ORTHOGONAL POLYNOMIALS.FOURIER SERIES.TRACE FORMULA

B.P.Osilenker

Seminar on Analysis, Differential Equations and Mathematical Physics

Rostov 14.XII.2023

The following issues will be outlined:

1. Systems of the continual-discrete Sobolev polynomials $\{\hat{q}_n(x)\}(\mathrm{n} \in \mathbb{Z}_+)$ and polynomials $\{p_n(x)\}$ $(\mathrm{n} \in \mathbb{Z}_+)$ with asymptotically N-periodic coefficients.

2. Trace formula and asymptotics of Forsythe's determinant for the continual-discrete Sobolev polynomials.

3. Trace formula and asymptotics of Turan's determinant for polynomials

   $\{p_n(x)\}$ $(\mathrm{n} \in \mathbb{Z}_+)$.

4. Fourier series on system $\{\hat{q}_n(x)\}$.

5. On multipliers of the Fourier Series in polynomials orthogonal in continuous-discrete Sobolev space

Orthogonal polynomials

Suppose θ is a positive Borel measure on a compact set of the real line. There is unique sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$

$$p_n(x) = k_n x^n + l_n x^{n-1} + \cdots, k_n > 0(\mathrm{n} \in \mathbb{Z}_+ = \{0,1,2,\ldots\})$$

such that

$$\int p_m(x) p_n(x) d\theta(x) = \delta_{m,n}\ (m, \mathrm{n} \in \mathbb{Z}_+).$$

These orthonormal polynomials satisfy a three -term recurrence relation(TTRR)

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)(\mathrm{n} \in \mathbb{Z}_+),$$

$$p_{-1}(x) = 0,\ \ p_0(x) = c,\ \ a_0 = 0,$$

where

$$a_n = \frac{k_n}{k_{n+1}} > 0, \qquad b_n = \frac{l_n}{k_n} - \frac{l_{n+1}}{k_{n+1}} \in \mathbb{R}\ (\mathrm{n} \in \mathbb{Z}_+).$$

Since the support of measure θ is compact, the recurrence coefficients a_{n+1} и b_n are bounded.

This TTRR can be written in matrix form

$$J\begin{bmatrix}p_0\\ \vdots\\ p_n\end{bmatrix} = x\begin{bmatrix}p_0\\ \vdots\\ p_n\end{bmatrix},$$

Jacobi matrix

$$J = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_1 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_2 & a_3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Conversely, by Favard's Theorem if $p_n(x)$ are given by the recursion relation with $a_{n+1} > 0$ and $b_n \in \mathbb{R}$, then there exists a positive Borel measure θ such that $p_n(x)(n \in \mathbb{Z}_+)$ is an orthonormal polynomial system with respect to the measure θ. If a_{n+1} and b_n are bounded, then the measure θ is unique and the support of θ is compact.

Using recursion one can calculate polynomial system $\{p_n(x)\}_{n=0}^{\infty}$.

If (P. Nevai class $\mathfrak{M}$)

$$\lim_{n\to\infty} a_n = \frac{1}{2}, \quad \lim_{n\to\infty} b_n = 0,$$

then the support of measure θ is $[-1,1] \cup S$, where S is a finite or infinite number of real mass points out of [-1,1].

Если $\theta'(x) > 0$ almost everywhere [-1,1], then associated Jacobi matrix belongs to class $\mathfrak{M}$ (E.A.Rakhmanov,1982).

Continual-discrete Sobolev polynomials

Let μ be a finite positive Borel measure on the interval [-1,1] with infinitely many points at the support and let the points $a_k, a_k \in \mathbb{R}$, k=1,2,…,m. For f and g in $L^2_\mu([-1,1])$ such that there exist the derivatives in $a_к$, we can introduce the inner product

$$\langle f,g\rangle := \int_{-1}^{1} f(x)g(x)d\mu(x) + \sum_{k=1}^{m}\sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k) g^{(i)}(a_k),$$

where $M_{k,i} \geq 0, M_{k,N_k} > 0 (i = 0,1,2, \dots, N_k; k = 1,2, \dots, m), \mu(\{a_k\}) = 0$, $\mu'(x) > 0$ $a.e.$

Linear spaces with this inner product is called a "**continual-discrete Sobolev spaces".**

Let $\{\hat{q}_n(x), n \in \mathbb{Z}_+ ; x \in [-1,1]\}$ be the sequence of polynomials of degree n with a positive leading coefficients orthonormal with respect to this inner product

$$< \hat{q}_n, \hat{q}_m > = \delta_{n,m} \quad (n,m\in\mathbb{Z}_+) .$$

$Let\ N_k^*$ be the positive integer number defined by

$$N_k^* = \{\begin{matrix} N_k + 1, & if & N_k\ is\ odd, \\ N_k + 2, & if & N_k\ is\ even \end{matrix} ,$$

$$w_N(x) := \prod_{k=1}^{m}(x - a_k)^{N_k^*}, N = \sum_{k=1}^{N} N_k^*, \ ; \pi_{N+1}(x) = \int_{-1}^{x} w_N(t)dt .$$

Orthonormal polynomials $\hat{q}_n(x)$ satisfy the following recurrence relations

$$w_N(x)\hat{q}_n(x) = \sum_{j=0}^{N} d_{n+j,j}\, \hat{q}_{n+j}(x) + \sum_{j=1}^{N} d_{n,j}\hat{q}_{n-j}(x)$$

and

$$\pi_{N+1}(x)\hat{q}_n(x) = \sum_{j=0}^{N+1} d_{n+j,j}\, \hat{q}_{n+j}(x) + \sum_{j=1}^{N+1} d_{n,j}\hat{q}_{n-j}(x)$$

$$(n\in \mathbb{Z}_+; \hat{q}_{-j} = 0, j = 1,2, \dots ; d_{n,s} = 0, n = 0,1, \dots , s - 1).$$

Example. Discrete symmetric Gegenbauer-Sobolev polynomials

$<f,g>_\alpha = \int_{-1}^{1} f(x)g(x)\omega_\alpha(x)dx + M[f(1)g(1) + f(-1)g(-1)] + N[f'(1)g'(1)+f'(1)g'(-1)] (M\geq 0, N\geq 0)$,

$$\omega_\alpha(x) = (1 - x^2)^\alpha (\alpha > -1)$$

$$\{\hat{q}_n^{(\alpha)}(x) \equiv \hat{q}_n^{(\alpha)}(x; M, N)\}(n \in \mathbb{Z}_+, x \in [-1,1])$$

$$<\hat{q}_n^{(\alpha)}, \hat{q}_m^{(\alpha)}>_\alpha = \delta_{n,m} \ (n,m\in\mathbb{Z}_+).$$

They have been introduced :H.Bavinck, Y.J.Meijer (1989-1990) and have been investigated the following authors:Marcellán F.,W.Van Assche, Foulquié Moreno A.,Koekoek R.,Koekoek J.,Arvesu J.,R.Alvarez-Nodarse Osilenker B.P. and so on.

Some of their properties differ from the properties of classical Gegenbauer polynomials $p_n^\alpha(x)$:

1. For n large enough,the orthogonal polynomials $\hat{q}_n^{(\alpha)}(x; M, N), N > 0,$ positive exactly (n-2)different ,real and simple zeros belonging to the interval (-1,1); the two remainder zeros are outside of the interval being one positive and the other one negative. All roots of $p_n^\alpha(x)$ in $(-1,1)$.

2. $p_n^\alpha(x)$ are eigenfunctions of the differential operators of a second-order. Polynomials $\hat{q}_n^{(\alpha)}(x; M, N)$ are eigenfunctions of the linear differential operator usually infinite degree.

$If\ \alpha = 0,1,2, \ldots M > 0, N > 0$: degree $4\alpha + 10$;M>0,N=0: 2α+4; M=0,N>0 : 2α+8 ; this degrees are least.

3. $|\hat{q}_m^{(\alpha)}(\pm 1)| \approx n^{-\alpha-\frac{3}{2}},|$ $(|\, p_n^\alpha(\pm 1)| \approx n^{\alpha+\frac{1}{2}}, \alpha > -\frac{1}{2})$ behaviour at the ends

(Difference in n^2).

4. $(x^3 - 3x)\hat{q}_n^{(\alpha)}(x) = a_{n+3}\hat{q}_{n+3}^{(\alpha)}(x) + b_{n+1}\hat{q}_{n+1}^{(\alpha)}(x) + b_n\hat{q}_{n-1}^{(\alpha)}(x) + a_n\hat{q}_{n-3}^{(\alpha)}(x)$ $(n \in \mathbb{Z}_+;\ \hat{q}_{-s}^{(\alpha)}(x)=0,\ s=1,2,\ldots;\ a_n = 0, n = 0,1,2; b_0 = 0)$,

$$a_n = \frac{1}{8} + \frac{C_1}{n} + O\left(\frac{1}{n^2}\right),\quad b_n = -\frac{9}{8} + \frac{C_2}{n} + O\left(\frac{1}{n^2}\right).$$

This recurrence relation has the lowest order.

Polynomials $p_n^\alpha(x)$ satisfy TTRR with

$a_n = \frac{1}{2} + O\left(\frac{1}{n^2}\right), b_n = O\left(\frac{1}{n^2}\right)$ $(n \to \infty)$.

Orthogonal polynomials with asymptotically N-periodic recurrence coefficients (AP_N)

Let $\{p_n(x)\}_{n=0}^\infty be\ a$ sequence of polynomials, θ is a measure on a compact set of the real line

$$\int p_m(x)p_n(x)d\theta(x) = \delta_{m,n}\ (m, n \in \mathbb{Z}_+).$$

These orthonormal polynomials satisfy a three -term recurrence relation

$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x)(n \in \mathbb{Z}_+)$,

$$p_{-1}(x) = 0,\ \ p_0(x) = 1,\ \ a_0 = 0.$$

We assume that two periodic sequences $a_{n+1}^0 > 0$ and $b_n^0 \in \mathbb{R}(n \in \mathbb{Z}_+)$ are given such that

$$a_{n+N}^0 = a_n^0(n = 1,2, \ldots), b_{n+N}^0 = b_n^0(n \in \mathbb{Z}_+)$$

(here N≥ 1 is the period), and that the recurrence coefficients a_{n+1} and b_n

satisfy

$$\lim_{n\to\infty} |a_n - a_n^0| = 0, \lim_{n\to\infty} |b_n - b_n^0| = 0.$$

We say that these orthonormal polynomials have *asymptotically N-periodic recurrence coefficients, or that the orthogonal polynomials* $\{p_n(x)\}(n \in \mathbb{Z}_+)$ *belong to the class* AP_N.

This class has been studied by many authors.

Denote the orthonormal polynomials with periodic recurrence coefficients a_n^0 and b_n^0 by $\{q_n(x)\}(n \in \mathbb{Z}_+)$. Then

$$xq_n(x) = a_{n+1}^0 q_{n+1}(x) + b_n^0 q_n(x) + a_n^0 q_{n-1}(x) \ (n \in \mathbb{Z}_+),$$

$$q_{-1}(x) = 0, q_0(x) = 1, a_0^0 = 0.$$

The k-associated polynomials of order k $\{q_n^{(k)}(x)\}(n \in \mathbb{Z}_+, k \in \mathbb{N})$ satisfy the "shifted recurrence relation"

$$xq_n^{(k)}(x) = a_{n+k+1}^0 q_{n+1}^{(k)}(x) + b_{n+k}^0 q_n^{(k)}(x) + a_{n+k}^0 q_{n-1}^{(k)}(x) (n \in \mathbb{Z}_+)$$

with

$$q_{-1}^{(k)}(x) = 0, q_0^{(k)}(x) = 1.$$

Define

$$\omega^N(x) = \rho(T(x)),$$

where

$$T(x) = \frac{1}{2}\{q_N(x) - \frac{a_N^0}{\boldsymbol{a_{N+1}^0}} q_{N-2}^{(1)}(x)\}$$

and

$$\rho(x) = x + \sqrt{x^2 - 1}.$$

Let us introduce

$$E = \{x \in [-1,1], |\omega^N(x)| = 1\}.$$

Then E consists of N intervals, where $-1 \le T(x) \le 1$, E corresponds to the essential spectrum of the orthonormal polynomials $p_n(x)$, and the measure θ of these orthonormal polynomials has support $E \cup E^*$, where $E^*$is a denumerable set, for which the accumulation points are on E.

Besides being of interest in its own right(from Function theory to algebraic geometry) orthogonal polynomials on several intervals is a topic of interest in numerical analysis, and in quantum chemistry.

In a survey , P. Nevai set up a problem: Extend the Trace formula to orthogonal polynomials on several intervals.

A solution of this problem is given in the particular case N=2 (Osilenker B.P.,1992)

Lemma 1 (J.S. Geronimo **,**W.Van Assche,1991 **)**If the orthonormal polynomial system $\{\boldsymbol{p_n}\}(n \in \mathbb{Z}_+)$ belongs to the class AP_N and the recurense coefficients $\{a_{n+1}, b_{\boldsymbol{n}}\}(n \in \mathbb{Z}_+)$ satisfy

$\sum_{n=0}^{\infty}(|a_{n+1} - a_{n+N+1}|+|b_n - b_{n+N}|) < \infty,$

$(N - bounded\ variation)$

then the measure θ is absolutely continuous in

$$E_0 := E\backslash\{\omega^{2N}(x) = 1\},$$

the weight function $\omega(x) = \theta'(x)$ is strictly positive and continuous in E_0.

Lemma 2 [Osilenker B.P., 1998) Suppose that the system of orthonormal polynomials $\{\boldsymbol{p_n}\}(n \in \mathbb{Z}_+)$ belongs to the class AP_N. Then for all $x \in E$, the recurrence relation

$$\pi_N(x)p_n(x) = \sum_{j=0}^{N} d_{n+j}^{(j,N)} p_{n+j}(x) + \sum_{j=0}^{N} d_n^{(j,N)} p_{n-j}(x)$$

holds with the boundary conditions

$$p_{-s}(x) = 0(s = 1,2, \dots), \quad d_n^{(j,N)} = 0(n = 0,1,2, \dots, j - 1),$$

where

$$\pi_N(x) := 2T(x) \prod_{k=1}^{N} a_k^0$$

with

$$\lim_{n\to\infty} d_{n+j}^{(j,N)} = \lim_{n\to\infty} d_n^{(j,N)} = 0 \ (j = 0,1, \dots, N - 1)$$

and

$$\lim_{n\to\infty} d_{n+N}^{(N,N)} = \lim_{n\to\infty} d_n^{(N,N)} = \prod_{k=1}^{N} a_k^0.$$

Generalized Trace formula and asymptotics of the Forsythe(Turan's) determinant

Trace Formula

This subject is very popular. There are many papers devoted this subject. Authors:I.M.Gelfand,B.M.Levitan,B.Simon,V.A.Sadovnichii, Aptekarev A.I., P.Nevai, Suetin S.P., Gesztesy G., Teschl G., J.Dombrowski, Osilenker B.P.,

and so on. I will consider this formula in the following direction.

Theorem A (J.Dombrowski, A,$Máté$,P.Nevai,1986) If the recurrence coefficients of the orthonormal polynomial system $\{p_n(x)\}(n \in \mathbb{Z}_+)$ belongs to the class $\mathfrak{M}$ and

$$\sum_{n=0}^{\infty}(|a_n - a_{n+1}| + |b_{n+1} - b_n|) < \infty,$$

("bounded variation"),then ("Trace formula")

$$\sum_{n=0}^{\infty}(a_{n+1}^2 - a_n^2)p_n^2(x) + a_{n+1}(b_{n+1} - b_n)p_n(x)p_{n+1}(x) = \frac{1}{2\pi}\frac{\sqrt{1-x^2}}{\omega(x)}$$

holds uniformly on all compact subsets in (-1,1). In addition, the measure θ is absolutely continuous in the open interval (-1,1), $\theta'(x) = \omega(x) > 0$ for all $x \in (-1,1)$, and $\theta'(x) = \omega(x)$ is continuous in (-1,1).

The Trace formula contains recurrence coefficients for TTRR and polynomials,which can be constructed by TTRR.

I will explain a title «Trace formula».

Let $\{e_n\}_{n=0}^{\infty}$ be orthonormal basis of a separable Hilbert space $\mathcal{H}$ and J is Jacobi

operator

$Je_n = a_{n+1}e_{n+1} + b_n e_n + a_n e_{n-1} (n \in \mathbb{Z}_+; e_{-1} = 0).$

A self-adjoint operator T in a Hilbert space $\mathcal{H}$ has a simple (Lebesgue) spectrum if there exists an element g∈ $\mathcal{H}$ (generating or cyclic), that is $T^k g$∈D(T) (k∈ $\mathbb{Z}_+$) and the closure of the linear hull of g,Tg,$T^2 g$, ... is coincide with $\mathcal{H}$.

Theorem (Stone M.H.,1930) Set of all self-adjoint bounded operators with a simple spectrum is coincided with a set of all operators generated by a bounded Jacobi matrices. Polynomials $\{p_n(x)\}(n \in \mathbb{Z}_+)$ associated with J-matrix form an orthonormal system respect to measure θ,which is a spectral measure of T , and U: $\mathcal{H} \leftrightarrow L^2_\theta$, $e_n \leftrightarrow p_n(x)$, $e_n = p_n(J)e_0$, $n \in \mathbb{N}$). U is unitary operator .

We introduced a skew-symmetric operator $\mathcal{B}$

$\mathcal{B}e_n = a_n e_{n-1} - a_{n+1}e_{n+1} (n \in \mathbb{Z}_+; e_{-1} = 0).$

Commutator

$$[J,\mathcal{B}]e_n = (J\mathcal{B} - \mathcal{B}J)e_n = 2(a_{n+1}^2 - a_n^2)e_n + 2a_{n+1}(b_{n+1} - b_n)e_{n+1}.$$

The trace of this commutator

$$\mathrm{Sp}[J,\mathcal{B}] = \sum_{n=0}^{\infty} ([J,\mathcal{B}]\, e_n, e_n) =$$

$=2\sum_{n=0}^{\infty}[(a_{n+1}^2 - a_n^2)(p_n(J)e_0, p_n(J)e_0) +$

$a_{n+1}(b_{n+1} - b_n)(p_{n+1}(J)e_0, p_n(J)e_0)] \overset{U}{\Leftrightarrow}$

$$2\sum_{n=0}^{\infty}[(a_{n+1}^2 - a_n^2)\, p_n^2(x) + a_{n+1}(b_{n+1} - b_n)p_n(x)p_{n+1}(x)].$$

Reformulating of the Theorem A.

Under the conditions of the Theorem A, the following formula (“Trace formula”)

$$\mathrm{Sp}[J,\mathcal{B}] = \frac{1}{\pi}\frac{\sqrt{1-x^2}}{\omega(x)}$$

holds uniformly on all compact subsets in (-1,1).

A commutator $[\mathrm{J}, \mathcal{B}]$ is a right-hand side of the Lax representation of semi-indefinite Toda lattice.

Tur$\acute{a}$n's determinant

They are defined by

$$T_n(x) = a_n[p_n^2(x) - p_{n-1}(x)p_{n+1}(x)\ ](\mathrm{n} \in \mathbb{Z}_+; x \in [-1,1]).$$

Theorem B (A.M$\acute{a}t\acute{e}$, P.Nevai,V.Totik,1985) Under the conditions of Theorem A, we have the following asymptotics of Tur$\acute{a}$n's determinant

$$\lim_{n\infty} T_{n+1}(x) = \frac{1}{\pi}\frac{\sqrt{1-x^2}}{\omega(x)}$$

uniformly on all compact subsets K in (-1,1).

The statements A and B were obtained by different methods. They play an important role in a problem of approximation of the orthogonality measure for orthogonal polynomials on an interval of the real line.

Our goal is to extend Theorems A and B to more generalized classes of orthogonal plynomials. We obtain analogs of Theorems A and B by a unifed method that allows to get the order of approximation.

Trace formula and asymptotics of the Forsythe(Turan's) determinant for continual-discrete systems

$$\langle \mathrm{f},\mathrm{g}\rangle := \int_{-1}^{1} \mathrm{f(x)g(x)d\mu(x)} + \sum_{\mathrm{k=1}}^{\mathrm{m}} \sum_{\mathrm{i=0}}^{\mathrm{N_k}} \mathrm{M_{k,i}} \mathrm{f}^{(\mathrm{i})}(\mathrm{a_k})\mathrm{g}^{(\mathrm{i})}(\mathrm{a_k})$$

$$\langle \hat{\mathrm{q}}_\mathrm{n}, \hat{\mathrm{q}}_\mathrm{m} \rangle = \delta_{\mathrm{n,m}}(\mathrm{n,m} \in \mathbb{Z}_+)$$

Define by $\mathcal{E}_m$=(-1,1)\ $\cup_{k=1}^{m}\{a_k\}$.

Let's put

$$\delta=\{\delta_n, \delta_n \in \mathbb{R}, \delta_n \neq 0, \lim_{n\to\infty} \delta_n = \delta \neq 0\ \}.$$

Define by

$$G_{n,r}^{(N)}(x;\delta) = \sum_{j=1}^{N+1} \sum_{k=n+1}^{n+j} d_{k,j}\,[\delta_{k-j}\hat{q}_k(\mathrm{x})\hat{q}_{\mathrm{k+r-j}}(\mathrm{x}) - \delta_k\hat{q}_{\mathrm{k-j}}(\mathrm{x})\hat{q}_{\mathrm{k+r}}(\mathrm{x})$$

δ-Forsyte's determinant

If $\delta_n = 1$ ($n \in \mathbb{Z}_+$), then

$$G_{n,r}^{(N)}(x) = \sum_{j=1}^{N+1} \sum_{k=n+1}^{n+j} d_{k,j}\, \Delta_{K,j,r}(x)$$

$$\Delta_{n,j,r}(x) = \hat{q}_n(x)\hat{q}_{n+r-j}(x) - \hat{q}_{n-j}(x)\hat{q}_{n+r}(x) = \begin{vmatrix} \hat{q}_n(x) & \hat{q}_{n+r}(x) \\ \hat{q}_{n-j}(x) & \hat{q}_{n+r-j}(x) \end{vmatrix}$$

$$(n \in \mathbb{Z}_+; j = 0,1,\dots,N+1;\ r = 0,1,2,\dots).$$

In particular, if j= r =1 we obtain the Turan's determinant $T_{n+1}(x)$ if we put $d_{n+1,1} = a_{n+1}$.

I use two approach: analog of the Diriclet kernel and the Fejer kernel and

obtain two different Trace formulas.

For a continual-discrete polynomials we have the recurrence relation

$$w_N(x)\hat{q}_n(x) = \sum_{j=0}^{N} d_{n+j,j}\, \hat{q}_{n+j}(x) + \sum_{j=1}^{N} d_{n,j}\hat{q}_{n-j}(x).$$

Theorem 1 . Suppose that

$$\sum_{j=1}^{N} \sum_{k=0}^{\infty} |\delta_k - \delta_{k+j}| < \infty.$$

Assume that the orthonormal polynomial system $\{\hat{q}_n(x)\}_{n=0}^{\infty}$ satisfies the following conditions:

(i)There exists a continuous function h(x) on $\mathcal{E}_m$ (majorant function) such that

$$|\hat{q}_n(x)| \le h(x)(x \in \mathcal{E}_m, n \in \mathbb{Z}_+).$$

(ii)The recurrence coefficients are of N-bound variation,i.e

$$\sum_{j=0}^{N} \sum_{r=0}^{N} \sum_{k=0}^{\infty} |d_{k,j} - d_{k+r,j}| < \infty.$$

(iii)The measure μ is absolutely continuous in $\mathcal{E}_m$, and $\mu'(x) = \omega(x)$

is strictly positive and continuous on $\mathcal{E}_m$.

Then the following statements are valid:

(1) Uniformly on all compact subsets K in $\mathcal{E}_m$ the asymptotics of the δ-Forsyte's determinant

$$\lim_{n\to\infty} G_{n,r}^{(N)}(x;\delta) = \frac{\delta}{\pi} U_{r-1}(x) w_N'(x) \frac{\sqrt{1-x^2}}{\omega(x)}$$

holds; an upper bound for the uniform error of approximation on all compact subsets K in $\mathcal{E}_m$ is given by

$$|G_{n,r}^{(N)}(x;\delta) - \frac{\delta}{\pi} U_{N-1}(x) w_N'(x) \frac{\sqrt{1-x^2}}{\omega(x)}| \le C\{\sum_{j=0}^{N} \sum_{k=n+1}^{\infty} |\delta_k - \delta_{k+1}| +$$

$$\sum_{j=0}^{N} \sum_{r=1}^{N} \sum_{k=n+1}^{\infty} |d_{k,j} - d_{k+r,j}|\}$$

where C>0 is positive constant independent of $n \in \mathbb{Z}_+$ and $x \in K$, $U_j(x) = \frac{\sin(j+1)\arccos x}{\sin(\arccos x)}$ $(-1 \le x \le 1; n \in \mathbb{Z}_+)$ is the Chebyshev polynomial of the second kind and degree j. .

(2) Uniformly on all compact subsets K in $\mathcal{E}_m$ the following generalized Trace formula

$$\sum_{j=0}^{N} \sum_{k=0}^{\infty} (\delta_k d_{k+r,j} - \delta_{k-j} d_{k,j}) \hat{q}_k(x) \hat{q}_{k-j+r}(x) +$$

$$\sum_{j=1}^{N} \sum_{k=0}^{\infty} (\delta_k d_{k+r+j,j} - \delta_{k+j} \alpha_{k+j,j}) \hat{q}_k(x) \hat{q}_{k+j+N}(x) = \frac{\delta}{\pi} U_{r-1}(x) w_N'(x) \frac{\sqrt{1-x^2}}{\omega(x)}$$

holds.

Remark. For a given class polynomials this statement solves the problem posed in the review

V.A.Sadovnichij, V.E.Podolskij . Traces of operators. Uspehi Math.Nauk, 61:5(2006), 89-156[in Russian]

Theorem 2 Let the orthonormal polynomial system $\{\hat{q}_n(x)\}_{n=0}^{\infty}$

has a majorant h(x), and the measure μ is absolutely continuous in $\mathcal{E}_m$, *and* $\mu'(x) = \omega(x)$ is strictly positive and continuous on $\mathcal{E}_m$.

and for recurrence coefficients $d_{k,j}$ the following relation

$$\sum_{l=1}^{N}\sum_{j=1}^{N} j\sum_{k=0}^{\infty}\left|d_{k,j}-d_{k+l,j}\right| + \sum_{l=0}^{N}\sum_{j=1}^{N} j\sum_{k=0}^{\infty}\left|d_{k,l}-d_{k+j,l}\right| < \infty$$

holds. Then uniformly on all compact subsets K in $\mathcal{E}_{\mathrm{m}}$ the following generalized Trace formula

$$\sum_{j=1}^{N} j\sum_{l=0}^{N}\sum_{k=0}^{\infty}\left(d_{k+j,j}d_{k+j+l,l}-d_{k+j+l,j}d_{k+l,l}\right)\hat{q}_k(x)\hat{q}_{k+j+l}(x)$$

$$\sum_{j=1}^{N} j\sum_{l=1}^{N}\sum_{k=0}^{\infty}\left(d_{k+j,j}d_{k+j,l}-d_{k+j-l,j}d_{k,l}\right)\hat{q}_k(x)\hat{q}_{k+j-l}(x) =$$

$$= \frac{1}{2\pi}\left[w_N'(x)\right]^2 \frac{\sqrt{1-x^2}}{\omega(x)}$$

holds.

Example. Discrete Sobolev polynomial $\{\hat{q}_n^{(\alpha)}(x)\}_{n=0}^{\infty}$

Corollary. Uniformly on all compact subsets K⊂(-1,1) the following generalized Trace formula holds:

$$\sum_{n=0}^{\infty}\left[3(a_{n+3}^2-a_n^2)+(b_{n+1}^2-b_n^2)\right]\left[\hat{q}_n^{(\alpha)}(x)\right]^2 +$$

$$4\sum_{n=0}^{\infty}\left(a_{n+3}b_{n+3}-a_{n+2}b_n\right)\right]\ \hat{q}_{n+2}^{(\alpha)}(x)\,\hat{q}_n^{(\alpha)}(x)$$

$$\sum_{n=0}^{\infty} 2\left(a_{n+3}b_{n+4}-a_{n+4}b_{n+1}\right)\hat{q}_{n+4}^{(\alpha)}(x)\,\hat{q}_n^{(\alpha)}(x) =$$

$$=\frac{9}{\pi}\frac{2^{2\alpha}\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)}(1-x^2)^{\frac{5}{2}-\alpha}.$$

Trace formula for the class AP_N

$\{\boldsymbol{p}_n\}(n\in\mathbb{Z}_+)$ belongs to the class AP_N

Let's put

δ=$\{\delta_n, \delta_n\in\mathbb{R}, \delta_n\neq 0, \lim_{n\to\infty}\delta_n=\delta\neq 0\}$

Define by

$$G_n^{(N)}(x;\delta) = \frac{1}{N}\sum_{k=n+1}^{n+N} \Delta_k^{(N)}(x;\delta)\,,$$

where

$$\Delta_k^{(N)}(x;\delta) = d_k^{(N,N)}\,[\delta_k p_k^2(x) - \delta_{k+N} p_{k-N}(x) p_{k+N}(x).$$

the averaged Turán's δ -determinant $G_n^{(N)}(x;\delta)$.

Theorem . Suppose that an orthonormal polynomial system $\{p_n\}(n \in \mathbb{Z}_+)$ with respect to a measure θ belongs to the class AP_N and recurrence coefficients are N-bounded variation. If the relation

$$\sum_{j=0}^{N}\sum_{k=0}^{\infty} |[\delta_{k+N} - \delta_{k+j+N}]\, d_{k+j}^{(j,N)}| < \infty,$$

is valid, then the following results hold:

1) for the averaged Turán's δ -determinant $G_n^{(N)}(x;\delta)$, uniformly on all compact subsets $K \subset E_0$ the relation

$$\lim_{n\to\infty} G_n^{(N)}(x;\delta) = \frac{2\delta}{N\pi}\left(\prod_{k=1}^{N} a_k^0\right)\frac{\sqrt{1-T^2(x)}|T'(x)|}{\mu'(x)}$$

holds;

2) the generalized trace formula

$$\sum_{j=0}^{N}\sum_{k=0}^{\infty}[\delta_{k+N}\, d_{k+N}^{(j,N)} - \delta_{k+N-j} d_k^{(j,N)}]p_k(x)p_{k+N-j}(x)+$$

$$\sum_{j=0}^{N}\sum_{k=0}^{\infty}[\delta_{k+N}\, d_{k+N+j}^{(j,N)} - \delta_{k+N+j} d_{k+j}^{(j,N)}]p_k(x)p_{k+N+j}(x) =$$

$$\frac{2\delta}{\pi}\left(\prod_{k=1}^{N} a_k^0\right)\frac{\sqrt{1-T^2(x)}|T'(x)|}{\mu'(x)}$$

is valid uniformly on all compact subsets $K \subset E_0$.

We now formulate the particular case of Theorem , when N=1 .

Let N=1. Here

$$d_n^{(1,1)} = a_n,\ \ d_n^{(0,1)} = \frac{1}{2} b_n, T(x) = x.$$

Corollary .

Let $\{\mathrm{p_n}\}$ is orthonomal polynomial system with respect to the measure θ belongs to class $\mathfrak{M}$. If for the sequence δ the relation

$$\sum_{n=0}^{\infty}|\delta_n-\delta_{n+1}|<\infty$$

holds, then uniformly on all compact subsets of (-1,1) the following formulas are valid:

1) $\lim_{\mathrm{n}\to\infty} a_{n+1}[\delta_{n+1}p_{n+1}^2(x)-\delta_{n+2}p_n(x)p_{n+2}(x)]=\frac{\delta}{\pi}\frac{\sqrt{1-x^2}}{\omega(x)}$; $(*)$

2)

$$\sum_{n=0}^{\infty}(\delta_{n+1}a_{n+1}-\delta_n a_n)p_n^2(x)+\sum_{n=0}^{\infty}\delta_{n+1}(b_n-b_{n+1})p_n(x)p_{n+1}(x)+$$

$\sum_{n=0}^{\infty}(\delta_{n+1}a_{n+2}-\delta_{n+2}a_{n+1})p_n(x)p_{n+2}(x)=\frac{\delta}{\pi}\frac{\sqrt{1-\mathrm{x}^2}}{\omega(\mathrm{x})}$. $(**)$

By setting $\delta_n=1(n=1,2,\dots)$ in $(*)$, we obtain Theorem B, and from $(**)$ for $\delta_n=a_n(n=1,2,\dots)$ one derives Theorem A.

Fourier series in continual-discrete polynomial system

Denote by $\mathfrak{R}_{\mathrm{p}}$ $(1\le\mathrm{p}<\infty)$, $\mathfrak{R}_1=\mathfrak{R}$, the set of functions

$$\mathfrak{R}_p=\left\{\begin{matrix} f,\int_{-1}^{1}|f(x)|^p d\mu(x)<\infty;\ f^{(i)}(a_k)\ \ exist \\ i=0,1,2,\dots,N_k\ ,a_k\in\mathbb{R}(k=1,2,\dots,m)\end{matrix}\right\}.$$

To each $f\in\mathfrak{R}$ we construct Fourier-Sobolev series

$$f(x)\sim\sum_{k=0}^{\infty}c_k(f)\hat{q}_k(x)(x\in[-1,1])$$

with Fourier coefficients

$$c_k(f)=<f,\hat{q}_k>=\int_{-1}^{1}f(x)\,\hat{q}_k(x)d\mu(x)+\sum_{s=1}^{m}\sum_{i=0}^{N_s}M_{s,i}f^{(i)}(a_s)\,\hat{q}_k^{(i)}(a_s).$$

We consider the trilinear T-regular method of summability defined by the matrix

$$\Lambda = \left\{\lambda_k^{(n)}, k = 0,1,\dots,n,n+1; \lambda_0^{(n)} = 1, \lambda_{n+1}^{(n)} = 0, n \in \mathbb{Z}_+\right\}.$$

Matrix Λ is called T - regular,if the following conditions are valid:

a) $lim_{n\to\infty}\lambda_k^{(n)} = 1(k \in \mathbb{Z}_+$is fixed);

b) $\sum_{k=0}^{n}\left|\lambda_k^{(n)} - \lambda_{k+1}^{(n)}\right| \le C\ (n \in \mathbb{Z}_+)$.

For example:Cesaro means $(C, \alpha > 0)$, Voronoj -N$\ddot{o}$rlund means,Riesz's means,Bernstein-Rogozinskij method for the Fourier -Jacobi series (G.I.Natanson) and so on.

For every $f \in \Re$ one form Λ-means

$\mathbb{U}_n f(\,x;\Lambda) := \sum_{k=0}^{n}\lambda_k^{(n)}\, c_k(f)\hat{q}_k(x)(n \in \mathbb{Z}_+; x \in [-1,1])$

The point $\mathrm{x} \in (\mathrm{a},\mathrm{b})$ is called a Lebesgue point of a function f, if the following relation

$$\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h}|f(t) - f(x)|d\mu(t) = 0$$

holds. As is known, the set of the Lebesgue points of $f \in L_\mu^1(a,b)$ is situated μ-almost-everywhere x∈(a,b).

Theorem 1. Suppose that an orthonormal polynomial system $\{\hat{q}_n\,(x)\}_{n=0}^{\infty}$ has a continuous in $\mathcal{E}_m$ majorant

$$|\hat{q}_n(x)| \le h(x)(x \in \mathcal{E}_m)$$

and the recurrence coefficients satisfy

$\sum_{j=1}^{N+1} j \sum_{l=0}^{N+1}\sum_{s=0}^{n}(|d_{s+j,j}$-$d_{s+j+l,j}|$ +$|d_{s+j,l} - d_{s+j+l,l}|) \le$ C,(bound variation)

where the constant C>0 is independent on $n \in \mathbb{Z}_+$, and for the entries of T-regular matrix Λ the following estimate

$$\sum_{k=0}^{n}\frac{(k+1)(n-k+1)}{n+1}\left[1 + ln\frac{n+1}{n-k+1}\right]\left|\Delta^2\lambda_k^{(n)}\right| \le C(n = 0,1,2\dots).$$

holds. Then the following statements are valid

(i) Let f∈$\Re_p, 1 \le p < \infty$, be satisfy

$\int_{-1}^{1} |f(x)|^p h^p(x) d\mu(x) < \infty, \int_{-1,}^{1} h^p(x) d\mu(x) < \infty.$ (!)

Then at every Lebesgue point x∈ $\mathcal{E}_m$ of function f , the Λ - means $\mathbb{U}_n f(x;\Lambda)$ of the Fourier-Sobolev series converges to f(x), that is

$$\lim_{n\to\infty} \mathbb{U}_n f(x;\Lambda) = f(x).$$

(ii)If,an addition, the measure μ is absolutely continuous,$\mu'(x) = \omega(x)$, and f is continuous on [-1,1], then this relation holds uniformly on all compact subsets K⊂(-1,1).

Define by $W_\omega^p(F) = \{f, ||f||_{W_\omega^p(F)} < \infty,$

$||f||_{W_\omega^p(F)} = ||f||_{L_\omega^p(F)} + \sum_{k=1}^{m} \sum_{i=0}^{N_k} M_{k,i} |f^{(i)}(a_k)|^p (1 \le p < \infty)\},$

where the subset F ⊂(-1,1).

Theorem 2. Let a polynomial system $\{\hat{q}_n(x)\}_{n=0}^{\infty}$ satisfy all conditions of Theorem 1 and,in addition,

$\sum_{j=0}^{\infty} \left|\hat{q}_j^{(i)}(a_k)\right| < \infty, ||h||_{L_\omega^q([-1,1])} < \infty (1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1).$ **(!!)**

Then for any function f, satisfying (!), we have

$$||U_n f(x;\Lambda)||_{W_\omega^p(K)} \le C_p ||f||_{W_\omega^p([-1,1])} < \infty$$

on an arbitrary compact subset K ⊂(-1,1),where the constant $C_p > 0$ is independent of n∈ $\mathbb{Z}_+$ and the function f.

Remarks 1.Sobolev-Gegenbaer polynomial system $\{\hat{q}_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfy the conditions of Theorem 1 and Theorem 2.

2. The Cesaro means of Fourier-Sobolev series

$$\sigma_n^\alpha(f;x) = \sum_{k=0}^{n} \frac{A_{n-k}^\alpha}{A_n^\alpha} c_k(f) \hat{q}_k(x) \ (n \in \mathbb{Z}_+; x \in [-1,1])$$

satisfy the conditions of Theorem 1 and Theorem 2.

For every function $f \in \Re$ let us denote by $u(r,x)$ of Poisson-Abel's means of the orthogonal Fourier series of f for O.N.P.S.$\{\hat{q}_n\}(n \in \boldsymbol{Z}_+)$, that is,

$$u(r,x) = \sum_{k=0}^{\infty} r^k c_k(f) \hat{q}_k(x) (0 < r < 1; x \in [-1,1]).$$

We will say that $u(r,x)$ is μ-*harmonic extension* of function $f(x)$ to the region

$$D = \{(r,x), 0 \le r < 1; -1 \le x \le 1\}.$$

We will call the track $\Gamma = \{(r,x), 0 \le r < 1; -1 < x < 1\}$ *is nontangential* at the point $M_0(1, x_0)(-1 < x_0 < 1)$, if

$$\Gamma = \{(r,x), 0 \le r < 1; -1 < x < 1; |x - x_0| < \gamma(1-r)\},$$

where the constant $\gamma > 0$ is independent of *r,x*. The Fourier series of orthonormal polynomials $\{\hat{q}_n\}(n \in \boldsymbol{Z}_+)$ will be A^**-summable* at the point $x_0 \in (-1,1)$ to the value β, if

$$u(r,x) \overset{A^*}{\to} \beta,$$

when the point $M(r,x)$ tends to the point $M_0(1,x_0)$ for any nontangential track Γ.

Theorem 3 . Suppose that orthonormal polynomial the system $\{\hat{q}_n\}(n \in \boldsymbol{Z}_+)$ have a majorant , measure μ is absolutely continuous($\mu'(x) = \omega(x)$) and the recurrence coefficients are bounded variation.If the function f$\in \Re$ satisfies

$$\int_{-1}^{1} |f(x)| h(x) \omega(x) dx < \infty,$$

then at every Lebesgue point $x_0 \in K$ the Fourier – Sobolev series is summable to the value for $f(x_0)$, when the point $M(r,x)$ tends to the point $M_0(1,x_0)$ for a nontangential track Γ, i.e. the Fourier– Sobolev series is A^*- summable to the value for $f(x_0)$.

On multipliers of the Fourier Series in polynomials orthogonal in continuous-discrete Sobolev space.

Fourier-Sobolev series

$$f(x) \sim \sum_{k=0}^{\infty} c_k(f) \hat{q}_k(x) \qquad (x \in [-1,1])$$

$$c_k(f) = < f, \hat{q}_k > = \int_{-1}^{1} f(x)\, \hat{q}_k(x) d\mu(x) + \sum_{s=1}^{m} \sum_{i=0}^{N_s} M_{s,i} f^{(i)}(a_s)\, \hat{q}_k^{(i)}(a_s).$$

We consider the following sequence of the real numbers

$$\Phi = \{\varphi_n, \mathrm{n} \in \mathbb{Z}_+; \varphi_0 = 1; \{\varphi_n\}_{\mathrm{n}=0}^{\infty} \in \mathrm{l}^{\infty}\}.$$

For any function $f \in \Re$ by their Fourier-Sobolev series we introduce the linear transformation T defined by relation

$$\mathrm{T}(\mathrm{f};\mathrm{x};\Phi)\sim\sum_{\mathrm{k}=0}^{\infty}\varphi_{\mathrm{k}}\mathrm{c}_{\mathrm{k}}(\mathrm{f})\hat{q}_{\mathrm{k}}(\mathrm{x}).$$

Transformation T is called the *multiplicatoral operator*, the sequence $\{\varphi_n\}_{n=0}^{\infty}$ *is* called the *multiplicator of convergence*, and series is called the *multiplicatoral series.*

The sequence $\Phi=\{\varphi_{\mathrm{n}},;\varphi_0=1,\varphi_n\in l^{\infty},n\in\mathbb{Z}_+\}$ is called *quasiconvex* if

$$\sum_{k=0}^{\infty}(k+1)|\,\Delta^2\varphi_k|<\infty,$$

where $\Delta\varphi_k=\varphi_k-\varphi_{k+1},\ \ \Delta^2\varphi_k=\Delta(\Delta\varphi_k)=\varphi_k-2\varphi_{k+1}+\varphi_{k+2}(k\in\mathbb{Z}_+)$

Theorem 1 . Let the orthonormal polynomial system $\{\hat{q}_n(x)\}_{n=0}^{\infty}$ be satisfy the following condition

$$|\hat{q}_n(t)|\le h(t)(t\in\mathcal{E}_m)$$

and for the recurrence coefficients the estimate

$$\sum_{\mathrm{j}=1}^{\mathrm{N}+1}\mathrm{j}\sum_{\mathrm{l}=0}^{\mathrm{N}+1}\sum_{\mathrm{s}=0}^{\infty}(|\mathrm{d}_{\mathrm{s+j,j}}-\mathrm{d}_{\mathrm{s+j+1,j}}|+|d_{s+j,l}-\mathrm{d}_{\mathrm{s+j+1,l}}|)<\infty$$

holds.If for quasiconvex sequence Φ the relation

$$\varphi_n=O\left(\frac{1}{\ln n}\right)(n\to\infty)\qquad(*)$$

holds, then the following statements are valid :

(i) let for each function $f\in\Re$ be fulfilled

$$\int_{-1}^{1}|f(t)|h(t)\omega(t)dt<\infty,$$

then at every Lebesgue's point $x\in\mathcal{E}_m$ (and, consequently, a.e.) the *multiplicatoral series* converges;

(ii) suppose function f is continuous in [-1,1] and the measure $\mathrm{d}\mu(\mathrm{x})$ is absolutely continuous($\mu'(x)=\omega(x)$) and bounded in $\mathcal{E}_m$, then *multiplicatoral series* is uniformly converges on all compact subsets $K\subset\mathcal{E}_m$.

Define by

$$T_nf(x;\Phi)=\sum_{\mathrm{k}=0}^{n}\varphi_{\mathrm{k}}\mathrm{c}_{\mathrm{k}}(\mathrm{f})\hat{q}_{\mathrm{k}}(\mathrm{x})\ (\mathrm{n}\in\mathbb{Z}_+,\mathrm{x}\in[-1,1])$$

the partial sums of the multiplicatoral series.

Theorem 2. Let an orthonormal polynomial system satisfies the hypotheses of Theorem 1 and (!!). Also let a quasi-convex sequence Φ satisfy the relation ($*$) and a function f$\in W_{\omega}^{p}([-1,1])(1<p<\infty)$ satisfy (!). Then,on all compact subsets K of $\mathcal{E}_m$

$$||T_n f(x;\Phi)||_{W_{\omega}^{p}(K)} \leq C_p||f||_{W_{\omega}^{p}([-1,1])} < \infty,$$

where the constant C_p>0 does not depend on n ,on the function f ,and on the sequence Φ.

Remark. Symmetric Gegenbauer-Sobolev orthonormal polynomials $\left\{\hat{q}_n^{(\alpha)}(x)\right\}$ satisfy the conditions of Theorem 1 and Theorem 2.

List of papers where these results have been published

1. B. P. Osilenker, Fourier series in orthogonal polynomials.World Scientific, Singapore,1999

2.B. P. Osilenker, “On multipliers for Fourier series in Sobolev orthogonal polynomials”, *Sb. Math.*, **213**:8(2022), 1058–1095.

3. B. P. Osilenker, “Generalized Trace Formula for Polynomials Orthogonal in Continuous-Discrete Sobolev Spaces”, *Funct. Anal. Appl.54*:4 (2020), 310-312.

4. B. P. Osilenker, “ A new approach to the generalized trace formula and asymptotics of Turan’s determinant for polynomials with asymptotically N-periodic recurrence coefficients”, J. Math. Sci.266:4(2022), 621-634.

5. B. P. Osilenker, “On Fourier Series in Generalized Eigenfunctions of a Discrete Sturm-Liouville Operator”, *Funct. Anal. Appl.*, **52**:2 (2018), 154–157

6. B. P. Osilenker, “On linear summability methods of Fourier series in polynomials orthogonal in a discrete Sobolev spaces”, *Siberian Math. J.*, **56**:2 (2015), 339–351.

7. B. P. Osilenker, “An Analog of the Trace Formula for Discrete Sturm–Liouville Operators with Asymptotically N-Periodic Coefficients”, *Funct. Anal. Appl.*, **31**:4(1997), 283–285.

8. B.P.Osilenker, “Fourier series in orthonormal matrix polynomials”, Soviet Math.(Izv. VUZ.),**32**:2(1988),71-83.

9. F.Marcellán, B.P.Osilenker,I.A.Rocha,”On Fourier series of Jacobi-Sobolev orthogonal polynomials”, J.Ineq.Appl.,7(5)(2002),673-699.

10. F.Marcellán, B.P.Osilenker,I.A.Rocha,”On Fourier series of Jacobi-Sobolev inner product”, J.Approx. Theory,117(2002),1-22.

11. B.P.Osilenker, “Generalized trace formula and asymptotics of the averaged Turan determinant for orthogonal polynomials”, J. Approx. Theory.141(2005),70-94.

12. Osilenker B.P. Trace formula for orthogonal polynomials with asymptotically 2-periodic recurrence coefficients// J. Approx. Theory 92 (1998), 442-462.

THANK YOU FOR YOUR ATTENTION