To the Spectral Theory of Hausdorff operators

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- To describe the spectra of generalized Hausdorff operators on $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$;

- To consider some generalizations to $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

This part of the talk is based on the following papers:

[1] Mirotin A. R., On the description of normal Hausdorff operators on Lebesgue spaces, *Funk. Anal. Appl.*, **53** (2019), no. 4, 261–269 arXiv:1902.07671v2.

[2] ———–, The structure of normal Hausdorff operators on Lebesgue spaces, *Forum Math.*, **32** (2020), no 1, pp. 111–119, arXiv:1812.02680v2.

- The last part of the talk will be devoted to the problem of compactness of Hausdorff operators.

  It is based on the paper

  [3] ———, A Hausdorff Operator with Commuting Family of Perturbation Matrices Is a Non-Riesz Operator, *Rus. J. Math. Phys.*, **27** (2020), no 4, pp. 484–490.

Classical definition of $\mathcal{H}$

Firstly Hausdorff operators in the form

$$(\mathcal{H}f)(x) = \int_{[0,1]} f(ux)d\mu(u),$$

(μ is some finite regular Borel measure on $[0, 1]$)

were introduced by **Garabedyan** and **Rogosinskii** as a continuous variable analog of the regular Hausdorff transformations for sequences and series
(see also G. H. Hardy's "Divergent Series").

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(see also G. H. Hardy's "Divergent Series").

For a probability μ this is a sort of an average of f.

General definition of $\mathcal{H}$ on $\mathbb{R}$

In the first part of the talk we shall consider Hausdorff operator of the form

Definition of $\mathcal{H}$ in $\mathbb{R}$.

$$\mathcal{H}_{K,a}f(x) = \int_{\Omega} K(u)f(a(u)x)d\mu(u),\ x \in \mathbb{R},$$

where $f : \mathbb{R} \to \mathbb{C}$, μ is some regular Borel measure on Ω $K \in L^1_{loc}(\Omega, \mu)$ ("a kernel"), $a : \Omega \to \mathbb{R} \setminus \{0\}$ is some measurable function ("a dilation family").

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Taking $\Omega = \mathbb{Z}$, μ a counting measure we get discrete Hausdorff operator

$$f \mapsto \sum_{u \in \mathbb{Z}} K(u)f(a(u)x).$$

Classical examples

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3)

$$(H_\gamma f)(x) = \frac{\gamma \mathrm{sgn}(x)}{|x|^\gamma}\int_0^x |t|^{\gamma-1} f(t)dt \ (\gamma > 0)$$ – the Hardy operator;

Classical examples 2

4)

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5) Let $\alpha > 0$.

$$(C_\alpha f)(x) = \alpha x^{-\alpha}\int_0^x (x-t)^{\alpha-1} f(t)dt - \text{ the Cesàro mean of order } \alpha;$$

$$(I^\alpha f)(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}(C_\alpha f)(x) - \text{ the Rieman-Liuville operator;}$$

6)

$$(H_1^* f)(x) = \begin{cases} \int_x^{\infty} \frac{f(t)}{t} dt, & x > 0 \\ -\int_{-\infty}^{x} \frac{f(t)}{t} dt, & x < 0 \end{cases} \quad -- \text{ the Copson operator;}$$

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7)

$$(C_q f)(x) := \frac{1}{x} \int_0^x f(t) d_q t := (1-q) \sum_{k=0}^\infty q^k f(q^k x)$$

– the q-calculus version of a Cesàro operator.
Here $x \in \mathbb{R}$, and q is real, $0 < |q| < 1$. This is a discrete Hausdorff operator.

For several special Hausdorff operators (mainly for discrete and continuous Cesàro operators) in various spaces some spectra were calculated by **P.R. Halmosh, A. Brown, and A. L. Shields; D. W. Boyd; G. M. Leibowitz; B. E. Rhoades; A. A. Albanese, J. Bonet, and W. J. Rickert** and others.

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The modern development of the theory of Hausdorff operators begins with the work of **Liflyand** and **Móricz** (2001) where the boundedness of a some class of Hausdorff operators on the Hardy space $H^1(\mathbb{R})$ were considered.

Boundedness of $\mathcal{H}_{K,a}$

The boundedness condition

Lemma 1. $|a(u)|^{-1/p}K(u) \in L^1(\mu) \Longrightarrow \mathcal{H}_{K,a} \in \mathcal{L}(L^p(\mathbb{R}))$ $(p \in [1, \infty])$,

$$\|\mathcal{H}_{K,a}\| \leq \int_\Omega |K(u)||a(u)|^{-1/p} d\mu(u).$$

This estimate is sharp: in the case where $K(u) > 0$ this condition is necessary and we get an equality for the norm.

In the first part of the talk *We shall assume that the condition of Lemma 1 holds*.

Scalar symbols for $p \geq 1$

To calculate the spectra, we must introduce two functions. Let

$$\Omega_\pm := \{u \in \Omega : a(u) \gtrless 0\},$$

Scalar symbols

$$\varphi(s) := \int_{\Omega_+} K(u)|a(u)|^{-1/p-is} d\mu(u) + \int_{\Omega_-} K(u)|a(u)|^{-1/p-is} d\mu(u)$$

$$= \int_{\Omega} K(u)|a(u)|^{-1/p-is} d\mu(u),$$

$$\varphi^*(s) := \int_{\Omega_+} K(u)|a(u)|^{-1/p-is} d\mu(u) - \int_{\Omega_-} K(u)|a(u)|^{-1/p-is} d\mu(u).$$

Spectra of $\mathcal{H}_{K,a}$

Theorem 1. Let $|a(u)|^{-1/2}K(u) \in L^1(\Omega)$. Then we have for $L^2(\mathbb{R})$
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(ii) $\sigma_p(\mathcal{H}_{K,a}) = \{\lambda \in \mathbb{C} : \mathrm{mes}(\varphi^{-1}(\{\lambda\}) \cup (\varphi^*)^{-1}(\{\lambda\})) > 0\}$.

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(iii) $\sigma_r(\mathcal{H}_{K,a}) = \varnothing$.

Several corollaries

Corollary 1.

$\|\mathcal{H}_{\Phi,a}\| = \max(\sup_{\mathbb{R}} |\varphi|, \sup_{\mathbb{R}} |\varphi^*|).$

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(iii) $\|\mathcal{H}_{K,a}\| = \sup_{\mathbb{R}} |\varphi|$
$=$ [for $K(u) > 0$] $= \int_\Omega K(u)a(u)^{-1/2}d\mu(u)$.

n-dimensional definition

Definition. Let (Ω, μ) be topological space endowed with a positive regular Borel measure μ, $K \in L^1_{\mathrm{loc}}(\mu)$, and $(A(u))_{u\in\Omega} \subset \mathrm{GL}(n, \mathbb{R})$ a μ-measurable family. The *Hausdorff operator* with the kernel K is defined by

$$\mathcal{H}_{K,A}f(x) := \int_\Omega K(u)f(A(u)x)d\mu(u).$$

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The n-dimensional "classical definition"($\mu = \mathrm{mes}$ in $\mathbb{R}^N$) is due to **Liflyand-Lerner** and **Brown-Moricz** (2002).

Boundedness in L^p

To our knowledge, all known results on Hausdorff operators refer to the boundedness of such operators in various settings only.

Lemma 2 $|\det A(u)|^{-1/p}K(u) \in L^1(\mu) \Longrightarrow \mathcal{H}_{K,A} \in \mathcal{L}(L^p(\mathbb{R}^n))$ $(p \in [1,\infty))$,

$$\|\mathcal{H}_{K,A}\| \leq \int_\Omega |K(u)||\det A(u)|^{-1/p} d\mu(u).$$

This estimate is sharp.

We shall assume that the condition of Lemma 2 holds.

The matrix symbol in L^p

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$$\varphi_{ij}(s) := \int\limits_{\{u|A(u):V_j \to V_i\}} K(u)|\lambda(u)|^{-1/p-\imath s} d\mu(u)\ (s \in \mathbb{R}^n)$$

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Definition

Define the *matrix symbol* of a Hausdorff operator $\mathcal{H}_{K,A}$ by

$$\Phi = (\varphi_{ij})_{i,j=1}^{2^n}$$

$\Phi \in \mathrm{Mat}_{2^n}(C^b(\mathbb{R}^n))$.

Theorem 2

Let $A(u)$ be commuting and self-adjoint. Then in $L^2(\mathbb{R}^n)$ we have
(i) $\sigma(\mathcal{H}_{K,A}) = \sigma(\Phi)$ [the spectrum of Φ in $\mathrm{Mat}_{2^n}(C^b(\mathbb{R}^n))$]

$$= \{\lambda \in \mathbb{C} : \inf_{s \in \mathbb{R}^n} |\det(\lambda - \Phi(s))| = 0\}.$$

Spectra of normal $\mathcal{H}_{K,A}$ in $L^2(\mathbb{R}^n)$

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(ii) Let $E(\lambda) := \{s \in \mathbb{R}^n : \det(\lambda - \Phi(s)) = 0\}$. Then

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(iii) $\sigma_r(\mathcal{H}_{K,A}) = \varnothing$.

Corollary 1.

$$\exists \mathcal{H}_{K,A}^{-1} \Leftrightarrow \inf_{s \in \mathbb{R}^n} |\det \Phi(s)| > 0.$$

Corollaries. Inverse and the norm.

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Corollary 2.

$$\|\mathcal{H}_{K,A}\| = \sup_{s\in\mathbb{R}^n} \|\Phi(s)\|_{op}$$

$$= \max\{|\lambda| : \inf_{s\in\mathbb{R}^n} |\det(\lambda - \Phi(s))| = 0\}.$$

Corollaries. Positive definiteness

Let $A(u) \geq O$.

Definition

$$\varphi(s) := \int_\Omega K(u)|\lambda(u)|^{-1/2-\imath s} d\mu(u) \ (s \in \mathbb{R}^n)$$

(the *scalar symbol*).

Corollary 3

$$A(u) \geq O \Longrightarrow \sigma(\mathcal{H}_{K,A}) = \mathrm{cl}(\varphi(\mathbb{R}^n)).$$

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Problem: Generalizations to $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$?.

Riesz operators

Problem: Can $\mathcal{H}_{K,A} \neq O$ be a compact operator? (E. Liflyand, 2007).

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Definition (Ruston, 1954).

A bounded operator T on some Banach space X is called a *Riesz operator* if it is non-invertible and its nonzero spectrum consists of eigenvalues of finite multiplicity with no limit points other then 0.

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T is a Riesz operator in $X \iff T - \lambda I \in \mathrm{Fred}(X)\ \forall \lambda \in \mathbb{C}, \lambda \neq 0.$

Non Riesz in $L^p(\mathbb{R}^n)$

Theorem 3.

Let $A(u) \subset GL(n, \mathbb{R})$ be a commuting family of self-adjoint matrices, and $(\det A(u))^{-1/p} K(u) \in L^1(\mu)$ $(1 \leq p \leq \infty)$. Then a Hausdorff operator $\mathcal{H}_{K,A} \neq O$ in $L^p(\mathbb{R}^n)$ is non-Riesz.

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Corollary 1.

$\mathcal{H}_{K,A} \ne O$ in $L^p(\mathbb{R}^n)$ is not a sum of a quasinilpotent and compact operator.

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Let $A(u) \subset GL(n, \mathbb{R})$ be a commuting family of self-adjoint matrices, and $(\det A(u))^{-1/p} K(u) \in L^1(\mu)$ $(1 \le p \le \infty)$. Then a Hausdorff operator $\mathcal{H}_{K,A} \ne O$ in $L^p(\mathbb{R}^n)$ is non-Riesz.

Corollary 1.

$\mathcal{H}_{K,A} \ne O$ in $L^p(\mathbb{R}^n)$ is not a sum of a quasinilpotent and compact operator.

Corollary 2.

Any Hausdorff operator $\mathcal{H}_{K,a} \ne O$ in $L^p(\mathbb{R})$ is non-Riesz (and so it is not a sum of a quasinilpotent and compact operator).

Thanks

THANK YOU FOR YOUR ATTENTION!