Toeplitz operators on quotient domains

E. K. Narayanan

Department of Mathematics, Indian Institute of Science, Bangalore

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Joint work with Gargi Ghosh

Hardy Space

Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \, r^n \, e^{in\theta} \quad z = re^{i\theta}$$

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 \, r^{2n}$$

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty \iff \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

Hardy Space

Hardy space:

$$H^2(\mathbb{D}) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$g \in L^2(\mathbb{T}), g(e^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n \, e^{in\theta}$$ Fourier series

$$||g||_2^2 = \sum_{n=-\infty}^{\infty} |b_n|^2 < \infty$$

$$\mathbb{H}^2(\mathbb{D}) \subset L^2(\mathbb{T})$$ closed subspace

Toeplitz operators

$P : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ Orthogonal projection

Let $u \in L^\infty(\mathbb{T})$, the Toeplitz operator

$$T_u : H^2(\mathbb{D}) \to H^2(\mathbb{D})$$

$$T_u f = P(uf)$$

u is called the symbol of the Toeplitz operator.

Easy to see: If $T_u = 0$ then $u = 0$.

Brown-Halmos

A. Brown and P. R. Halmos initiated the study of Toeplitz operators in 1963-1964.

Theorem

$T_u T_v = T_\psi$ if and only if either $\overline{u}$ or v is in $H^\infty(\mathbb{D})$, in which case $\psi = uv$

Important consequence is the zero product theorem:
If $T_u T_v = 0$ then either u or v is identically zero

Halmos also asked, suppose $T_{u_1} T_{u_2} \cdots T_{u_k} = 0$, must some $u_j = 0$?
Yes! Proved by A. Alemanand D. Vukotić (2009, Duke Math J)

Bergman Space

$$\mathbb{A}^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ hol } \int_{\mathbb{D}} |f(z)|^2 \, dV(z) < \infty \right\}$$

where $dV(z)$ is the two dimensional Lebesgue measure on $\mathbb{R}^2$.

$\mathbb{A}^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$.

Toeplitz operators on Bergman spaces

$\mathbb{A}^2(\mathbb{D}) \subset L^2(\mathbb{D})$ $\quad$ $\mathbb{A}^2(\mathbb{D})$ closed subspace

$$P : L^2(\mathbb{D}) \to \mathbb{A}^2(\mathbb{D}), \quad \text{Bergman projection}$$

For $u \in L^\infty(\mathbb{D})$, Toeplitz operator $T_u : \mathbb{A}^2(\mathbb{D}) \to \mathbb{A}^2(\mathbb{D})$ by

$$T_u f = P(uf).$$

Theorem (Ahern-Cuckovič (2001), Ahern(2004))

If u and v are bounded harmonic functions on $\mathbb{D}$ such that $T_u T_v = 0$ then u or v is zero.

Zero product problem is open on $\mathbb{A}^2(\mathbb{D})$ in its full generality even now.

Higher dimensional cases

Let Ω be a bounded domain in $\mathbb{C}^d$.
Polydisc $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ and the unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$, are important examples of domains.
Bergman space $\mathbb{A}^2(\Omega)$: closed subspace of holomorphic functions in $L^2(\Omega)$.
$P : L^2(\Omega) \to \mathbb{A}^2(\Omega)$ orthogonal projection

If $u \in L^\infty(\Omega)$, the Toeplitz operator $T_u : \mathbb{A}^2(\Omega) \to \mathbb{A}^2(\Omega)$ is defined by

$$T_u f = P(uf), \; f \in \mathbb{A}^2(\Omega).$$

We will assume that holomorphic polynomials are dense in $\mathbb{A}^2(\Omega)$. One may also consider weighted Bergman spaces and Toeplitz operators on them.

Zero product-polydisc

B. R. Choe, H. Koo,Y. J. Lee proved (2007)

Theorem

Suppose u and v are bounded $d-$harmonic functions in $\mathbb{D}^d$, continuous on $\mathbb{D}^d \cup W$ for some relatively open subset W of $\mathbb{T}^d$. If $T_u T_v = 0$ then either $u = 0$ or $v = 0$.

Similar result holds for the unit ball $\mathbb{B}^d$ in $\mathbb{C}^n$ too.

Pseudoreflections and Quotient domains

Pseudoreflection: $\sigma \in GL_d(\mathbb{C})$ has finite order in $GL_d(\mathbb{C})$ and $(\sigma - I)$ has rank one

Pseudoreflection group G : finite group generated by pseudoreflections
Examples: finite cyclic groups, symmetric group, dihedral groups

$\Omega \subset \mathbb{C}^d$ be a bounded domain.
If Ω is G-invariant, form the quotient domain Ω/G.

Examples

- Symmetrized polydisc $\mathbb{G}_d = \{(\mathbf{s}_1(z), \cdots \mathbf{s}_d(z))\}$ where $\mathbf{s}_j$ is the j-th symmetric function Here $\Omega = \mathbb{D}^d$, $G =$ the symmetric group, $\Omega/G = \mathbb{G}_d$
- Rudin's domains: $\mathbb{B}^d/G$
- Monomial polyhedrons

Chevalley-Shephard-Todd theorem

Theorem (CST)

The invariant ring $\mathbb{C}[z_1, z_2, \cdots z_d]^G$ is equal to $\mathbb{C}[\theta_1, \theta_2, \cdots \theta_d]$ where θ_i are algebraically independent homogenous polynomials if and only if G is a pseudoreflection group

Example: Symmetric polynomials in d-variables are polynomials in $\mathbf{s}_1, \mathbf{s}_2, \cdots \mathbf{s_d}$.

The map $\boldsymbol{\theta} : \mathbb{C}^d \to \mathbb{C}^d$, $\boldsymbol{\theta}(z) = \big(\theta_1(z), \cdots \theta_d(z)\big)$ is called a basic polynomial map associated to G.

Quotient domains

Let $\Omega \subset \mathbb{C}^d$ be a G-invariant domain where G is a finite pseudoreflection group. It is known that

- $\boldsymbol{\theta}(\Omega)$ is a domain
- $\boldsymbol{\theta} : \Omega \to \boldsymbol{\theta}(\Omega)$ is proper
- Ω/G is biholomorphic to $\boldsymbol{\theta}(\Omega)$

We study Toeplitz operators on $\mathbb{A}^2(\boldsymbol{\theta}(\Omega))$

Analytic CST Theorem

G:finite pseudoreflection group
$\Omega \subset \mathbb{C}^d$, a G-invariant bounded domain.
$\widehat{G}$-irreducible representations of G

The polynomials ring $\mathbb{C}[z_1, \cdots z_d]$ decomposes under the G-action

$$\mathbb{C}[z_1, \cdots z_d] = \oplus_{\rho \in \widehat{G}} \mathcal{R}_\rho$$

$\mathcal{R}_\rho$: ρ-isotypic component
$\mathcal{R}_\rho$ is module over the ring $\mathbb{C}[z_1, \cdots z_d]^G$
$\mathcal{R}_\rho$ is generated by $(deg\rho)^2$ homogenous polynomials $\{\ell_{\rho,j} : \ 1 \leq j \leq (deg\rho)^2\}$.

Analytic CST Theorem

Analytic version of CST proved by S. Biswas, S. Datta, G. Ghosh and S. Shyam Roy (Advances in Math 2022)

Theorem

For every holomorphic function f on Ω there exists unique G-invariant holomorphic functions $f_{\rho,j}$ such that

$$f = \sum_{\rho \in \widehat{G}} \sum_{j=1}^{(deg\rho)^2} f_{\rho,j} \, \ell_{\rho,j}$$

sgn representation

$\mathrm{sgn} : G \to \mathbb{S}^1$, $\mathrm{sgn}(g) = \det(g)^{-1}$. The homogenous polynomial $\ell_{\mathrm{sgn}}(z) = J_\theta(z)$ where $J_\theta(z)$ is the determinant of the complex Jacobian.

$$\mathbb{A}^2(\Omega) = \oplus_{\rho \in \widehat{G}} \mathbb{A}^2_\rho(\Omega)$$

Theorem (G. Ghosh)

The map $\Gamma : \mathbb{A}^2(\theta(\Omega)) \to \mathbb{A}^2_{\mathrm{sgn}}(\Omega)$ *by*

$$\Gamma f(z) = \frac{1}{\sqrt{|G|}} f \circ \theta(z)\, J_\theta(z) \; z \in \Omega$$

is unitary.

$\Omega = \mathbb{D}^d, \theta(\Omega) = \mathbb{G}_d, J_\theta(z) = \prod_{i<j}(z_i - z_j)$

zero product theorem on $\theta(\Omega)$

u, v be bounded functions on $\theta(\Omega)$
$\widetilde{u} = u \circ \theta, \widetilde{v} = v \circ \theta$ bounded on Ω
Note that $\widetilde{u}$ and $\widetilde{v}$ are G-invariant
T_u, T_v Toeplitz operators on $\mathbb{A}^2(\theta(\Omega))$ and
$T_{\widetilde{u}}, T_{\widetilde{v}}$ on $\mathbb{A}^2(\Omega)$

Theorem (zero product)

$T_u T_v = 0$ on $\mathbb{A}^2(\theta(\Omega))$ if and only if $T_{\widetilde{u}} T_{\widetilde{v}} = 0$ on $\mathbb{A}^2(\Omega)$. Consequently, zero product theorem on $\mathbb{A}^2(\Omega)$ implies zero product theorem on $\mathbb{A}^2(\theta(\Omega))$

In particular, zero product theorem on the polydisc implies a zero product theorem on the symmetrized polydisc.

Sketch of the proof

- $\mathbb{A}^2(\Omega) = \mathbb{A}^2_{\text{trivial}}(\Omega) \oplus \cdots \oplus \mathbb{A}^2_{\text{sgn}}(\Omega), \quad \mathbb{A}^2_{\text{sgn}} \sim \mathbb{A}^2(\boldsymbol{\theta}(\Omega))$

- $T_u T_v = 0$ on $\mathbb{A}^2(\boldsymbol{\theta}(\Omega))$ implies $T_{\widetilde{u}} T_{\widetilde{v}} = 0$ on $\mathbb{A}^2_{\text{sgn}}(\Omega)$ via Γ map

- Analytic CST and G-invariance of $\widetilde{u}$ and $\widetilde{v}$ implies $T_{\widetilde{u}} T_{\widetilde{v}} = 0$ on all of $\mathbb{A}^2(\Omega)$

Thank you!