Nonlinear composition operators in generalized Morrey spaces

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First of all I would like to thank the organizers for giving me the opportunity to talk today.

Today we consider a function

$$f : \mathbb{R} \to \mathbb{R}$$

an open subset Ω of $\mathbb{R}^n$ and we set

$$T_f[g] \equiv f \circ g \qquad \forall g \in \mathbb{R}^\Omega$$

and we ask for which Borel measurable functions f the map T_f

maps a generalized Morrey space to itself,

is continuous, uniformly continuous, α-Hölder continuous, Lipschitz continuous.

in a generalized Morrey space of functions in Ω.

For extensive references on nonlinear composition operators, we refer to the monographs of

J. Appell and P.P. Zabreiko. (1990) Nonlinear Superposition Operators. Cambridge University Press, Cambridge.

T. Runst and W. Sickel, *Sobolev Spaces of Fractional order, Nemytskij Operators*, and Nonlinear Partial Differential Equations, De Gruyter, Berlin (1996).

R.M. Dudley and R. Norvaiša, *Concrete functional calculus*. Springer Monographs in Mathematics. Springer, New York, 2011.

Today we will NOT talk about the Koopman composition operator

$$\begin{aligned} C_g : \quad \mathbb{R}^{\Omega} \to & \quad \mathbb{R}^{\Omega} \\ f \mapsto & \quad f \circ g \end{aligned}$$

for some $g : \Omega \to \Omega$ as in

N. Hatano, M. Ikeda, I. Ishikawa, and Y. Sawano, *Boundedness of composition operators on Morrey spaces and weak Morrey spaces*, J. Inequal. Appl. 2021, Paper No. 69 (case $\Omega = \mathbb{R}^n$).

We recall the definition of generalized Morrey space:

$\mathbb{B}_n(x,r) = \{y \in \mathbb{R}^n : \ |x-y| < r\}$

Ω an open subset of $\mathbb{R}^n$

$M(\Omega)$ = set of measurable functions from Ω to $\mathbb{R}$

w : $]0,+\infty[\to [0,+\infty[$ a 'weight function'

$p \in [1,+\infty[$

If $g : \ \Omega \to \mathbb{R}$ is measurable, $\rho \in]0,+\infty]$, we set

$|g|_{\rho,w,p,\Omega} \equiv \sup_{(x,r)\in\Omega\times]0,\rho[} w(r)\|g\|_{L_p(\mathbb{B}_n(x,r)\cap\Omega)}$

The generalized Morrey space with weight w and exponent p is the space

$\mathcal{M}_p^w(\Omega) \equiv \left\{g \in M(\Omega) : \ |g|_{+\infty,w,p,\Omega} < +\infty\right\}$

with the norm

$\|g\|_{\mathcal{M}_p^w(\Omega)} \equiv |g|_{+\infty,w,p,\Omega} \qquad \forall g \in \mathcal{M}_p^w(\Omega)$

The classical weights for $0 < \lambda < n/p$:

$$r^{-\lambda} \quad \forall r \in]0, +\infty[\,,$$

$$w_{\lambda,1}(r) \equiv \begin{cases} r^{-\lambda} & \text{if } r \in]0,1[\,, \\ 0 & \text{if } r \in [1,+\infty[\,, \end{cases}$$

$$w_{\lambda}(r) \equiv \begin{cases} r^{-\lambda}, & \forall r \in]0,1[\\ 1 & \forall r \in [1,+\infty[. \end{cases}$$

$\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ is the classical homogeneous Morrey space of exponents λ and p,

$\mathcal{M}_p^{w_{\lambda,1}}(\Omega)$ is the classical inhomogeneous Morrey space of exponents λ and p

• T. Mizuhara, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo (1991), 183 –189.

$$w(r) = \varphi(r)^{-1/p}(r)$$

for some $\varphi :]0, +\infty[\to]0, +\infty[$.

• E. Nakai, Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces. Math. Nachr. 166 (1994), 95 – 103.

$w(r)$ is replaced by $\varphi(x, r)^{-1/p}$ for some

$\varphi : \mathbb{R}^n \times]0, +\infty[\to]0, +\infty[$.

E. Nakai. *A characterization of pointwise multipliers on the Morrey spaces*, Sci. Math., **3** (2000), 445–454.

$$w(r) = \varphi(r)^{-1} m_n(\mathbb{B}_n(0, r))^{-1/p}$$

for some $\varphi :]0, +\infty[\to]0, +\infty[$.

• Y. Sawano J. Indones. Math. Soc. **25** (2019), 210–281.

$$w(r) = \varphi(r) m_n(\mathbb{B}_n(0,r))^{-1/p}$$

for some $\varphi :]0, +\infty[\to]0, +\infty[$.

• A. Gogatishvili, R. Mustafayev. Eurasian Math. J., **2** (2011), 134–138.

definition of today

• V. I. Burenkov, Eurasian Math. J. 3 (2012), no. 3, 11–32.

• V. I. Burenkov, II. Eurasian Math. J. 4 (2013), no. 1, 21–45.

definition of today.

Other authors reserve the word weight for a weight put on the measure in Ω as in

• N. Samko. Proc. A. Razmadze Math. Inst., **148** (2008), 51–68.

The vanishing generalized Morrey space with weight w and exponent p is the subspace

$$\mathcal{M}_p^{w,0}(\Omega) \equiv \left\{ g \in \mathcal{M}_p^w(\Omega) : \lim_{\rho \to 0} |g|_{\rho,w,p,\Omega} = 0 \right\}$$

of $\mathcal{M}_p^w(\Omega)$.

If w is not identically equal to 0, $\mathcal{M}_p^{w,0}(\Omega)$ is well known to be closed in $\mathcal{M}_p^w(\Omega)$.

Our assumptions on the weight $w :]0, +\infty[\to [0, +\infty[$

- w is not identically equal to 0

- w is decreasing

- $\lim_{r\to 0} w(r) r^{n/p} = 0$

- there exists $\rho_0 \in]0, 1]$ such that

$w(r)(r)^{n/p}$ is continuous and increasing for $r \in]0, \rho_0[$

$\exists c > 0$ such that $w(r) \leq c w(1/\alpha) w(\alpha r)$ $(*)$

for all $\alpha > 1/\rho_0$, $0 < r < \rho_0$ such that $\alpha r < \rho_0$

• All the above assumptions are satified by the classical weights with $0 < \lambda < n/p$, $p \in [1, +\infty[$.

• If Ω is bounded then

$$\mathcal{M}_p^{r^{-\lambda}}(\Omega) = \mathcal{M}_p^{w_{\lambda,1}}(\Omega) = \mathcal{M}_p^{w_\lambda}(\Omega)$$

with equivalent norms.

The analysis of T_f in Lebesgue spaces depends on whether

$$m_n(\Omega) < +\infty \qquad \text{or} \qquad m_n(\Omega) = +\infty.$$

Here for generalized Morrey spaces the analysis depends on whether

$$1 \in \mathcal{M}_p^w(\Omega) \qquad \text{or} \qquad 1 \notin \mathcal{M}_p^w(\Omega)$$

and for vanishing generalized Morrey spaces on whether

$$1 \in \mathcal{M}_p^{w,0}(\Omega) \qquad \text{or} \qquad 1 \notin \mathcal{M}_p^{w,0}(\Omega)$$

Under the assumptions of today's talk w:

$$1 \in \mathcal{M}_p^w(\Omega) \Rightarrow 1 \in \mathcal{M}_p^{w,0}(\Omega)$$

So the two of the memberships are equivalent.

The **'action problem'** of T_f:

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, $p \in [1, +\infty[$.

• If $1 \in \mathcal{M}_p^w(\Omega)$, then

$T_f[\mathcal{M}_p^w(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$ if and only if

$T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$ if and only if

$T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^{w,0}(\Omega)$ if and only

there exist a, $b \in [0, +\infty[$ such that

$|f(t)| \leq a|t| + b \qquad \forall\, t \in \mathbb{R}$, i.e. f is sub-affine

• If $1 \notin \mathcal{M}_p^w(\Omega)$, then we have

$T_f[\mathcal{M}_p^w(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$ if and only if

$T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$ if and only if

$T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^{w,0}(\Omega)$ if and only

there exists $a \in [0, +\infty[$ such that

$|f(t)| \leq a|t| \qquad \forall\, t \in \mathbb{R}$, i.e. f is sub-linear.

For the sufficiency: N. Kydyrmina & M. L. Eurasian Mathematical Journal, **7**, No. 2 (2016), pp. 50–67. [where Sobolev-Morrey spaces have been considered]

The proof of the necessity is based on a generalization of the proof for Lebesgue spaces of G. Bourdaud that exploits a Lemma of Y. Katznelson

that says that the acting condition of T_f implies a property of boundedness of T_f

on bounded sets of g's with uniformly bounded (small) support and with small norm.

The problem of **uniform continuity** of T_f:

• characterize those Borel functions $f : \mathbb{R} \to \mathbb{R}$ such that T_f is uniformly continuous.

• If $1 \in \mathcal{M}_p^w(\Omega)$, then

$T_f : \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is uniformly continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is uniformly continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is uniformly continuous if and only if

$f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

[uniformly continuous functions are always sub-affine]

And how about case $1 \notin \mathcal{M}_p^w(\Omega)$?

Here the answer is more surprizing:

If $1 \notin \mathcal{M}_p^w(\Omega)$ and if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is uniformly continuous,

then f is Lipschitz continuous and $f(0) = 0$.

On the other hand we shall see that the Lipschitz continuity of f and $f(0) = 0$ is sufficient for the Lipschitz continuity of T_f.

The problem of α**-Hölder continuity** of T_f for $\alpha \in]0,1[$:

Here the point is that

• If $T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w}(\Omega)$ is α-Hölder continuous

then $|f|_\alpha \leq \|\chi_E\|_{\mathcal{M}_p^w(\Omega)}^{\alpha-1} |T_f|_\alpha$

for all measurable subsets E of Ω of finite nonzero measure.

In particular, f is α-Hölder continuous.

If f is not constant, then $1 \in \mathcal{M}_p^{w,0}(\Omega)$

and $|f|_\alpha \|1\|_{\mathcal{M}_p^w(\Omega)}^{1-\alpha} \leq |T_f|_\alpha$

• If f is Borel measurable but NOT constant, then

$T_f : \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is α-Hölder continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is α-Hölder continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is α-Hölder continuous if and only if

$f : \mathbb{R} \to \mathbb{R}$ is α-Hölder continuous and $1 \in \mathcal{M}_p^{w,0}(\Omega)$.

• If the above equivalent conditions hold,

then $|T_f|_\alpha \leq |f|_\alpha \|1\|_{\mathcal{M}_p^w(\Omega)}^{1-\alpha}$

The problem of **Lipschitz continuity** of T_f:

• If $1 \in \mathcal{M}_p^w(\Omega)$, then

$T_f : \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is Lipschitz continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is Lipschitz continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is Lipschitz continuous if and only if

$f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

• If $1 \notin \mathcal{M}_p^w(\Omega)$, then

$T_f : \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is Lipschitz continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is Lipschitz continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is Lipschitz continuous if and only if

$f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $f(0) = 0$.

The problem of **continuity** of T_f:

Here unfortunately we have only sufficient conditions and necessary conditions.

A necessary condition for continuity:

• If $T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is continuous,

then f is continuous and there exist a, $b \in [0, +\infty[$ such that

$|f(t)| \leq a|t| + b \qquad \forall\, t \in \mathbb{R}$

• If $1 \notin \mathcal{M}_p^w(\Omega)$ and if $T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is continuous,

then f is continuous and there exists $a \in [0, +\infty[$ such that

$|f(t)| \leq a|t| \qquad \forall\, t \in \mathbb{R}$

A sufficient condition for continuity:

If $m_n(\Omega) < +\infty$ (a case in which $1 \in \mathcal{M}_p^{w,0}(\Omega)$) and if

$$c_f \equiv \sup \left\{ \frac{|f(x)-f(y)|}{1+|x-y|} \,:\, x, y \in \mathbb{R} \right\} < +\infty \,,$$

then

$T_f \,:\, \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is continuous.

As shown in L & Bourdaud and Sickel (2002) condition $c_f < +\infty$ is a necessary and sufficient condition for the action of T_f in $BMO(\mathbb{R}^n)$.

A sufficient condition for continuity in generalized vanishing Morrey spaces:

• If f is continuous and if there exists $a \in [0, +\infty[$ such that

$|f(t)| \leq a|t| \qquad \forall\, t \in \mathbb{R}$, then

T_f :

$$(\mathcal{M}_p^{w,0}(\Omega) \cap L_p(\Omega), \|\cdot\|_{\mathcal{M}_p^w(\Omega)\cap L_p(\Omega)}) \to \mathcal{M}_p^{w,0}(\Omega).$$

is continuous

• If $1 \in \mathcal{M}_p^{w,0}(\Omega)$, f is continuous and if there exist a, $b \in [0, +\infty[$ such that

$|f(t)| \leq a|t| + b \qquad \forall\, t \in \mathbb{R}$, then

T_f :

$$(\mathcal{M}_p^{w,0}(\Omega) \cap L_p(\Omega), \|\cdot\|_{\mathcal{M}_p^w(\Omega)\cap L_p(\Omega)}) \to \mathcal{M}_p^{w,0}(\Omega).$$

is continuous.

A necessary and sufficient condition for continuity in generalized vanishing Morrey spaces

under the special assumption

$\eta_w \equiv \inf_{r\in]0,+\infty[} w(r) > 0$

• If $1 \in \mathcal{M}_p^{w,0}(\Omega)$, then

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w}(\Omega)$ is continuous if and only if

f is continuous and there exist $a, b \in [0, +\infty[$ such that

$|f(t)| \leq a|t| + b \qquad \forall\, t \in \mathbb{R}$.

Unfortunately the only classical weight for which

$\eta_w > 0$ is $w_\lambda(r) \equiv \begin{cases} r^{-\lambda}, & \forall r \in]0,1[\\ 1 & \forall r \in [1,+\infty[. \end{cases}$

If Ω is bounded:

$$\mathcal{M}_p^{r^{-\lambda}}(\Omega) = \mathcal{M}_p^{w_{\lambda,1}}(\Omega) = \mathcal{M}_p^{w_\lambda}(\Omega)$$

with equivalent norms

under the special assumption $\eta_w > 0$:

• If $1 \notin \mathcal{M}_p^{w,0}(\Omega)$, then

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w}(\Omega)$ is continuous if and only if

f is continuous and there exists $a \in [0, +\infty[$ such that

$$|f(t)| \leq a|t| \qquad \forall\, t \in \mathbb{R}$$

THANK YOU FOR YOUR ATTENTION!

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