On the Bishop-Phelps property

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Krein-Milman properties

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A real topological vector space enjoys the Krein-Milman property provided that every bounded, closed, convex subset has an extreme point.

Strong Krein-Milman property

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Convergence linear topologies

$\sigma(F, \mathcal{G})$

Let M be a topological module over a topological ring R and I a nonempty set. Take a submodule F of M^I and $\mathcal{G} \subseteq \mathcal{P}(I)$ upward directed so that $f(G)$ is bounded in M for all $G \in \mathcal{G}$ and all $f \in F$. For every $G \in \mathcal{G}$ and every 0-neighborhood $U \subseteq M$, the sets

$$\mathcal{U}(G, U) := \{f \in F : f(G) \subseteq U\}$$

form a base of 0-neighbourhoods for a module topology on F called the convergence linear topology of F generated by $\mathcal{G}$.

Examples of convergence linear topologies

Remark

If R is a practical topological ring, in other words, $0 \in \mathrm{cl}(\mathcal{U}(R))$, then every finite subset of M is bounded.

Example

If R is practical and $\mathcal{G}$ is the set of finite subsets of I, then $\sigma(F, \mathcal{G})$ is the pointwise convergence topology on F or, equivalently, the inherited product topology on F from M^I.

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Remark

The topological dual module of M, M^*, is a right R-module. Thus, it can be endowed with a convergence linear topology since $M^* \subseteq R^M$.

Example

If R is practical, then the finite subsets of M define a convergence linear topology on M^* which is the pointwise convergence topology on M^*, also known as the w^*-topology.

Example

If we take the family $\mathscr{B}_M$ of all bounded subsets of M, then we obtain a stronger convergence linear topology on M^* called the $\mathscr{B}_M$-topology.

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Convergence linear topologies

Notation

Let X be a real or complex topological vector space. Then:

- $\mathscr{B}_X$ *is the family of bounded subsets of* X.
- $\mathscr{BCC}_X$ *is the family of bounded, closed, convex subsets of* X.

Lemma

If X is a Hausdorff locally convex real or complex topological vector space, then $\overline{\text{co}}(A)$ is bounded for every bounded subset A of X. Therefore, $\sigma\left(X^*, \mathscr{B}_X\right) = \sigma\left(X^*, \mathscr{BCC}_X\right)$.

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Bishop-Phelps property

Notation

Let X be a real topological vector space and C a nonempty subset of X. The set of functionals attaining their sup on C is denoted by $\mathrm{SA}(C) := \{f \in X^* : \sup f(C)$ *is attained on* $C\}$.

Bishop-Phelps property

A real topological vector space X is said to have the Bishop-Phelps property if for every bounded, closed, convex subset B of X, the set $\mathrm{SA}(B)$ is dense in X^* for the $\sigma\left(X^*, \mathscr{BCC}_X\right)$-topology.

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Equivalence of Krein-Milman properties

Theorem

Let X be a real topological vector space. Let Y be a closed complemented subspace of X. If X has the Bishop-Phelps property, then so does Y.

Theorem

Let X be a Hausdorff locally convex real topological vector space satisfying the Bishop-Phelps property. If $\mathrm{ext}(A) \neq \varnothing$ for all bounded, closed, convex subset A of X, then $A = \overline{\mathrm{co}}(\mathrm{ext}(A))$ for all bounded, closed, convex subset A of X.

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Examples

Remark

Every real Banach space satisfies the Bishop-Phelps property in view of the famous Bishop-Phelps Theorem.

Theorem

Let X be a real vector space. If X is endowed with the finest locally convex vector topology, then X enjoys the Bishop-Phelps property.

Example

There exists a real normed space X which is not subreflexive, therefore, $\mathrm{SA}(\mathrm{B}_X)$ is not dense in the norm-topology of X^*, which is precisely the $\sigma\left(X^*, \mathscr{BCC}_X\right)$-topology of X^*. As a consequence, X does not satisfy the Bishop-Phelps property.

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Inf-attaining functionals

Notation

Let X be a real topological vector space and C a nonempty subset of X. The set of functionals attaining their inf on C is denoted by $\mathrm{IA}(C) := \{f \in X^* : \inf f(C)$ *is attained on* $C\}$.

Lemma

Let X be a real topological vector space. Let $A \subseteq X$ and $f \in X^*$ such that $f(A)$ is bounded above. Then there exists $B \subseteq X$ closed and convex such that $A \subseteq B$ and $f \in \mathrm{SA}(B)$. If X is Hausdorff and locally convex and A is bounded, then B can be choosen to be also bounded.

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A characterization of reflexivity

Theorem

Let X be a Hausdorff locally convex real topological vector space. If there exist a bounded, closed, and convex subset C of X and $f \in X^* \setminus \mathrm{IA}(C)$, then there exists another bounded, closed, and convex subset D of X such that $C \subseteq D$ and $f \in \mathrm{SA}(D) \setminus \mathrm{IA}(D)$.

Corollary

A real Banach space X is reflexive if and only if $\mathrm{IA}(C) = \mathrm{SA}(C)$ for all bounded, closed, and convex subset C of X. In this situation, $\mathrm{IA}(C) = \mathrm{SA}(C) = X^*$ for all bounded, closed, and convex subset C of X.

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Supporting vectors

Definition

Let $T : X \to Y$ be a continuous linear operator between real or complex normed spaces. The set of supporting vectors of T is defined as $\operatorname{suppv}(T) = \{x \in X : \|T(x)\| = \|T\|\|x\|\}$.

Remark

Let X, Y be real or complex normed spaces and $T : X \to Y$ a continuous linear operator. If $\operatorname{suppv}(T) \neq \{0\}$, then $\operatorname{suppv}(T^*) \neq \{0\}$.

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Supporting vectors of dual operators

Lemma

Let X, Y be real or complex normed spaces and $T : X \to Y$ a continuous linear operator. Then

$$\operatorname{suppv}(T^*) \supseteq \left\{y^* \in Y^* : T^{-1}\left(\operatorname{suppv}(y^*)\right) \cap \operatorname{suppv}(T) \neq \{0\}\right\}.$$

If X is reflexive, then

$$\operatorname{suppv}(T^*) \setminus \{0\} \subseteq \left\{y^* \in Y^* : T^{-1}\left(\operatorname{suppv}(y^*)\right) \cap \operatorname{suppv}(T) \neq \{0\}\right\}.$$

Another characterizations of reflexivity

Corollary

A real or complex Banach space X *is reflexive if and only if for every Banach space* Y *and for every continuous linear operator* $T : X \to Y$ *such that* $\operatorname{suppv}(T) \neq \{0\}$,

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