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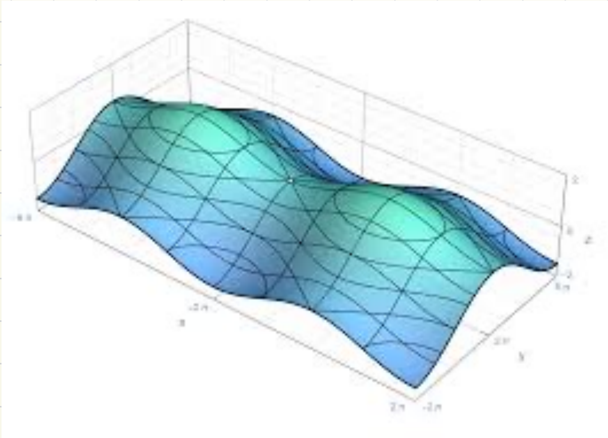
INTERNATIONAL BIWEEKLY SEMINAR ON ANALYSIS ETC.

SEPTEMBER 19, 2024

ROSTO, DON

Inverse Problems for Screens



by

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Joint research

with

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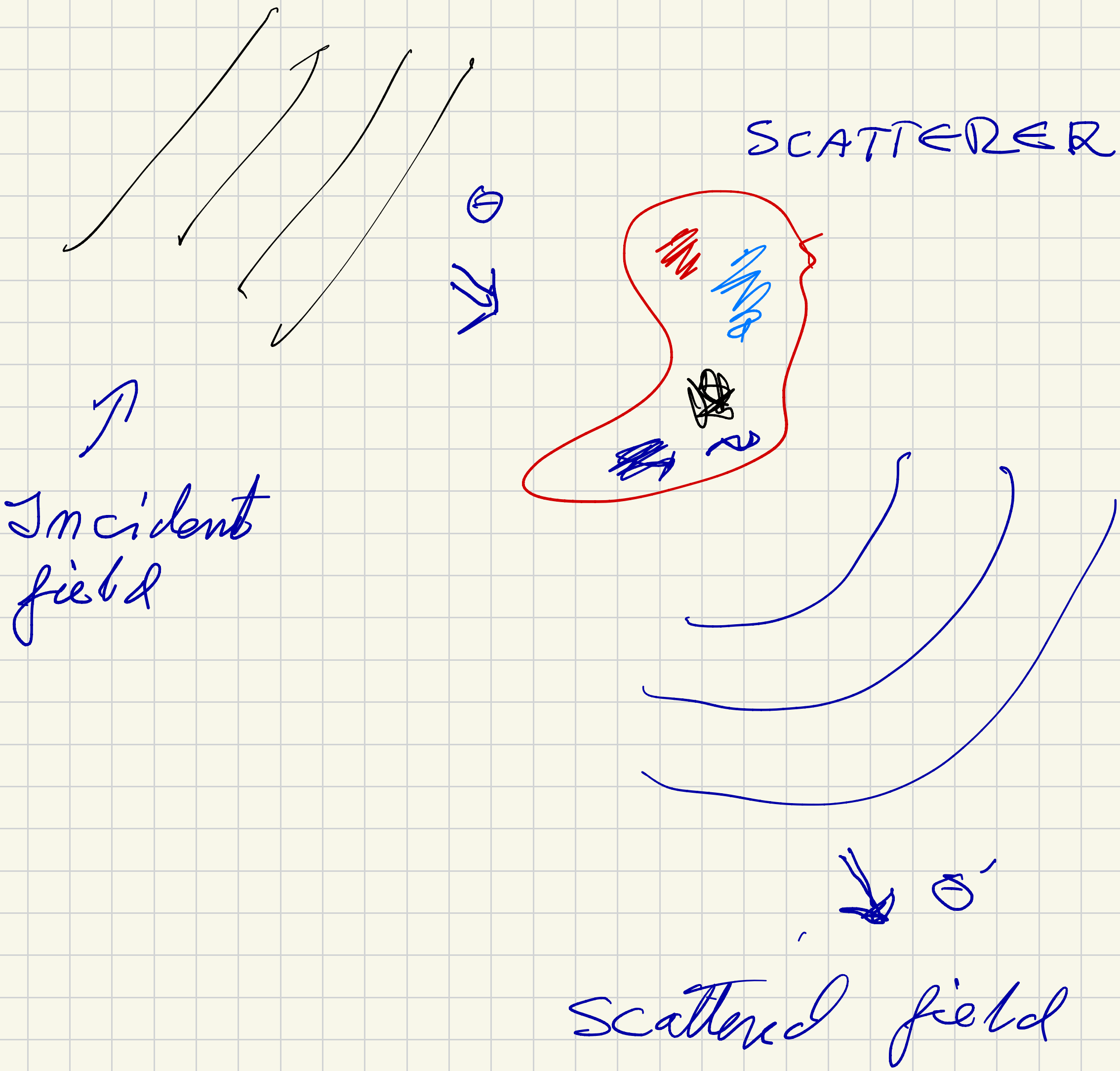
Lappeenranta University of Technology or LUT-university

and

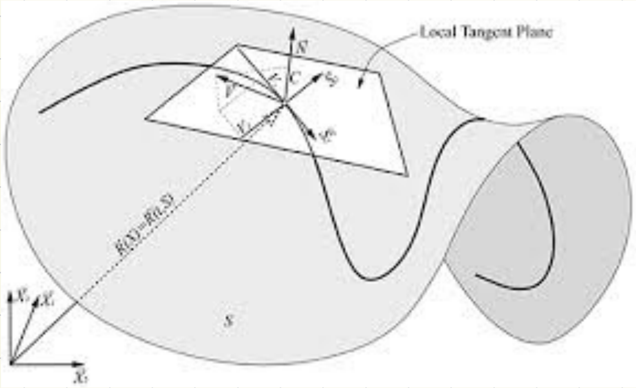
Petri Ola
Helsinki university

SCATTERING



<u>Definition 1</u> A n-dimensional screen is a bounded smooth n-dimensional submanifold, with boundary, on $\mathbb{R}^{n+1}$.

Hence a 2D screen is bounded surface with boundary in $\mathbb{R}^3$.



THE SCHIFFER'S PROBLEM

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Direct Scattering Problem

Let u be C^2-function. We say that u is a solution of the direct scattering problem for incident field u^i if $\exists\, u^s$, the scattered field, such that $u = u^i + u^s$ satisfy

i) $u|_{\partial\Omega} = 0$

ii) $(\Delta + k^2)u^i = 0$

iii) $(\Delta + k^2)u = 0$ in $\mathbb{R}^n \setminus \Omega$

iv) u^s satisfies the Sommerfeld's radiation condion i.e. u^s is an outgoing field.

If S is a screen, the direct scattering problem is defined in the same way except that the condition (i) is replaced by

(i′) $$u\Big|_S = 0.$$

Remark Physically the above problem describes the acoustic scattering from a sound-soft screen.

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(i′) $$u\big|_S = 0.$$

<u>Remark</u> Physically the above problems describe the scattering of acoustic waves from a sound-soft obstacle or screen

A. Sommerfeld

1868 – 1951

$\textcircled{3\frac{1}{2}}$

The equation

$$(\Delta + k^2) u = 0$$

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Hermann Helmholtz

1821 – 1894

Inverse Scattering Problem

If $u = u^i + u^s$, $u = u(x, k, \theta)$ is a solution of the Direct Scattering problem, then

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} u^\infty(\hat{x})$$

$$+ O(|x|^{-\frac{(n+1)}{2}})$$

where $\hat{x} = \frac{x}{|x|}$ the direction of x.

Note that

$$u^\infty(\hat{x}) = u^\infty(\hat{x}, \theta, k)$$

is called the far field or the scattering amplitude.

Fixed energy inverse scattering problem

ISP Determine Ω or S from $u^\infty(\hat{x}, \theta, k)$ for given values of $\hat{x}, \theta$ when $k > 0$ is fixed.
(Fixed energy inverse scattering problem)

Inverse Scattering Problem

ISP Determine Ω or S from $u^\infty(\hat{x}, \theta, k)$ for certain values of $\hat{x}, \theta$ when $k > 0$ is fixed.

Schiffer Theorem The measurements

$$M_{\theta_j} = \{u^\infty(\hat{x}, \theta_j) \mid \hat{x} \in S^{n-1}\}$$

$j \in \mathbb{N}$, uniquely determine Ω for all choices of different θ_j, $j \in \mathbb{N}$

This is true also for screens.

THE SCHIFFER PROBLEM

We call M_Θ one measurement.

Schiffer Theorem $\Longleftrightarrow$ any infinite number of measurement determine Ω or S uniquely

WHAT ABOUT ONE MEASUREMENT?

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WHAT ABOUT ONE MEASUREMENT?
THIS IS SCHIFFER'S
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CONJECTURE

In $\mathbb{R}^n$ one single measurement $\{u^\infty(\hat{x}, \theta) \mid \theta \text{ fixed and } \hat{x} \in S^{n-1}\}$ always determines any obstacle or any screen uniquely.

- The Schiffer's Problem exist's since 1960'.
- See the book of Lax and Phillips in scattering theory 1962

THE CASE $n = 2$

In $\mathbb{R}^2$ the screens are one-dimensional and hence arcs. If $k \neq 0$ the solution of the direct scattering problem can be written as

$$u^s(x) = \int_\Gamma H_0^{(1)}(k|x-y|)\,\varphi(y)\,ds(y)$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. Here $H_0^{(1)}$ is the Hankel function of the first kind.

Properties of $H_0^{(1)}$

1) $(\Delta + k^2)\, H_0^{1}(k|x|) = \pi\, \delta(x)$

↗ Dirac delta

2) $H_0^{(1)}(r) = \pi \log r + \text{bounded}$

for $r > 0$

The solution of the direct scattering problem for a screen Γ is easy :

Integration by parts gives

(1) $$u^s(x) = \int_\Gamma H_0^{(1)}(k|x-y|)\,\rho(y)\,ds(y)$$

where

$$\rho(x) = \left[\frac{\partial u}{\partial \nu}(x)\right], \quad x \in \mathbb{R}^2 \setminus \Gamma$$

↖ jump on Γ

Letting $x \to \Gamma$ (Recall $u = u^i + u^s$, vanish on Γ)

(2) $$-u^i(x) = \int_\Gamma H_0^{(1)}(k|x-y|)\,\rho(y)\,ds(y)$$

for $x \in \Gamma$

Solution: Solve ρ from (2), insert it to (1) to obtain $u^s(x)$ everywhere in $\mathbb{R}^2$

Sub-case $k = 0$

Physically this corresponds to the case of determining of cracks in electrostatic measurements.

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Physically this corresponds to the case of determining of screens by electrostatic measurements.

More exactly: Assume Γ is a screen in $\mathbb{R}^2$ and

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma$$

$$u\big|_\Gamma = 0$$

$$(*) \begin{cases} u = \text{const} + u_0 \\ \text{where} \quad \int\limits_{\mathbb{R}^2 \setminus \Gamma} |\nabla u_0|^2 + \int\limits_\Gamma |u_0|^2 \, ds < \infty \end{cases}$$

Note that $(*)$ can be replaced by assuming: u is bounded

The inverse electrostatic problem is the following:

Given Γ and u as above # determine Γ from the Cauchy-data of u on ∂B where B is a (large) ball containing Γ

KORJAA

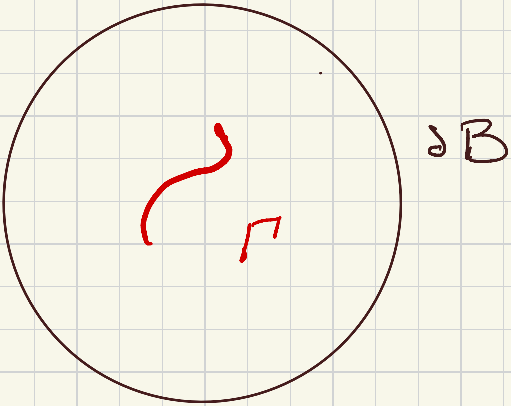
The inverse electrostatic problem is the following:

Given Γ and u as above determine Γ from the Cauchy-data of u on ∂B where B is a (large) ball containing Γ.

<u>Note</u> that in Electrical Impedance Tomography or in Calderón problem it is assumed that the Cauchy data is known for ALL measurements.

Case $k = 0$, $n = 2$

Recall



$D_\Gamma = \left(u, \frac{\partial u}{\partial \nu}\right)$ known on ∂B for one u

The task: Determine Γ

Uniqueness of the Inverse Problem:

Assume $D_{\Gamma_1} = D_{\Gamma_2}$. Does it follow

$\Gamma_1 = \Gamma_2$?

Case $k = 0$, $n = 2$.

Recall u_Γ is real analytic in $\mathbb{R}^2 \setminus \Gamma$.

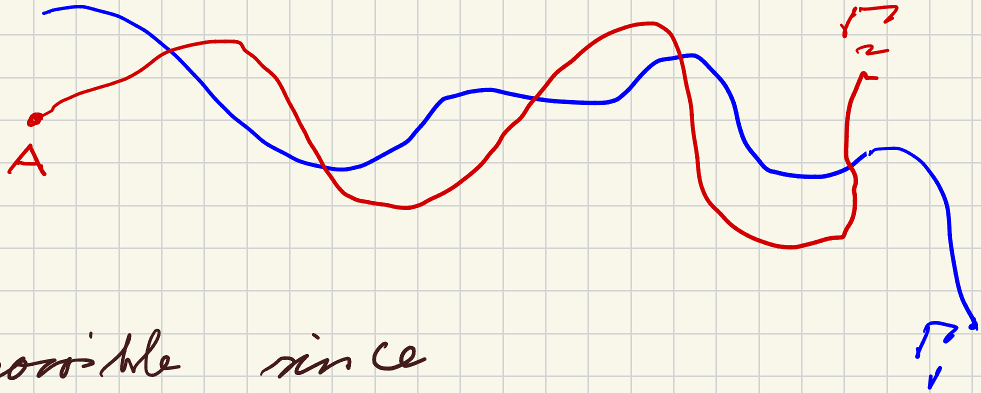
<u>Theorem 1</u> (Blåsten, Ola, P. 2024)
If Γ is any smooth crack then u_Γ is singular at both of it's tips

<u>Corollary 2</u> D_Γ uniquely determines the crack.

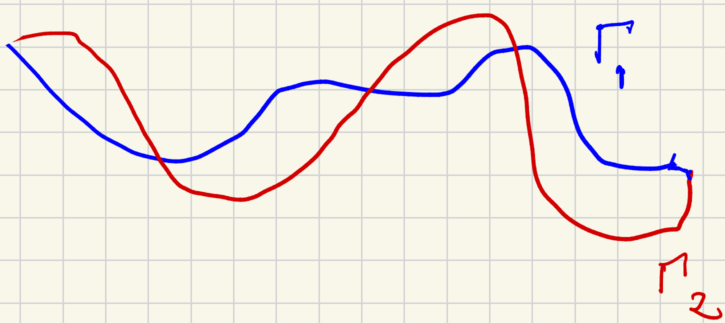
The beef here is the proof of Thm 1. However we next show how Cor 2 follows readily from Thm 1.

Case $n = 2$, $k = 0$

'Proof' The most difficult case

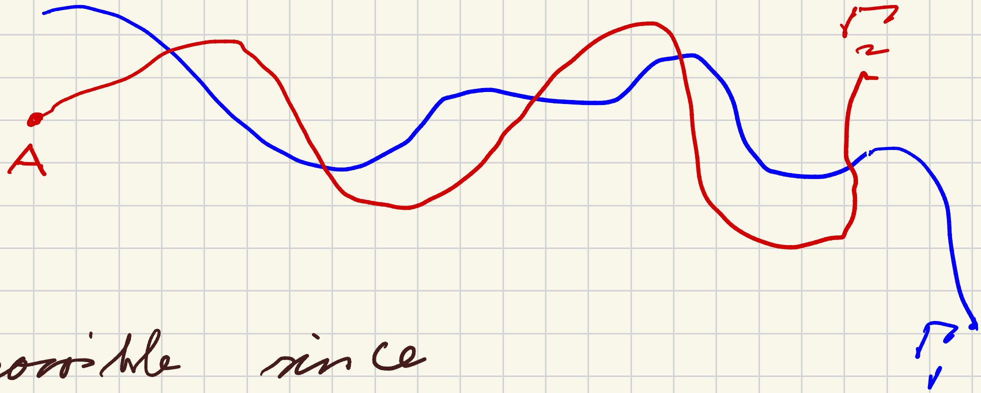


Impossible since
u_{Γ_1} is analytic near point A
but by Theorem 1 u_{Γ_2} is singular
at the tip A
of Γ_2. Thus only this is possible



Case $n=2$, $k=0$

The most difficult case



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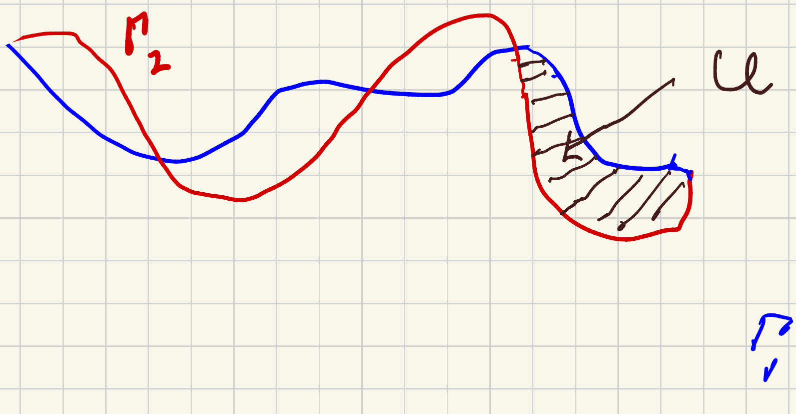


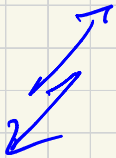
Since u_{Γ_i} is 0 at Γ_i we have

u_i, $i = 1, 2$ is harmonic in U

and $u_i \big|_{\partial U} = 0$, $i = 1, 2$.

Maximum Principle $\Rightarrow$ $u_i = 0$

outside Γ_i



This ends up the proof of Corollary 2

$\square$

The idea of the proof of Theorem 1

Assume first that Γ is FLAT:

$$\Gamma = [-1, 1]$$

We need to show that $u(z)$ is singular at $z = -1$ and $z = +1$.

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where ρ can be solved from

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Physically u is the electric potential (voltage) and $\rho(t)$ the charge density on I.

By differentiating (4) with respect to variable t we get

(5) $$\text{p.v.}\int_{-1}^{1} \frac{1}{t-s}\, g(t)\, dt = 0 \quad , \quad t \in I$$

Hence g is in the kernel of the local Hilbert transform $\mathcal{H}_I$.

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For $I = \mathbb{R}$, $\mathcal{H} = \mathcal{H}_{\mathbb{R}}$ is an isomorphism

$$\mathcal{H}^2 = -I \qquad \text{and}$$

$u + iv$, $u, v \in L^p$, $1 < p < \infty$ has a holom. extension from $\mathbb{R}$ to $\mathbb{C}_+$ $\Longleftrightarrow$

$$v = \mathcal{H} u$$

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$u + iv$, $u, v \in L^p$, $1 < p < \infty$ has a cont. extension from $\mathbb{R}$ to $\mathbb{C}_+$ $\Longleftrightarrow$

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$\Longrightarrow$ Theory of Hardy-spaces $H^p(\mathbb{R})$

Hardy Spaces H^p

Def $f : \mathbb{C}_+ \longrightarrow \mathbb{C}$ belongs to $H^p(\mathbb{C}_+) = H^p(\mathbb{R})$ if f is bounded near ∞-point and $f|_{\mathbb{R}} \in L^p(\mathbb{R})$ and f is complex-analytic.

Hardy Spaces H^p

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Theorem holds

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Theorem $f = u + iv \in \mathcal{H}^p(\mathbb{C}_+)$
iff $v = \mathcal{H} u$

Question: is u singular near the tips
if $g \neq 0$ solves

$$(*) \quad \begin{aligned} &\operatorname{supp} g \subset I \\ &\operatorname{supp} \mathcal{H} g \subset \mathbb{R} \setminus I \end{aligned}$$

Look at the function

$$f(z) = \frac{1}{\sqrt{1-z^2}} \quad \text{in } \mathbb{C} \setminus i\mathbb{R}_-$$

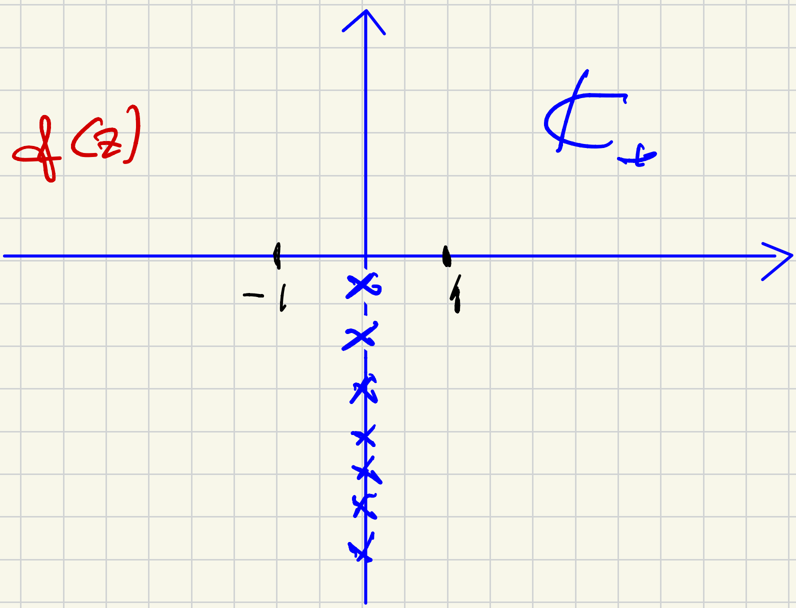
Then

$$f \in H^p(\mathbb{C}_+) \quad , \quad 1 < p < 2$$

$$f = g + i\,\tilde{g}$$

$$g(t) = \frac{1}{\sqrt{1-t^2}}\,\chi_I(t) \quad , \qquad |t| < 1$$

$$\tilde{g}(t) = \frac{1}{\sqrt{t^2-1}}\,\chi_{I^c}(t) \quad , \qquad |t| > 1$$



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belongs to

$$H^p(\mathbb{C}_+)$$

$$f = g + i\tilde{g}$$

$$g(t) = \frac{1}{\sqrt{1-t^2}} \chi_I(t), \quad I = [-1,1]$$

$$\tilde{g}(t) = \frac{1}{\sqrt{t^2-1}} \chi_{I^c}(t), \quad I^c = \mathbb{R} \setminus I$$

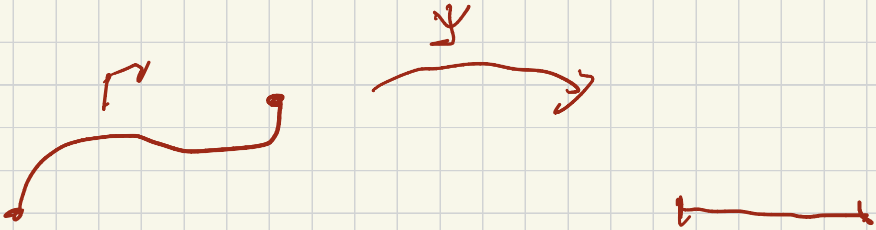
The g is the solution of

$$\operatorname{supp} g \subset I$$

$$\mathcal{H} g = 0 \quad \text{on } I$$

and all non-zero solutions are singular at the edges of I.

what about curved screens ?



Use Riemann Mapping Theorem in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \Rightarrow \exists\, \Psi$,

$\Psi : \hat{\mathbb{C}} \setminus \Gamma \longrightarrow \hat{\mathbb{C}} \setminus I$,

Ψ biholomorphic.

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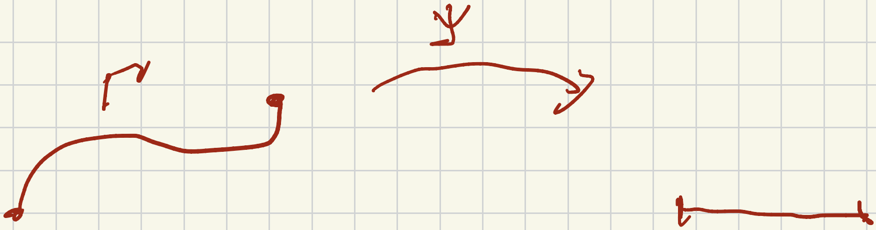

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Ψ biholomorphic.

Caratheodory's <u>prime-end</u> <u>theory</u> $\Rightarrow \exists\ C^1$ extension of Ψ to $\Gamma \longrightarrow I$.

Singularities in I shift over to Γ by the inverse of Ψ.

If $\Psi : \Omega_1 \to \Omega_2$ is a smooth Riemann-map then

$\exists$ smooth injective extension to $\partial\Omega_1$

$\iff$ Ω_1 and Ω_2 are Jordan domains

unfortunately $\Gamma^c = \hat{\mathbb{C}} \setminus \Gamma$ is **not** a Jordan domain!

If $\psi : \Omega_1 \to \Omega_2$ is the Riemann-map then

$\exists$ smooth injective extension to $\partial\Omega_1$

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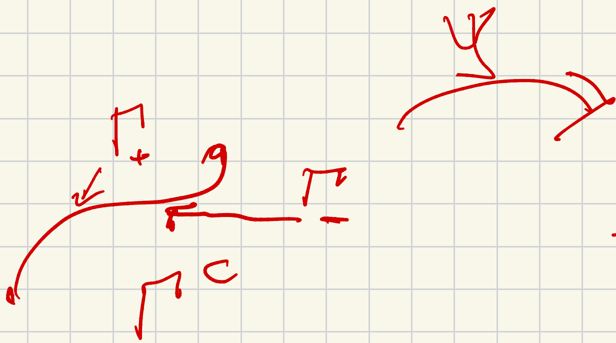
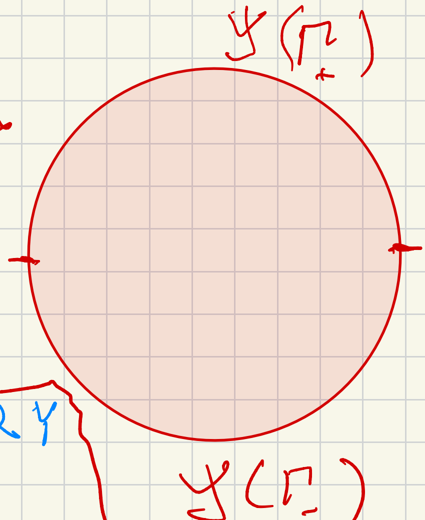
Unfortunately $\Gamma^c = \hat{\mathbb{C}} \setminus \Gamma$ is not a Jordan domain





PRIME END THEORY
SAVES THE DAY !

2D screens in $\mathbb{R}^3$

We say that a screen $\Sigma \subset \mathbb{R}^3$ is *flat* if $\exists$ plane $T \subset \mathbb{R}^3$ s.t. $\Sigma \subset T$.

The answer to Schiffer's problem is positive for both the acoustic scattering (Blåsten - P. - Sadique 2021) and for electrodynamic scattering (Maxwell's equations) (P. - Ola - Sadique (2023))

Theorem Both the screen Σ and the supporting plane T are uniquely determined by one scattering measurement.

Thank you for your attention! :)