DIRECT AND INVERSE SPHERICALLY SYMMETRIC TRANSMISSION EIGENVALUE PROBLEMS: PAST, PRESENT, AND FUTURE

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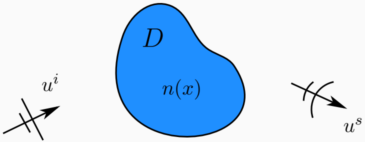
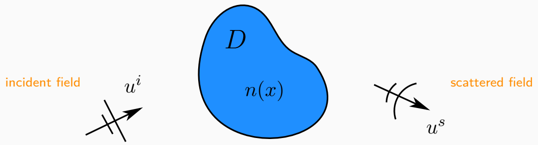
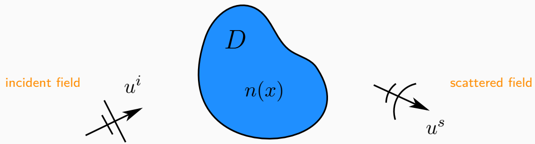
Outline

INTRODUCTION


D
n(x)
u^i
u^s

The acoustic scattering problem

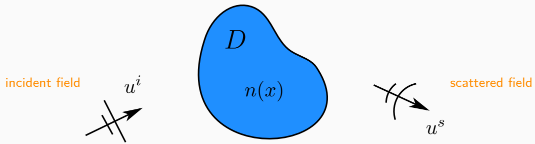




Acoustic scattering by inhomogeneous medium $D \subset \mathbb{R}^d,\ d = 2, 3$:

$$\begin{aligned} \Delta u + k^2 n(x) u = 0 & \qquad \text{in } \mathbb{R}^d \\ u = u^i + u^s & \qquad \text{in } \mathbb{R}^d \\ \lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 & \end{aligned}$$

$k > 0$ is the wavenumber and $n(x)$ the refractive index,



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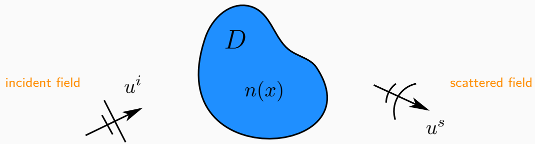
$n(x) \in \mathcal{L}^\infty(D),\ n(x) = 1$ for $x \in \mathbb{R}^d \setminus D$



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Direct Problem: existence, uniqueness and stability for $u \in H^2_{loc}(\mathbb{R}^d)$
Inverse Problem: recover $n(x)$, and ∂D from scattering data

▷ for incident plane wave: $u^i(x) = e^{ikx\cdot\hat{\theta}}$, $\hat{\theta}$: direction of propagation
the scattered field satisfies:

$$u^s(x) = e^{ikr} r^{-\frac{d-1}{2}} u_\infty(\hat{x}) + \mathrm{O}\left(r^{-\frac{d+1}{2}}\right), \quad \text{in } \mathbb{R}^d,\ r = |x| \to \infty$$

where $\hat{x} = x/|x|$ and u_∞ is the far-field pattern:

$$u_\infty(\hat{x}) := \frac{k^2}{4\pi} \int_{\mathbb{R}^d} (n(y) - 1)\, e^{-ik\hat{x}\cdot y} u(y) \mathrm{d}y, \quad \hat{x} \in \Omega = \{x \in \mathbb{R}^d :\ |x| = 1\}$$

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▷ the far-filed patterns define the far-field operator: $F : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x};\hat{\theta}) g(\hat{\theta}) \mathrm{d}s(\hat{\theta}) \qquad \hat{x} \in \Omega,$$

Theorem (Colton-Kress [1])

The far-field operator is injective and has dense range if and only if k is not a transmission eigenvalue and (w, v) solves the Interior Transmission Problem, for v a *Herglotz wave function*.

[1] Inverse Acoustic and Electromagnetic Scattering Theory, 4^{th} Edn., Springer, 2019.

Interior Transmission Eigenvalue Problem:

Find $k \in \mathbb{C}$ and $v, w \in \mathcal{L}^2(D)$ such that $w - v \in H_0^2(D)$:

$$
\begin{aligned}
\Delta w + k^2 n(x) w = 0 \qquad & \text{in } D, \\
\Delta v + k^2 v = 0 \qquad & \text{in } D, \\
w = v \qquad & \text{on } \partial D, \\
\tfrac{\partial w}{\partial \nu} = \tfrac{\partial v}{\partial \nu} \qquad & \text{on } \partial D.
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* ITEP is a non-standard and non-self-adjoint eigenvalue problem
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[2] Cakoni and Colton, A Qualitative Approach to Inverse Scattering Theory, Springer, 2014.

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$k \in \mathbb{C}$ such that $(v, w) \neq 0$ are the Transmission Eigenvalues (TE)

- ∗ ITEP is a non-standard and non-self-adjoint eigenvalue problem
- ∗ If k is not a TE $\Rightarrow$ use far-field to recover ∂D with sampling methods [2]
- ∗ TE are related to non-scattering wavenumbers [3]

[2] Cakoni and Colton, A Qualitative Approach to Inverse Scattering Theory, Springer, 2014.

[3] Cakoni, Colton and Haddar, Transmission Eigenvalues, Not. Am. Math. Soc., 2021.

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▷ Motivation for the inverse problem:

- real transmission eigenvalues can be measured from scattering data:
  Cakoni-Colton-Haddar [5]
- transmission eigenvalues carry information about material properties:
  Cakoni-Çayören-Colton [6], Harris [7]

[4] The existence of an infinite discrete set of transmission eigenvalues, *SIAM J. Math. Anal.*, 2010

[5] On the determination of Dirichlet or transmission eigenvalues from far field data, *C. R. Acad. Sci.*, 2010

[6] Transmission eigenvalues and the nondestructive testing of dielectrics, Inv. Probl., 2008.

[7] Non-destructive testing of anisotropic materials, Ph.D. thesis, Univ. of Delaware, 2015.

THE SPHERICALLY SYMMETRIC TRANSMISSION EIGENVALUE PROBLEM

Literature Review:

* Colton and Kress [8], Chapter 10
* Kirsch [9], Chapter 7.6
* Cakoni and Colton [10], Chapter 9
* Cakoni, Colton and Haddar [11], Chapter 9

[8] Inverse Acoustic and Electromagnetic Scattering Theory, 4^{th} Edn., Springer, 2019.
[9] An Introduction to the Mathematical Theory of Inverse Problems, 3^{rd} Edn., Springer, 2021.
[10] A Qualitative Approach to Inverse Scattering Theory, Springer, 2014.
[11] Inverse Scattering Theory and Transmission Eigenvalues, 2^{nd} Edn., SIAM, 2022.

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* Cakoni, Colton and Haddar [11], Chapter 9

* Pallikarakis (invited review article) [12]

[8] Inverse Acoustic and Electromagnetic Scattering Theory, 4^{th} Edn., Springer, 2019.

[9] An Introduction to the Mathematical Theory of Inverse Problems, 3^{rd} Edn., Springer, 2021.

[10] A Qualitative Approach to Inverse Scattering Theory, Springer, 2014.

[11] Inverse Scattering Theory and Transmission Eigenvalues, 2^{nd} Edn., SIAM, 2022.

[12] A review on the direct and inverse transmission eigenvalue problem for the spherically symmetric refractive index, Bol. Soc. Mat. Mex., 2024.

PROBLEM 1: THE SPHERICALLY SYMMETRIC TEP, $\ell \geq 0$

Consider the ITEP for $D =$ unit ball of $\mathbb{R}^3$, in spherical coordinates:

$$v(r,\theta) = \alpha_\ell j_\ell(kr) P_\ell(\cos\theta), \;\; w(r,\theta) = \beta_\ell \frac{y_\ell(r)}{r} P_\ell(\cos\theta)$$

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▷ I.V.P. for the radial component, $\ell \geq 0$:

$$y_\ell^{''}(r) + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y_\ell(r) = 0, \quad \lim_{r\to 0}\left(\frac{y_\ell(r)}{r} - j_\ell(kr)\right) = 0$$

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$k \in \mathbb{C}$ is a transmission eigenvalue iff

$$D_\ell(k) := det \begin{pmatrix} y_\ell(1) & -j_\ell(k) \\ \frac{d}{dr}\left(\frac{y_\ell(r)}{r}\right)_{r=1} & -k j_\ell'(k) \end{pmatrix} = 0, \qquad \ell = 0, 1, \ldots$$

$*$ $D_\ell(k)$ is the characteristic function of the problem

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(We will assume that n is not absorbing, i.e. $Im(n) = 0$.)

PROBLEM 2: THE SPHERICALLY SYMMETRIC TEP, $\ell = 0$

Consider the ITEP for axially symmetric eigenfunctions, i.e., $\ell = 0$:

$$v_0(r) = \alpha_0 j_0(kr),\ w_0(r) = \beta_0 \frac{y_0(r)}{r}$$

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$$y_0^{''}(r) + k^2 n(r) y_0(r) = 0, \quad y_0(0) = 0, \; y'(0) = 1,$$

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$k \in \mathbb{C}$ is a special transmission eigenvalue iff

$$D_0(k) := \frac{\sin k}{k} y'(1) - y_0(0) \cos k = 0$$

▷ $D_0(k)$ is the characteristic function of the problem

Existence, discreteness and distribution for TE:

- for sufficiently smooth refractive indices $n(r)$
- under restrictions on the size of $n \not\equiv 1$ and of the quantity $\delta := \int_0^1 \sqrt{n(t)}\mathrm{d}t$
- depending on the values of $n(1),\ n'(1)$ and $n''(1)$

Direct Problem 1: Existence for $\ell \geq 0$

Existence and discreteness for TE: by Colton and Monk [13], Colton and Päivärinta [14]

[13] The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium, Q. J. Mech. Appl. Math., 1988

[14] Far-field patterns for electromagnetic waves in an inhomogeneous medium, SIAM J. Math. Anal.,1990

Existence and discreteness for TE: by Colton and Monk [13], Colton and Päivärinta [14]

▷ Application of the Liouville Transformation:

$$\xi(r) := \int_0^r \sqrt{n(t)}\mathrm{d}t, \qquad y_\ell(r) = z_\ell(\xi) n(r)^{-1/4},$$

introduces the Schrödinger equation:

$$\ddot{z}_\ell(\xi) + \left(k^2 - \frac{\ell(\ell+1)}{\xi^2} - g(\xi)\right) z_\ell(\xi) = 0, \qquad 0 < \xi < \delta := \int_0^1 \sqrt{n(t)}\mathrm{d}t$$

where

$$g(\xi) := \frac{\ell(\ell+1)}{r^2 n(r)} - \frac{\ell(\ell+1)}{\xi^2} + \frac{n''(r)}{4n(r)^2} - \frac{5}{16}\frac{n'(r)^2}{n(r)^3}.$$

[13] The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium, Q. J. Mech. Appl. Math., 1988

[14] Far-field patterns for electromagnetic waves in an inhomogeneous medium, SIAM J. Math. Anal.,1990

Direct Problem 1: Existence for $\ell \geq 0$

▷ for large values of k and $\delta \neq 1$:

$$D_\ell(k) = \frac{1}{kn(0)^{\ell/2+1/4}} \sin\left(k\left(1-\delta\right)\right) + \mathrm{O}\left(\frac{\ln k}{k^2}\right)$$

▷ for large values of k and $\delta \neq 1$:

$$D_\ell(k) = \frac{1}{k n(0)^{\ell/2+1/4}} \sin\left(k\left(1-\delta\right)\right) + \mathrm{O}\left(\frac{\ln k}{k^2}\right)$$

Theorem (Colton - Päivärinta)

Assume that $n(x) = n(r) \in C^2$ is spherically stratified. Also, $n(r) = 1 \text{ for } r \geq 1$, and $n(r) > 1 \text{ or } 0 < n(r) < 1 \text{ for } 0 \leq r < 1$. Then there exists an infinite set of transmission eigenvalues.

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Theorem (Colton - Monk - Sun [15])

Assume that $n(x) = n(r) \in C^2[0,1]$ is spherically stratified. If $n(r) \not\equiv 1$ then there exists an infinite set of transmission eigenvalues.

[15] Analytical and computational methods for transmission eigenvalues, Inv. Probl., 2010.

Existence and discreteness for TE: by Colton and Monk [16], Colton, Päivärinta and Sylvester [17]

[16] The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium, Q. J. Mech. Appl. Math., 1988
[17] The interior transmission problem, Inv. Prob. Imag., 2007

Existence and discreteness for TE: by Colton and Monk [16], Colton, Päivärinta and Sylvester [17]

▷ Application of the Liouville Transformation introduces the Schrödinger equation:

$$\ddot{z}_0(\xi) + (k^2 - p(\xi))z_0(\xi) = 0, \;\; z_0(0) = 0, \qquad 0 < \xi < \delta,$$
$$\left(\frac{\cos k}{n(1)^{1/4}} + \frac{n'(1)}{4n(1)^{5/4}} \frac{\sin k}{k}\right) z_0(\delta) - n(1)^{1/4} \frac{\sin k}{k} \dot{z}_0(\delta) = 0,$$

where

$$p(\xi) := \frac{n''(r)}{4n(r)^2} - \frac{5}{16} \frac{(n'(r))^2}{n(r)^3}.$$

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Direct Problem 2: Existence for $\ell = 0$

▷ for large values of k:

$$D_0(k) = \frac{1}{k}\left[\left(\frac{n(1)}{n(0)}\right)^{1/4}\cos(k\delta)\sin k - \frac{1}{(n(1)n(0))^{1/4}}\sin(k\delta)\cos k\right] + \mathrm{O}\left(\frac{1}{k^2}\right).$$

▷ for large values of k and $n(1) = 1,\ \delta \neq 1$:

$$D_0(k) = \frac{1}{kn(0)^{1/4}}\sin\left(k\left(1-\delta\right)\right) + \mathrm{O}\left(\frac{1}{k^2}\right)$$

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Theorem (Colton - Päivärinta - Sylvester)

Assume that $n \in C^2[0,1]$ and either $n(1) \neq 1$ or $n(1) = 1$ and $\delta \neq 1$. Then there exists an infinite discrete set of transmission eigenvalues with axially symmetric eigenfunctions.

▷ Existence of complex eigenvalues k:

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Theorem (Colton - Leung - Meng [18])

Suppose the refractive index $n \in C^2[0,1]$ with $n(1) = 1,\ n'(1) = 0$ and $\delta \neq 1$. Then if $n''(1) \neq 0$, there exist infinitely many non-real and infinitely many real eigenvalues.

Theorem (Colton - Leung - Meng)

Let the refractive index $n \in C^2[0,1]$. Suppose $\delta = 1$ and $n(1) \neq 1$. Then there are at most finitely many complex transmission eigenvalues. However if both $\delta = 1$ and $n(1) = 1$, then it is possible to have only finitely many real eigenvalues.

[18] Distribution of complex transmission eigenvalues for spherically stratified media, Inv. Prob., 2015

Direct Problem 2: Complex eigenvalues for $\ell = 0$

▷ Existence of complex eigenvalues k:

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Theorem (Colton - Leung[19])

Assume that $n \in C^2[0,1]$ and either $1 < \sqrt{n(1)} < \delta$ or $\delta < \sqrt{n(1)} < 1$. Then there exist infinitely many real and infinitely many complex transmission eigenvalues.

Theorem (Colton - Leung)

Suppose that $n \in C^2[0,1]$ and $n(1) = 1$, $n'(1) = 0$, $n(r)$ is non-constant near $r = 1$ and $\delta \neq 1$. Then there exist infinitely many real and infinitely many complex eigenvalues.

[19] The existence of complex transmission eigenvalues for spherically stratified media, Appl. Anal., 2017

▷ Distribution of complex eigenvalues k:

Direct Problem 2: Eigenvalues distribution for $\ell = 0$

▷ Distribution of complex eigenvalues k:

Theorem (Colton - Leung)

Assume that $n(1) \neq 1$ and $n \in C^2[0,1]$. Then if complex eigenvalues exist, all of them lie in a strip parallel to the real axis.

Theorem (Colton - Leung - Meng)

Assume that $n \in C^2[0,1]$ with $n(1) = 1$ and $\delta \neq 1$. If either $n'(1) \neq 0$ or $n''(1) \neq 0$, the TE do not lie inside a fixed strip parallel to the real axis.

Direct Problem 2: Eigenvalues distribution for $\ell = 0$

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Theorem (Xu - Xu - Yang[20])

Assume that $n \in C^2[0,1]$, and $\delta = 1$. If either (i) $n(1) \neq 1$, or (ii) $n(1) = 1,\ n'(1) \neq 0$, or (iii) $n(1) = 1,\ n'(1) = 0,\ n''(1) \neq 0$ and $\int_0^1 p(s)\mathrm{d}s = 0$. If complex TE exist, lie in a strip parallel to the real axis.

[20] Distribution of transmission eigenvalues and inverse spectral analysis with partial information on the refractive index, Math. Meth. Appl. Sci., 2016

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[20] Distribution of transmission eigenvalues and inverse spectral analysis with partial information on the refractive index, Math. Meth. Appl. Sci., 2016

∗ Distribution of transmission eigenvalues:

Petkov - Vodev[21], Sylvester[22]

[21] Localization of the interior transmission eigenvalues for a ball, Inv. Probl. Imag., 2017.

[22] Transmission eigenvalues in one dimension, Inv. Probl.,2013.

∗ Distribution of transmission eigenvalues:

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McLaughlin - Polyakov[23], Xu - Xu -Wang, Xu - Yang -Buterin - Yurko [24], Wang - Shieh[25]

[21] Localization of the interior transmission eigenvalues for a ball, Inv. Probl. Imag., 2017.

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[23] On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, J. Differ. Equ., 1994.

[24] Estimates of complex eigenvalues and an inverse spectral problem for the transmission eigenvalue problem. Electron, J. Qual. Theory Differ. Equ, 2019.

[25] The inverse interior transmission eigenvalue problem with mixed spectral data, Appl. Math. Comput., 2019.

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∗ Absorbing media:

Cakoni - Colton - Haddar [26]

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21 Localization of the interior transmission eigenvalues for a ball, Inv. Probl. Imag., 2017.

22 Transmission eigenvalues in one dimension, Inv. Probl.,2013.

23 On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, J. Differ. Equ., 1994.

24 Estimates of complex eigenvalues and an inverse spectral problem for the transmission eigenvalue problem. Electron, J. Qual. Theory Differ. Equ, 2019.

25 The inverse interior transmission eigenvalue problem with mixed spectral data, Appl. Math. Comput., 2019.

26 The interior transmission eigenvalue problem for absorbing media, Inv. Probl., 2012.

Uniqueness results for the inverse spectral problem:

* for sufficiently smooth refractive indices $n(r)$
* under restrictions on the size of $n \not\equiv 1$ and of the quantity $\delta := \int_0^1 \sqrt{n(t)}\mathrm{d}t$
* depending on the values of $n(1),\ n'(1)$ and $n''(1)$

Inverse Problem 1: Uniqueness for $\ell \geq 0$

Inverse Spectral Problem:

Can the knowledge of the spectrum $\{k_j\}_{j=1}^{\infty}$, $(\forall \ell \geq 0)$,

uniquely determine $n(r)$?

Inverse Spectral Problem:

Can the knowledge of the spectrum $\{k_j\}_{j=1}^{\infty}$, $(\forall \ell \geq 0)$, uniquely determine $n(r)$?

Theorem (Cakoni - Colton - Gintides [27])

If $n(0)$ is known, then $n(r)$ is uniquely determined from all TE $(\forall l \geq 0)$, if $n \in C^2[0, \infty)$ and $n > 1$ or $0 < n < 1$.

[27] The interior transmission eigenvalue problem, SIAM J. Math. Anal., 2010

Inverse Problem 2: Uniqueness for $\ell = 0$

Inverse Spectral Problem:

Uniquely determine $n(r)$ from the special spectrum $\{k_j\}_{j=1}^{\infty}, \ (\text{for } \ell = 0)$.

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Uniquely determine $n(r)$ from the special spectrum $\{k_j\}_{j=1}^{\infty}$, (for $\ell = 0$).

Theorem (McLaughlin - Polyakov [28])

Let $n(r) > 0, n \in C^1(\mathbb{R}), n'' \in \mathcal{L}^2[0,1]$, and $n(1) = 1$, $n'(1) = 0$.
If $\delta \leq 1/3$, then $n(r)$ is uniquely determined from an appropriate subsequence of the spectrum. For $\delta > 1/3$, $\delta \neq 1$ n is determined in a subinterval.

[28] On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, J. Differ. Equ., 1994.

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Theorem (Aktosun - Gintides - Papanicolaou [29])

If $\delta < 1$, then the spherically symmetric TE uniquely determine $n(r)$.

[28] On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, J. Differ. Equ., 1994.

[29] The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation, Inv. Prob., 2011

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[29] The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation, Inv. Prob., 2011

[30] Complex eigenvalues and the inverse spectral problem for transmission eigenvalues, Inv. Probl., 2013

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▷ The $\delta = 1$ case:

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Theorem (Aktosun - Gintides - Papanicolaou)

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Theorem (Buterin - Yang - Yurko [31])

If $\delta = 1$, then the spherically symmetric TE without the constant coefficient γ cannot determine $n(r)$ uniquely.

[31] On an open question in the inverse transmission eigenvalue problem, Inv. Probl., 2015

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Theorem (Buterin - Yang - Yurko [31])

If $\delta = 1$, then the spherically symmetric TE without the constant coefficient γ cannot determine $n(r)$ uniquely.

Theorem (Wei - Xu [32])

If $\delta = 1$, and either (i) $n(1) \neq 1$ is known, or (ii) $n(1) = 1$ and $n \in C^{(m)}(1-\alpha, 1]$ for some $\alpha > 0$ and some $m \in \mathbb{N}$ which satisfies $n^{(j)} = 0$ for $j = 1, \ldots, m-1$ and $n^{(m)}(1) \neq 0$ is known, then the spherically symmetric TE uniquely determine $n(r)$.

[31] On an open question in the inverse transmission eigenvalue problem, Inv. Probl., 2015

[32] Inverse spectral analysis for the transmission eigenvalue problem, Inv. Probl., 2013.

Inverse Problem 2: Uniqueness for $\ell = 0$

- ▷ The $\delta > 1$ case:

Inverse Problem 2: Uniqueness for $\ell = 0$

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▷ The $\delta > 1$ case:

Comments (Aktosun - Gintides - Papanicolaou) and (Wei - Xu)

If $\delta > 1$, then the spherically symmetric TE cannot determine $n(r)$ uniquely.

Theorem (Xu - Yang -Buterin - Yurko [33])

Assume that $n \in W_2^{m+3}[0,1]$ for some $m \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Also, let $n(1) = 1,\ n^{(u)}(1) = 0$ for $u = \overline{1, m+1}$ and $n^{(m+2)}(1) \neq 0$. If $\delta > 1$ and $n(r)$ is known in $[\epsilon, 1]$ with ϵ satisfying

$$\int_\epsilon^1 \sqrt{n(t)}\mathrm{d}t = \frac{A-1}{2},$$

$\Rightarrow n(r)$ is uniquely determined by all zeros of $D_0(k)$, including multiplicity.

[33] Estimates of complex eigenvalues and an inverse spectral problem for the transmission eigenvalue problem. Electron, J. Qual. Theory Differ. Equ, 2019.

∗ Stability of the inverse problem:

Bondarenko - Buterin [34], Buterin - Choque-Rivero - Kuznetsova [35], Xu - Ma - Yang [36]

[34] On a local solvability and stability of the inverse transmission eigenvalue problem, Inv. Probl, 2017.

[35] On a regularization approach to the inverse transmission eigenvalue problem, Inv. Probl., 2020.

[36] On the stability of the inverse transmission eigenvalue problem from the data of McLaughlin and Polyakov, J. Differ. Equ., 2022.

Inverse Problem: Further Reading

∗ Stability of the inverse problem:

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∗ Isospectral sets:

Yang - Buterin [37]

[34] On a local solvability and stability of the inverse transmission eigenvalue problem, Inv. Probl, 2017.

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[36] On the stability of the inverse transmission eigenvalue problem from the data of McLaughlin and Polyakov, J. Differ. Equ., 2022.

[37] Isospectral sets for transmission eigenvalue problem, J. Inverse Ill-Posed Probl., 2020.

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∗ Absorbing media:

Chen [38]

[34] On a local solvability and stability of the inverse transmission eigenvalue problem, Inv. Probl, 2017.

[35] On a regularization approach to the inverse transmission eigenvalue problem, Inv. Probl., 2020.

[36] On the stability of the inverse transmission eigenvalue problem from the data of McLaughlin and Polyakov, J. Differ. Equ., 2022.

[37] Isospectral sets for transmission eigenvalue problem, J. Inverse Ill-Posed Probl., 2020.

[38] A uniqueness theorem on the eigenvalues of spherically symmetric interior transmission problem in absorbing medium, Complex Var. Elliptic Equ., 2015.

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∗ Fixed angular-momentum $\ell \geq 1$:

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[34] On a local solvability and stability of the inverse transmission eigenvalue problem, Inv. Probl, 2017.

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[39] Inverse transmission eigenvalue problem for fixed angular momentum, Inv. Probl. Imag., 2023.

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* Fixed angular-momentum $\ell \geq 1$:

Xu - Yang - Xu [39]

* Modified transmission eigenvalues:

Cogar - Colton - Leung [40], Gintides - Pallikarakis - Stratouras [41]

[34] On a local solvability and stability of the inverse transmission eigenvalue problem, Inv. Probl, 2017.

[35] On a regularization approach to the inverse transmission eigenvalue problem, Inv. Probl., 2020.

[36] On the stability of the inverse transmission eigenvalue problem from the data of McLaughlin and Polyakov, J. Differ. Equ., 2022.

[37] Isospectral sets for transmission eigenvalue problem, J. Inverse Ill-Posed Probl., 2020.

[38] A uniqueness theorem on the eigenvalues of spherically symmetric interior transmission problem in absorbing medium, Complex Var. Elliptic Equ., 2015.

[39] Inverse transmission eigenvalue problem for fixed angular momentum, Inv. Probl. Imag., 2023.

[40] The inverse spectral problem for transmission eigenvalues, Inv. Probl., 2016.

[41] Uniqueness of a spherically symmetric refractive index from modified transmission eigenvalues, Inv. Probl., 2022.

THE DISCONTINUOUS SPHERICALLY SYMMETRIC PROBLEM

Spherically symmetric and discontinuous $n(r)$

▷ Based on results of Gintides - Pallikarakis [42] and Pallikarakis [43]

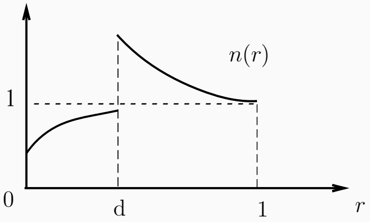
[42] The inverse transmission eigenvalue problem for a discontinuous refractive index, Inv. Probl., 2017.

[43] The inverse spectral problem for the reconstruction of the refractive index from the interior transmission problem, Ph.D. thesis, NTUA, 2017.

- Based on results of Gintides - Pallikarakis [42] and Pallikarakis [43]

Discontinuous refractive index:

$$n(r) \text{ is } C^2 \text{ in } [0,d) \text{ and } (d,1],\ n(r)=1 \text{ for } r \geq 1,\ n'(1)=0$$



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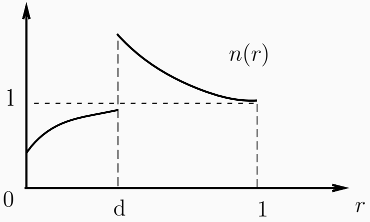
Jump Conditions:

$n(d^+) = a\, n(d^-)$

$n'(d^+) = a^{-1} n'(d^-) b\, n(d^-)$

$a>0,\ |a-1|+|b|>0,$

$d\in(0,1)$



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Transformed Problem:

▷ Liouville transf.: $z(\xi) := n(r)^{1/4} y_\ell(r),\ \xi(r) := \int_0^r \sqrt{n(\rho)}d\rho$

$$\frac{d^2 z(\xi)}{d\xi^2} + \left(k^2 - \frac{\ell(\ell+1)}{\xi^2} - g(\xi)\right) z(\xi) = 0, \quad 0 < \xi \neq \tilde{d}$$

$$g(\xi) = \frac{\ell(\ell+1)}{r^2 n(r)} - \frac{\ell(\ell+1)}{\xi^2} + \frac{n''(r)}{4n(r)^2} - \frac{5n'(r)^2}{16n(r)^3}$$

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where

$$0 < \xi < \delta := \int_0^1 \sqrt{n(t)} dt \ \text{ and } \ \tilde{d} := \int_0^d \sqrt{n(t)} dt, \quad \tilde{d} \in (0, A)$$

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▷ z is discontinuous at $\xi = \tilde{d}$, with conditions:

$$z(\tilde{d}^+) = \tilde{a}\ z(\tilde{d}^-), \quad \frac{dz(\tilde{d}^+)}{d\xi} = \tilde{a}^{-1} \frac{dz(\tilde{d}^-)}{d\xi} + \tilde{b}\ n(\tilde{d}^-)$$

where: $|a-1| + |b| > 0 \ \Rightarrow \ |\tilde{a}-1| + |\tilde{b}| > 0$
$\tilde{a} = a^{1/4}, \quad \tilde{b} = \frac{1}{4} n(d^-)^{3/4} n(d^+)^{-5/4} b + \frac{1}{4} n'(d^-) n(d^-)^{3/4} n(d^+)^{-9/4} (1 - a^2)$

Asymptotics of the characteristic functions:

▷ for $n(r) \in C^2[0,\infty)$,

$$\begin{aligned} \text{if } \ell = 0, \quad D_0(k) &= \frac{1}{kn(0)^{1/4}} \sin k(1-\delta) + O\left(\frac{1}{k^2}\right), \ k \to \infty \\ \text{if } \ell \geq 1, \quad D_\ell(k) &= \frac{1}{kn(0)^{\ell/2+1/4}} \sin k(1-\delta) + O\left(\frac{\ln k}{k^2}\right), \ k \to \infty \end{aligned}$$

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Proposition

▷ for $n(r) \in C^2[0,d) \cup C^2(d,1]$:

$$D_0(k) = \frac{1}{kn(0)^{1/4}} \left[\frac{\tilde{a}^2+1}{2\tilde{a}} \sin k(1-\delta) + \frac{1-\tilde{a}^2}{2\tilde{a}} \sin k\left(1-\delta+2\tilde{d}\right)\right] + O\left(\frac{1}{k^2}\right)$$

$$D_\ell(k) = \frac{1}{kn(0)^{\ell/2+1/4}} \left[\frac{\tilde{a}^2+1}{2\tilde{a}} \sin k(1-\delta) + (-1)^\ell \frac{1-\tilde{a}^2}{2\tilde{a}} \sin k\left(1-\delta+2\tilde{d}\right)\right] + O\left(\frac{\ln k}{k^2}\right),$$

⇒ there exists and infinite discrete set of transmission eigenvalues.

Uniqueness Results:

Theorem - uniqueness from eigenvalues for $\ell = 0$

Constants $\tilde{d}, \tilde{a}$ are uniquely determined by the special $(\ell = 0)$ transmission eigenvalues, if $|\tilde{a} - 1| + |\tilde{b}| > 0$ and:

(1). $\tilde{d} \in (0, \delta)$, for $0 < \delta < 1$

(2). $\tilde{d} \in \left(0, \frac{\delta-1}{2}\right)$ or $\tilde{d} \in \left(\frac{\delta-1}{2}, \delta - 1\right) \cup (\delta - 1, \delta)$ for $\delta > 1$

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Theorem - uniqueness from all eigenvalues for $\ell \geq 0$

Assume that $n(r)$ is C^2 or p-w C^2 and satisfies the jump conditions, where $0 < n < 1$ or $n > 1$. If $n(0)$ is known, then $n(r)$ is uniquely determined by all transmission eigenvalues.

SOME INTERESTING OPEN PROBLEMS

∗ Direct Problem:

Some interesting Open Problems

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  - Existence and distribution of eigenvalues for less smooth refractive indices, e.g. $n \in C$.

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* Numerical Methods:

[44] Distribution of complex transmission eigenvalues for spherically stratified media, Inv. Probl., 2015.

[45] Analytical and computational methods for transmission eigenvalues, Inv. Probl., 2010.

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  - The direct problem can be solved analytically in specific simple cases (Colton - Leung - Meng [44]) or for constant index (Colton - Monk - Sun [45]).

[44] Distribution of complex transmission eigenvalues for spherically stratified media, Inv. Probl., 2015.

[45] Analytical and computational methods for transmission eigenvalues, Inv. Probl., 2010.

[46] Reconstruction of a spherically symmetric speed of sound, SIAM J. Appl. Math., 1994.

[47] Reconstruction for a class of the inverse transmission eigenvalue problem, Math. Meth. Appl. Sci., 2019.

Some interesting Open Problems

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  - Existence and distribution of eigenvalues for less smooth refractive indices, e.g. $n \in C$.
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SOLVING TEP WITH NEUMANN SERIES OF BESSEL FUNCTIONS

▷ A numerical method for the direct and inverse TEP using NSBF: Kravchenko - Murcia-Lozano - Pallikarakis[48]

48 Neumann series of Bessel functions in direct and inverse spherically symmetric transmission eigenvalue problems, *(working paper)*.

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* Let $q \in \mathcal{L}^2(0, L)$ and $L > 0$. Consider the Sturm-Liouville equation

$$-y'' + q(x)y = \rho^2 y,\ 0 < x < L, \text{ and } \rho \in \mathbb{C}.$$

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∗ Solutions $S(\rho, x)$, $\phi(\rho, x)$ and $T(\rho, x)$ with initial conditions:

$$S(\rho,0) = 0, \; S'(\rho,0) = 1, \quad T(\rho,L) = 0, \; T'(\rho,L) = 1, \quad \phi(\rho,0) = 1, \; \phi'(\rho,0) = 0.$$

satisfying the identity:

$$T(\rho,x) = \phi(\rho,L)S(\rho,x) - \phi(\rho,x)S(\rho,L).$$

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Theorem (Kravchenko - Navarro - Torba [49], Kravchenko[50])

* The solutions $S(\rho, x)$ and $\phi(\rho, x)$ have the representation

$$S(\rho,x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho}\sum_{n=0}^{\infty} s_n(x) j_{2n+1}(\rho x), \qquad \phi(\rho,x) = \cos(\rho x) + \sum_{n=0}^{\infty} g_n(x) j_{2n}(\rho x),$$

$j_n(z)$ are spherical Bessel functions of order n.

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* The remainders of the partial sums

$$S_N(\rho, x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho}\sum_{n=0}^{N} s_n(x) j_{2n+1}(\rho x), \qquad \phi_N(\rho, x) = \cos(\rho x) + \sum_{n=0}^{N} g_n(x) j_{2n}(\rho x).$$

satisfy

$$\left|\rho S(\rho, x) - \rho S_N(\rho, x)\right| \leq \frac{\tilde{\varepsilon}_N(x)\sinh(Cx)}{C} \text{ and } \left|\phi(\rho, x) - \phi_N(\rho, x)\right| \leq \frac{\tilde{\varepsilon}_N(x)\sinh(Cx)}{C},$$

$\forall \rho$ in $|\rho| \leq C$, $C > 0$, where $\tilde{\varepsilon}_N(x) > 0$, tending to zero for $N \to \infty$.

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Solving TEP with NSBF

The spherically symmetric TEP (for $\ell = 0$) can be written as:

$$-\ddot{z}(\zeta) + p(\zeta)z(\zeta) = k^2 z(\zeta), \quad 0 < \zeta < \delta,$$

$$z(k,\delta) = 0, \quad \dot{z}(k,\delta) = -n^{-1/4}(0),$$

$$D_0(k) = \left(\frac{\cos k}{n^{1/4}(1)} + \frac{n'(1)\sin k}{4n^{5/4}(1)\,k} \right) z(k,0) + n^{1/4}(1)\frac{\sin k}{k}\dot{z}(k,0).$$

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Proposition

The characteristic function $D_0(k)$ is equivalent to

$$D_0(k) = a(k)\phi(k,\delta) + b(k)S(k,\delta), \quad k \in \mathbb{C},$$

where $\phi(k,\zeta)$ and $S(k,\zeta)$ are fundamental solutions of the Sturm-Liouville equation and

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Approximate the characteristic function by

$$D_{0,N}(k) = a(k)\cos(k\delta) + a(k)\sum_{n=0}^{N} g_n(\delta)j_{2n}(k\delta) + b(k)\frac{\sin(k\delta)}{k} + \frac{b(k)}{k}\sum_{n=0}^{N} s_n(\delta)j_{2n+1}(k\delta).$$

Refractive Index:

$n(r) = \frac{16}{(r+1)^2(3-r)^2}$.

The corresponding potential under the Liouville transform is $p(\zeta(r)) = 1/4$, and $\zeta \in [0, \log(3)]$.

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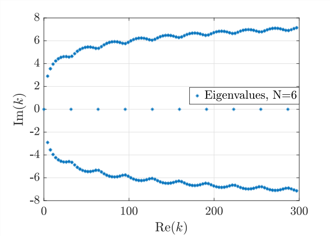

Direct Problem: Real and complex transmission eigenvalues.

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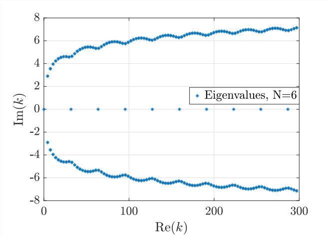

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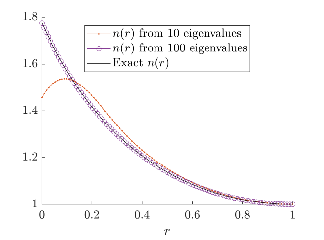
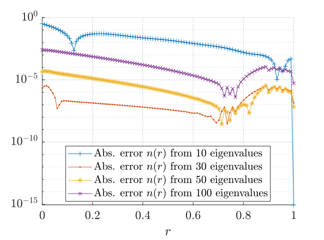


Inverse Problem: Reconstructions of the refractive index.



Inverse Problem: Abs error of the reconstructions.

References

1. N. Pallikarakis, **A review on the direct and inverse transmission eigenvalue problem for the spherically symmetric refractive index**, *Boletín de la Sociedad Matemática Mexicana*, 30, 2024.
2. D. Gintides and N. Pallikarakis, **The inverse transmission eigenvalue problem for a discontinuous refractive index**, *Inverse Problems*, 33, 2017.
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Thank You!!!