The characteristic Lie algebra of Klein-Gordon equation and higher symmetries

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Seminar on Analysis, Diff Eqs and Math Physics
Regional Math Center of SFU,
Rostov-on-Don, April 17, 2022

Based on

D.V. Millionshchikov, S.V. Smirnov, **Characteristic algebras and integrable systems of exponential type**, Ufa Math. Journal, **13**:2 (2021), 44–73.

40 years from the publication of the **preprint by A.B. Shabat and R. Yamilov "Exponential systems and Cartan matrices"** (Ufa, USSR, 1981).

Sad events: **Alexey Shabat** and **Ravil Yamilov** passed away in 2020, **Anatoly Zhiber** passed away in March 2022.

In my talk, I want to honor their memory and **pay special tribute to their key role** in the development of the symmetry approach to partial differential equations.

The great contribution to the subject by Ufa Math School:

Vladimir Sokolov,

Vsevolod Adler,

Ismail Habibullin,

Rustem Garifullin, Maria Kuznetsova, Regina Murtazina et al.

The wave equation

The wave equation (linear!)

$$u_{tt} - u_{zz} = 0.$$

In charactreristic variables $x = \frac{z+t}{2}, y = \frac{z-t}{2}$ it can be rewritten as

$$u_{xy} = 0. \qquad (1)$$

The general solution formula

$$u = \Psi(x) + \Phi(y),$$

where $\Psi(\cdot), \Phi(\cdot)$ are arbitrary functions on one variable.
We have an obvious property of $F_1 = u_x$ for an arbitrary solution u

$$\frac{\partial}{\partial y}(F_1) = u_{xy} = 0.$$

The Liouville equation

$$u_{xy} = e^u.$$

The general solution formula

$$u = \ln \frac{2\Psi'(x)\Phi'(y)}{(\Psi(x)+\Phi(y))^2},$$

where $\Psi(\cdot), \Phi(\cdot)$ are arbitrary functions on one variable.

One can see that $F_2 = u_{xx} - \frac{1}{2}u_x^2$ has the property $\frac{\partial}{\partial y}(F_2) = 0$.

$$\frac{\partial F_2}{\partial y} = u_{xyx} - u_x u_{xy} = (e^u)_x - u_x e^u = 0.$$

Klein-Gordon equation

Consider the Klein-Gordon equation

$$u_{xy} = f(u).$$

Obtained from the classical equation

$$u_{tt} - u_{zz} = f(u).$$

by changing variables

$$x = \frac{z+t}{2}, y = \frac{z-t}{2}.$$

Definition

A function $F(u, u_x, u_{xx}, u_{xxx}, \dots) = F(u, u_1, u_2, u_3, \dots)$ is called a x-integral of the Klein-Gordon equation if $\frac{\partial F}{\partial y} = 0$.

Obviously,

$$q_2 = u_2 - u_1^2 = u_{xx} - \frac{1}{2} u_x^2.$$

is x-integral of the Liouville equation $u_{xy} = e^u$

$$\frac{\partial F}{\partial y} = u_{xxy} - u_x u_{xy} = (e^u)_x - u_x e^u = 0.$$

A. B. Shabat, R. I. Yamilov, A.N. Leznov, V.G. Smirnov, V.M. Savelyev - articles from the early 1980s

Systems of exponential type

$$w^j_{xy} = \exp\left(\sum_{k=1}^{r} a_{jk} w^k\right), \quad j = 1, 2, \ldots, r, \tag{2}$$

Example $r = 2$.

$$\begin{cases} w^1_{xy} = e^{(a_{11}w^1 + a_{12}w^2)}, \\ w^2_{xy} = e^{(a_{21}w^1 + a_{22}w^2)}, \end{cases}, \; A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3}$$

If A is a non degenerate Cartan matrix (example $r = 2$)

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

then it is integrated explicitly as the Liouville equation.

Consider the degenerate Cartan matrix A.
Let's write its last line as a linear combination of the first $n-1$ lines:

$$l_n = \lambda_1 l_1 + \cdots + \lambda_{n-1} l_{n-1}.$$

Let's introduce new variables

$$u^i = a_{i1} w^1 + \cdots + a_{in} w^n, \quad i = 1, 2, \ldots n-1.$$

New reduced system

$$u^i_{xy} = a_{i1} e^{u_1} + a_{i2} e^{u_2} + \cdots + a_{in-1} e^{u_{n-1}} + a_{in} e^{\lambda_1 u_1 + \cdots + \lambda_{n-1} u_{n-1}}, i = 1, \ldots, n-1.$$

will contain one less equation: $n-1$.

Consider the degenerate Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

According to our method, we get:

$$w^1_{xy} = e^{2w^1-2w^2} = e^{u_1}, \quad w^2_{xy} = e^{-2w^1+2w^2} = e^{-u^1}$$

$$u^1_{xy} = 2w^1_{xy} - 2w^2_{xy} = 2e^{u^1} - 2e^{-u^1}.$$

Making a simple change of variables $x, y \to \tilde{x}, \tilde{y}$, and also denoting $u = u^1$, we obtain hyperbolic sine-Gordon equation

$$u_{\tilde{x}\tilde{y}} = \sinh u.$$

Conclusion: degenerate 2×2 Cartan matrices of affine algebras $A_1^{(1)}, A_2^{(2)}$

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

Lead to the sinh-Gordon and Tzizeica equations.

$$u_{xy} = \sinh u, \quad u_{xy} = e^u + e^{-2u}$$

They are special cases of the Klein-Gordon equation (in the characteristic variables x, y)

$$u_{xy} = f(u)$$

Definition

A hyperbolic equation

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{4}$$

is called Darboux integrable if admits both x-, y-integrals.

Theorem (Shabat, Zhiber, 1979)

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The Klein-Gordon equation $u_{xy} = f(u)$ is
1) Darboux integrable if $f(u)$ can be reduced to e^u;
2) admits a nontrivial Lie-Bäcklund group if $f(u)$ can be reduced to

$$e^u, e^u - e^{-u}, e^u + e^{-2u}.$$

Remark

This classification was later generalized by Sokolov and Mikhailov.

Consider functions

$$g(x,y) = g(u, u_x, u_{xx}, u_{xxx}, \dots) = g(u, u_1, u_2, u_3, \dots)$$

where $u(x,y)$ satisfies $u_{xy} = f(u)$ and denote

$$u_1 = u_x, u_2 = u_{xx}, u_3 = u_{xxx}, \dots .$$

We have the operator of the **full partial derivative** $D = \frac{\partial}{\partial x}$

$$D = \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial u_1}{\partial x}\frac{\partial}{\partial u_1} + \dots = u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2}\dots$$

Consider the operator $\frac{\partial}{\partial y}$ and denote by $X_0 = \frac{\partial}{\partial u}$

$$\frac{\partial}{\partial y} = u_y \underbrace{\frac{\partial}{\partial u}}_{X_0} + \underbrace{(u_x)_y \frac{\partial}{\partial u_1} + (u_{xx})_y \frac{\partial}{\partial u_2} + (u_{xxx})_y \frac{\partial}{\partial u_3} + \dots}_{X_f} =$$

$$= u_y X_0 + X_f = u_y X_0 + \underbrace{f \frac{\partial}{\partial u_1} + D(f) \frac{\partial}{\partial u_2} + D^2(f) \frac{\partial}{\partial u_3} \dots}_{X_f}$$

If F is annihilated by X_0 and X_f then F **is an x-integral of the Klein-Gordon equation**.

Characteristic Lie algebra

Definition (Leznov, Smirnov, Shabat, Yamilov)

A Lie algebra $\chi(f)$ generated by vector fields

$$X_f, X_0 = \frac{\partial}{\partial u},$$

is called characteristic Lie algebra of the Klein-Gordon equation.

It is a natural generalization of the notion of characteristic vector field of a hyperbolic PDE that was first proposed by Goursat in 1899.

Characteristic Lie algebra of the Liouville equation

Lemma

$$[X_0, X_f] = X_{f'_u}$$

Hence, the characteristic Lie algebra $\chi(e^u)$ of the Liouville equation is isomorphic to two-dimensional solvable Lie algebra

$$[X_0, X_{e^u}] = X_{e^u}.$$

The sinh-Gordon equation $u_{xy} = \sinh u$

Characteristic Lie algebra $\chi(\sinh u)$
It follows that

$$X_2 = [X_0, X_1] = X_{\cosh}$$

$$[X_0, X_2] = X_{\sinh} = X_1$$

$adX_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the basis X_1, X_2.

Consider its eigen-vectors $X_1' = X_1 + X_2 = X_{e^u}$ and $X_2' = X_1 - X_2 = X_{e^{-u}}$ (eigen-values $\lambda = \pm 1$).

Theorem (M., Algebras and Repres. Theory, 2018)

The characteristic Lie algebra of the sinh-Gordon equation

$$u_{xy} = \sinh u$$

is isomorphic to the Borel subalgebra $\tilde{\mathfrak{n}}_1$ *of the affine Lie algebra* $A_1^{(1)}$ *(Kac-Moody algebra).*

Define by the recurrence operators

$$X'_{3k+1} = -[X'_1, X'_{3k}], X'_{3k+2} = [X'_2, X'_{3k}], X'_{3k+3} = [X'_1, X'_{3k+2}], k \geq 1.$$

All of them $X'_{3k}, X'_{3k+1}, X'_{3k+2}, k \geq 0$, are non trivial and satisfy the relations of the loop algebra $\mathcal{L}(\mathfrak{s}l_2)$

$$[X'_i, X'_j] = c_{i,j} X'_{i+j},\ c_{i,j} = \begin{cases} 1, \text{if } j-i \equiv 1 \bmod 3; \\ 0, \text{if } j-i \equiv 0 \bmod 3; \\ -1, \text{if } j-i \equiv -1 \bmod 3. \end{cases} \qquad (5)$$

Theorem (M., 2017)

The characteristic Lie algebra of the Tzitzeica equation

$$u_{xy} = e^{u} + e^{-2u}$$

is isomorphic to the Borel subalgebra $\tilde{\mathfrak{n}}_2$ *of the twisted affine Lie algebra* $A_2^{(2)}$.

Consider the linearization of the equation $u_{xy} = f(x, y, u, u_x, u_y)$

$$\left(D\frac{\partial}{\partial y} - f_{u_x} D - f_{u_y} \frac{\partial}{\partial y} - f_u \right) g = 0, \; g = g(x, y, u, u_x, u_y, u_{xy}, \dots) \tag{6}$$

Theorem (Goursat,…, Sokolov, Zhiber)

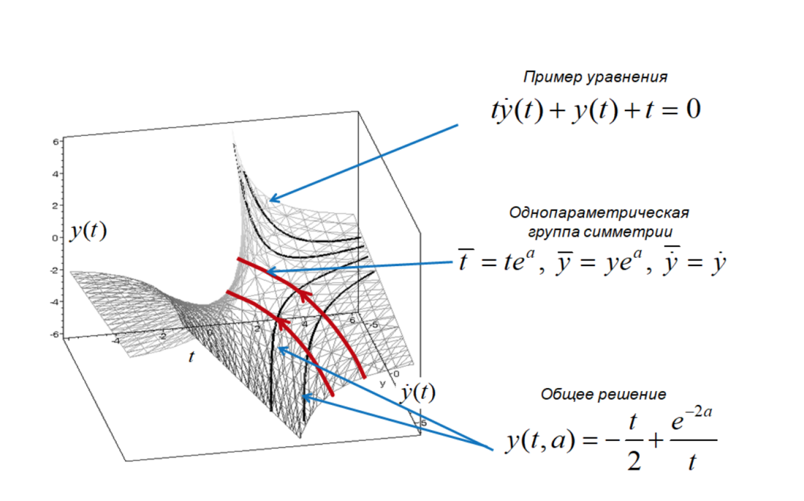
Nonlinear Equation $u_{xy} = f(x, y, u, u_x, u_y)$ is Darboux integrable if and only if the sequence of Laplace invariants of the linearized equation (6) terminates on both sides.

Zhiber and Murtazina showed that the sequence of Laplace invariants for (6) terminates on both sides when the characteristic Lie algebras are finite-dimensional.

Types of integrability of hyperbolic systems

- Darboux integrability;
- the existence of the so-called higher symmetries;
- integrability by the inverse scattering method;

Geometric illustration for symmetry group



Higher symmetries Shabat, Zhiber, Sokolov, Habibullin et al.

Consider the Klein-Gordon equation $u_{xy} = f(u)$.

Symmetry transformation $u_t = F(u_x, u_{xx}, u_{xxx}, \dots)$

The symmetry equation is again a linearized equation

$$\left(D\frac{\partial}{\partial y} - f_u\right) F = 0,\ F = F(u_x, u_{xx}, \dots)$$

Integrals of the Liouville equation

Denote $X_{e^u} = e^u X$, where

$$X = \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2} + (u_1^2 + u_2)\frac{\partial}{\partial u_2} + \dots = \sum_{k=1}^{\infty} B_{k-1}(u_1, \dots, u_{k-1}) \frac{\partial}{\partial u_k}$$

where B_k are complete Bell polynomials.
$F_2 = u_2 - \frac{1}{2}u_1^2$ is an x-integral of the Liouville equation.
Define new polynomials

$$F_3 = D(F_2) = u_3 - u_1 u_2,\ F_4 = D^2(F_2) = u_4 - u_2^2 - u_1 u_3,\ \dots$$

Theorem (Shabat, Zhiber, 1979)

x-integrals of the Liouvuille equation is an subalgebra $Ker\ X \subset \mathbb{K}[u_1, u_2, u_3, \dots]$ isomorphic $\mathbb{K}[F_2, F_3, \dots, F_k, \dots]$, where $F_k = D^{k-2}(F_2), k \geq 2$.

Higher symmetries of the Liouville equation

The defining equation of higher symmetry of the Liouville equation reduces to the algebraic equation

$$(D + u_1)XF = XDF = F$$

Hence the symmetry $F = F(u_1, u_2, \dots)$ is an eigen-vector of the operator XD with $\lambda = 1$.

Theorem (Shabat, Zhiber, 1979)

An arbitrary symmetry F (eigen-vector of XD with $\lambda = 1$) can be written

$$F = (D + u_1)(Q),\ Q \in Ker\ X = \mathbb{K}[F_2, F_3, \dots].$$

Example. $F^{(3)} = (D + u_1)F_2 = F_3 + u_1 F_2 =$
$u_3 - u_1 u_2 + u_1(u_2 - \frac{1}{2}u_1^2) = u_3 - \frac{1}{2}u_1^3.$

Symmetries of hyperbolic sinh-Gordon

Example: the symmetry equation for sinh-Gordon

$$D\frac{\partial}{\partial y}F = DX_{\sinh u}F = \cosh uF,\ F(u_1, u_2, u_3, \dots).$$

Zhiber found (late 70s) recurrent formulas for its solutions (symmetries)

$$X_1'X_2'F^{(n)} = F^{(n-2)},\ F^{(1)} = u_1 = u_x,\ F^{(n+2)} = (D^2 - u_1^2 + u_1D^{-1}u_2)F^{(n)}.$$

where X_1', X_2' are generators of its characteristic Lie algebra.

$$F^{(1)} = u_1,\ F^{(3)} = u_3 - \frac{1}{2}u_1^3,\ F^{(5)} = u_5 - \frac{5}{2}u_1^2u_3 - \frac{5}{2}u_1u_2^2 + \frac{3}{8}u_1^5, \dots$$

sinh-Gordon symmetries

A polynomial F is a symmetry of the sinh-Gordon-equation if

$$D\frac{\partial}{\partial y}F = DX_{\sinh u}F = \cosh uF.$$

Zhiber found the recurrent formulas for its symmetries

$$X_1'X_2'F^{(n)} = F^{(n-2)},\ F^{(1)} = u_1 = u_x,\ F^{(n+2)} = (D^2 - u_1^2 + u_1 D^{-1} u_2)F^{(n)}.$$

where X_1', X_2' are the generators of the charactrestic Lie algebra.

$$F^{(1)} = u_1,\ F^{(3)} = u_3 - \frac{1}{2}u_1^3,\ F^{(5)} = u_5 - \frac{5}{2}u_1^2 u_3 - \frac{5}{2}u_1 u_2^2 + \frac{3}{8}u_1^5, \ldots$$

Growth of Lie algebras

Suppose that an infinite-dimensional Lie algebra $\mathfrak{g}$ is generated by a finite-dimensional subspace V_1. For $n > 1$, let V_n denote the $\mathbb{K}$-linear span of all products in elements of V_1 of length at most n with arbitrary arrangements of brackets.

$$V_1 \subset V_2 \subset \dots \subset V_n \subset \dots, \ \cup_{i=1}^{+\infty} V_i = \mathfrak{g}.$$

Define the **growth function** of a Lie algebra

$$F_{\mathfrak{g}}^{V_1}(n) = \dim V_n$$

It depends **on the choice** of the generating set V_1 !
Two ways to get a correct definition of the **growth**:
1) to introduce an equivalence relation on growth fuctions;
2) to define the Gelfand-Kirillov dimension

$$GKdim\mathfrak{g} = \limsup_{n \to +\infty} \frac{\log \dim V_n}{\log n}.$$

Definition

A Lie algebra $\mathfrak{g}$ is called $\mathbb{N}$-graded, if $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \; i, j \in \mathbb{N}.$$

Example

$\mathfrak{g} = \mathcal{L}(m)$ – free Lie algebra of m generators $a_1, \dots, a_m$.
$\mathfrak{g}_k = \langle [a_{i_1}, [a_{i_2}, [\dots, \dots]], a_{i_k}] \rangle$ – k-.
The weight of a homogeneous word $[a_{i_1}, [a_{i_2}, [\dots, \dots]], a_{i_k}]$ is equal to k.

Definition

A $\mathbb{N}$-grading $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ is called **natural or Carnot grading**, if

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \ \forall i \in \mathbb{N}.$$

A free Lie algebra $\mathcal{L}(m)$ is naturally (Carnot) graded.

Example (naturally graded Lie algebra $\mathfrak{m}_0$)

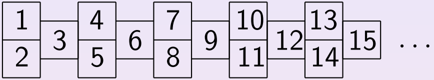
$\mathfrak{m}_0 = \langle e_1, e_2, e_3, e_4, \dots \rangle$ – defined by relations:

$$[e_1, e_i] = e_{i+1}, i \geq 2, \ [e_i, e_k] = 0.$$

e_1, e_2 – generators of weight 1, the weight of $e_3 = [e_1, e_2]$ is 2, the weight of $e_4 = [e_1, [e_1, e_2]]$ is 3 and so on.

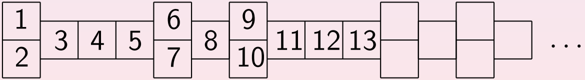
Natural grading of $A_1^{(1)}, A_2^{(2)}$

$\mathfrak{n}_1$:



$$
\begin{array}{cccc}
e_1 & & e_4 = [e_1,[e_1,e_2]] & \\
 & e_3 = [e_1,e_2] & & \dots \\
e_2 & & e_5 = [e_2,[e_1,e_2]] & \\
1 & 2 & 3 & \dots
\end{array}
$$

$\mathfrak{n}_2$:



Naturally graded Lie algebras of width 3/2

Theorem (M., 2019, Corollary)

There is an **uncountable** *family of naturally graded Lie algebras (central extensions of $\mathfrak{m}_0$) satisfying the condition*

$$\dim \mathfrak{g}_i + \dim \mathfrak{g}_{i+1} \leq 3, \quad i \geq 1.$$

Growth function of a naturally graded (Carnot) Lie algebra

Consider $V_1 = \mathfrak{g}_1$ and define a **special growth function**

$$F^{gr}_{\mathfrak{g}}(n) = \dim V_n = \dim \mathfrak{g}_1 + \dots + \dim \mathfrak{g}_n = \dim (\mathfrak{g}/\mathfrak{g}^{n+1}).$$

- fastest growth – free Lie algebra $L(X)$ with m generators.

$$F_{L(X)}(n) \sim \frac{1}{n} m^n$$

- slowest growth – $F_{\mathfrak{m}_0}(n) = n+1$.
- $\mathfrak{g} = \mathfrak{n}_1$

$$F_{\mathfrak{n}_1}(n) = \left[\frac{3n+1}{2}\right] \sim \frac{3n}{2}$$

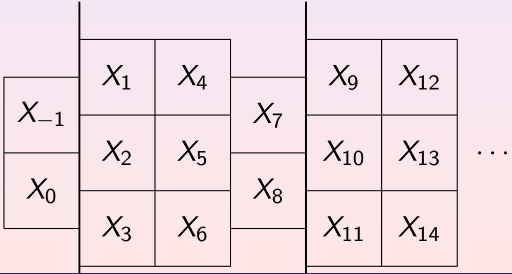
- $\mathfrak{g} = \mathfrak{n}_2$

$$F_{\mathfrak{n}_2}(n) \sim \frac{4n}{3}$$

Some new results

Theorem (Sergey Smirnov, 2021)

The characteristic Lie algebra of an exponential system with a matrix $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ *has growth function* $\frac{8}{3}$*. Recurrent formulas for higher symmetries are found.*



Thank you!