroup on $(X, \|\ \ \|)$. Then we define

$$C_{\mu,\nu}^{T}(x) := \int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1}T(t)(x)dt, \quad x \in X,\ mu, \nu > 0.$$

Let $\Re\alpha > 0$, consider the Cesàro-Hardy operator of order α,

$$\mathcal{C}_\alpha f(s) := \frac{\alpha}{s^\alpha}\int_0^s (s-u)^{\alpha-1} f(u)\,du$$
$$= \alpha\int_0^\infty (1-e^{-t})^{\alpha-1} e^{-t(1-\frac{1}{p})} T_p^+(t) f(s)\,dt, \qquad s > 0,$$

and the dual Cesàro-Hardy operator of order α given by

$$\mathcal{C}_\alpha^* f(s) := \alpha\int_s^\infty \frac{(u-s)^{\alpha-1}}{u^\alpha} f(u)\,du$$
$$= \alpha\int_0^\infty (1-e^{-t})^{\alpha-1} e^{-\frac{t}{p}} T_p^-(t) f(s)\,dt, \qquad s > 0,$$

where $(T_p^+(t))_{t\geq 0}$ and $T_p^-(t))_{t\geq 0}$ are defined above.

9. Cesàro-like operators on ℓ^p

Now we consider the generalized discrete Cesàro operator of order $\alpha > 0$ given for

$$
\begin{aligned}
\mathfrak{C}_\alpha a(n) &:= \frac{1}{k^{\alpha+1}(n)} \sum_{j=0}^{n} k^{\alpha}(n-j)a(j), \\
\mathfrak{C}_\alpha^* a(n) &:= \sum_{j=n}^{\infty} \frac{1}{k^{\alpha+1}(j)} k^{\alpha}(j-n)a(j),
\end{aligned}
$$

for $n \in \mathbb{N}_0$, $a \in \ell^p$ and $(k^\alpha(n))_{n\geq 0}$ are the Cesàro numbers. Remind that for $a \in \ell^p$ and $n \in \mathbb{N}_0$, we have that

$$
\begin{aligned}
\mathfrak{C}_\alpha a(n) &= \alpha \int_0^\infty (1-e^{-t})^{\alpha-1} e^{-t(1-\frac{1}{p})} \mathfrak{T}_p(t) a(n)\, dt, \quad 1 < p \leq \infty, \\
\mathfrak{C}_\alpha^* a(n) &= \alpha \int_0^\infty (1-e^{-t})^{\alpha-1} e^{-\frac{t}{p}} \mathfrak{S}_p(t) a(n)\, dt, \quad 1 \leq p < \infty.
\end{aligned}
$$

9. Cesàro-like operators on ℓ^p

Corollary

Take $\alpha > 0$. Then

(i) $\mathcal{P} \circ \mathcal{C}_\alpha^* = \mathfrak{C}_\alpha^* \circ \mathcal{P}$ *for* $1 \leq p < \infty$.

(ii) $\mathcal{P}^* \circ \mathfrak{C}_\alpha = \mathcal{C}_\alpha \circ \mathcal{P}^*$ *for* $1 < p \leq \infty$.

9. Cesàro-like operators on ℓ^p

Corollary

Take $\alpha > 0$. Then

(i) $\mathcal{P} \circ \mathcal{C}_\alpha^* = \mathfrak{C}_\alpha^* \circ \mathcal{P}$ *for* $1 \le p < \infty$.

(ii) $\mathcal{P}^* \circ \mathfrak{C}_\alpha = \mathcal{C}_\alpha \circ \mathcal{P}^*$ *for* $1 < p \le \infty$.

Remark Take $\alpha > 0$, Then $\partial\sigma(\mathfrak{C}_\alpha) = \sigma(\mathcal{C}_\alpha)$ due to

$$\sigma(\mathcal{C}_\alpha) = \overline{\left\{ \frac{\Gamma(\alpha+1)\Gamma(it+1-\frac{1}{p})}{\Gamma(\alpha+it+1-\frac{1}{p})} \; : \; t \in \mathbb{R} \right\}}$$

$$\sigma(\mathfrak{C}_\alpha) = \overline{\left\{ \frac{\Gamma(\alpha+1)\Gamma(z+1-\frac{1}{p})}{\Gamma(\alpha+z+1-\frac{1}{p})} \; : \; z \in \mathbb{C}_+ \cup i\mathbb{R} \right\}},$$

for $1 < p \le \infty$.

9. Cesàro-like operators on ℓ^p

Theorem

Take $1 \leq p < \infty$, $a = (a(n))_{n\geq 0} \in \ell^p$ and C_0-semigroups $(\mathfrak{T}_{\Delta,p}(t))_{t>0}$ and $(\mathfrak{S}_{\nabla,p}(t))_{t>0}$ on ℓ^p. Now we define

$$\begin{aligned} \mathfrak{c}^{\Delta,p}_{\mu,\nu} a(l) &:= \int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1}\mathfrak{T}_{\Delta,p}(t)a(l)dt, \\ \mathfrak{c}^{\nabla,p}_{\mu,\nu} a(l) &:= \int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1}\mathfrak{S}_{\Delta,p}(t)a(l)dt, \end{aligned}$$

for $\Re\mu, \Re\nu > 0$ and $l \geq 0$. Then

$$\mathfrak{c}^{\Delta,p}_{\mu,\nu} a(l) = \sum_{j=0}^{l} \binom{l}{j} \sum_{n=j}^{\infty} \frac{a(n)}{(n-j)!}\mathbb{B}_1(\mu + \frac{1}{p} + j, \nu + n + l - 2j),$$

$$\mathfrak{c}^{\nabla,p}_{\mu,\nu} a(l) =$$

$$\sum_{j=0}^{l} \frac{1}{(l-j)!} \sum_{n=j}^{\infty} \binom{n}{j} a(n)\mathbb{B}_1(\mu + 1 - \frac{1}{p} + j, \nu + n + l - 2j).$$

9. Cesàro-like operators on ℓ^p

Theorem

Operators $\mathfrak{c}_{\mu,\nu}^{\Delta,p}$ *and* $\mathfrak{c}_{\mu,\nu}^{\nabla,p}$ *are bounded on* ℓ^p*,*

$$\|\mathfrak{c}_{\mu,\nu}^{\Delta,p}\|,\ \|\mathfrak{c}_{\mu,\nu}^{\nabla,p}\| \leq \mathbb{B}(\Re\mu, \Re\nu),$$

and $(\mathfrak{c}_{\mu,\nu}^{\Delta,p})^* = \mathfrak{c}_{\mu,\nu}^{\nabla,p'}$ *for* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$*.*

The spectrum of $\mathfrak{c}_{\mu,\nu}^{\Delta,p}$ *and* $\mathfrak{c}_{\mu,\nu}^{\nabla,p}$ *on* $\mathcal{B}(\ell^p)$ *is the set*

$$\sigma(\mathfrak{c}_{\mu,\nu}^{\Delta,p}) = \sigma(\mathfrak{c}_{\mu,\nu}^{\nabla,p}) = \overline{\left\{\mathbb{B}(\mu + z, \nu)\ :\ \Re z \geq 0\right\}}.$$

9. Cesàro-like operators on ℓ^p

On $L^p(\mathbb{R}^+)$ with $1 \leq p < \infty$, we consider $S_p = (S_p(t))_{t>0}$ and $R_p = (R_p(t))_{t>0}$ and define Cesàro-like operators by

$$
\begin{aligned}
C^{S_p}_{\mu,\nu} f(r) &= \frac{1}{|r-1|^{\mu+\frac{1}{p}+\nu-1}} \int_{\Gamma_{1,r}} |s-1|^{\mu+\frac{1}{p}-1} |r-s|^{\nu-1} f(s)ds, \\
C^{R_p}_{\mu,\nu} f(r) &= |r-1|^{\mu-\frac{1}{p}} \int_{\Gamma'_{1,r}} \frac{|r-s|^{\nu-1} 1}{|s-1|^{\mu-\frac{1}{p}+\nu}} f(s)ds
\end{aligned}
$$

for $r > 0$ and $\Gamma_{1,r} := (1, r)$ for $r > 1$ and $\Gamma_{1,r} := (r, 1)$ in the case $0 < r < 1$ and $\Gamma'_{1,r} := (r, +\infty)$ for $r > 1$ and $\Gamma'_{1,r} := (r, 1)$ in the case $0 < r < 1$

Corollary

Take $\alpha > 0$, $\Re\mu$, and $\Re\nu > 0$. Then

(i) $\mathcal{P} \circ C^{R_p}_{\mu,\nu} = \mathfrak{c}^{\nabla,p}_{\mu,\nu} \circ \mathcal{P}$ *for* $1 \leq p < \infty$.

(ii) $\mathcal{P}^* \circ \mathfrak{c}^{\Delta,p}_{\mu,\nu} = C^{S_p}_{\mu,\nu} \circ \mathcal{P}^*$ *for* $1 \leq p < \infty$.

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