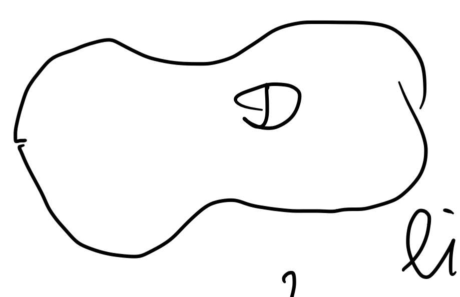
Semiclassical Asymptotics on Stratified Manifolds

Shallow water eq.



$D(x) \in C^\infty(\mathcal{D})$
$D(x) = 0$ on $\partial\mathcal{D}$
$\nabla D(x) \neq 0$ on $\partial\mathcal{D}$

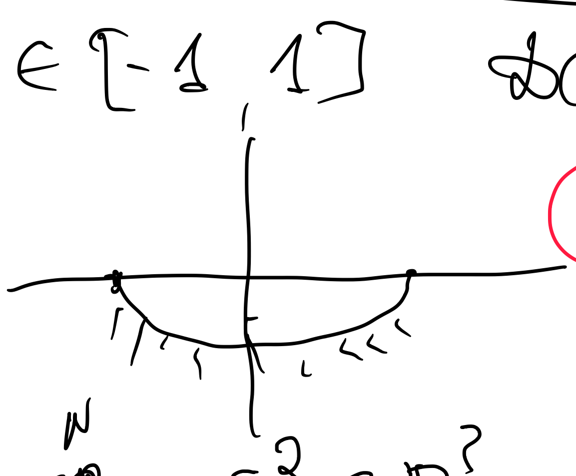
linear appr.

$$\begin{cases} -\dfrac{\partial^2\eta}{\partial t^2} + \nabla(D(x)\nabla\eta) = 0 \\ \eta|_{t=0} = \eta_0 \\ \eta_t|_{t=0} = \eta_1 \end{cases}$$

η surface elevation

$x \in \mathbb{R}^3$ $\left(-h^2\Delta_x - \dfrac{1}{|x|}\right)\psi = \lambda\psi$ $x = 0$

SWE $x \in [-1, 1]$ $\mathcal{D}(x) = 1 - x^2$



$P\eta = \lambda\eta$

$P = -\dfrac{d}{dx}(1 - x^2)\dfrac{d}{dx}$

$S^2 \subset \mathbb{R}^3$ $G = S^1$

$G \backslash S^2 = [-1, 1]$

$C^\infty(S^2) \supset C^\infty(S^2)^G = C^\infty([-1, 1])$

$Q : C^\infty(S^2) \to C^\infty(S^2)$

Q commutes with G-action

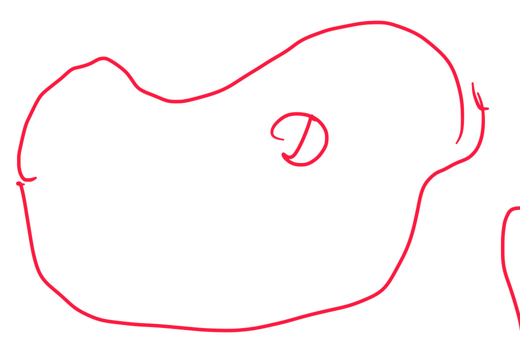
$\downarrow$

$\tilde{Q} : C^\infty([-1, 1]) \circlearrowleft$

$Q = -\Delta$ Laplace operator on S^2

$\hat{Q} = P$

$Q\psi = \lambda\psi$
ψ is G-invariant



$\mathcal{D} = \mathbb{S}^1 \backslash M$ dim M

$\mathbb{R}^4 \ni (u_1, u_2, u_3, u_4)$

$z_1 = u_1 + iu_2$, $z_2 = u_3 + iu_4$

$\mathbb{R}^4 \to \mathbb{R}^3 \ni (x_1, x_2, x_3)$

$$\boxed{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} z_1\bar{z}_2 \\ |z_1|^2 - |z_2|^2 \end{pmatrix}}$$

(restriction to $\mathbb{S}^3$

$\mathbb{S}^3 \xrightarrow{\mathbb{S}^1} \mathbb{S}^2$ Hopf Bundle

$\mathbb{S}^1 : \mathbb{R}^4$

$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} e^{i\varphi}z_1 \\ e^{i\varphi}z_2 \end{pmatrix}$

$\left(-h^2\Delta_x - \dfrac{1}{|x|}\right)\psi = \lambda\psi$

(*) $\underline{\left(-h^2|x|\Delta_x - \lambda|x|\right)}\psi = \psi$

$$\boxed{\hat{P} = \frac{1}{4}\left(-h^2\Delta_u - \lambda u^2\right)}$$

General construction

M C^∞-mfld w/o ∂.

G compact Lie group

$G : M$ $(g, m) \longmapsto gm$

$X = G \backslash M$ the orbit space.

$$\left.\begin{array}{l} C^\infty(X) = (C^\infty(M))^G \\ H^s(X) = (H^s(M))^G \\ L^2(X) = (L^2(M))^G \end{array}\right\} \mathcal{F}(X) \text{ general funct. spcs}$$

(h)-ΨDO on M $\hat{\mathcal{P}} : C^\infty(M) \circlearrowleft$ $\hat{\mathcal{P}}$ is G-invariant

$\hat{P} = \hat{\mathcal{P}}|_{C^\infty(X)} =$ h-ΨDO on X

$\hat{P}\psi = \lambda\psi$ asymptotic solutions

$\hat{\mathcal{P}} = \mathcal{H}\left(x, -ih\dfrac{\partial}{\partial x}\right)$ $h \to +0$

Maslov's canonical operator

$\mathcal{K}_{(\Lambda, d\mu)}$ canonical operator

$\Lambda \subset T^*M$

gives G-invariant functions?

$\Lambda \subset T^*M$ Lagrangian mfld

$d\mu$ - volume form on Λ

$K = K_{(\Lambda, d\mu)} : C_0^\infty(\Lambda) \to C_h^\infty(M)$

Two commutation formulas

1) $\hat{\mathcal{H}}KA = K(H_0|_\Lambda \cdot A) + O(h)$

H_0 principal symbol of $\hat{\mathcal{H}}$

2) $H_0|_\Lambda = 0$ ~~(or const)~~

$$\boxed{\hat{\mathcal{H}}KA = -ihK(\Pi A) + O(h^2)}$$

$\Pi = V(H_0) + Q$ $Q = 0$ function

Hamiltonian vector field

G is connected

$f \in C^\infty(M)$ is G-invariant

$\Updownarrow$

$v_\xi f = 0$ for any $\xi \in \mathfrak{g}$ Lie algebra

$-ihv_\xi = \hat{H}_\xi$

$H_\xi = \omega^1(v_\xi)$ $\omega^1 = p\,dx$ on T^*M

$0 = \hat{H}_\xi KA = K(H_\xi|_\Lambda A) + O(h)$

$H_\xi = 0$ $\forall \xi \in \mathfrak{g}$

<u>moment map</u> $\pi : T^*M \to \mathfrak{g}^*$

$\pi(x, p)(\xi) = H_\xi(x, p)$

$\Lambda \subset \pi^{-1}(0)$ $\underline{\Lambda \text{ is } G\text{-invariant}}$

$G : T^*M$

Weinstein-Marsden
Symplectic reduction

$L = G \backslash \Lambda \subset \Phi = T^*M // G$

$\tilde{\Phi} =$ the set of regular points in $\pi^{-1}(0)$

$\Phi = G \backslash \tilde{\Phi}$

$\hat{H}_\xi KA = -ihK(\underline{V(H_\xi)A}) + O(h^2)$

~~A must be G-invariant~~

A is a function on L