Instability theory of kink and anti-kink profiles for the sine-Gordon equation on a $\mathscr{Y}$-junction graph

Ramón G. Plaza
Institute of Applied Mathematics and Systems (IIMAS)
National Autonomous University of Mexico (UNAM)

June 27, 2024

International Biweekly Online Seminar on Analysis, Differential Equations and Mathematical Physics
Regional Mathematical Center, Southern Federal University - Russian Federation.

Joint work with: Jaime Angulo Pava (Univ. São Paulo)

Sponsors:

- DGAPA-UNAM, program PAPIIT, grants no. IN-100318, IN-104922.
- CNPq/Brazil and FAPERJ/Brazil program PRONEX-E - 26/010.001258/2016.

Table of contents

The sine-Gordon equation on graphs

The sine-Gordon equation

sine-Gordon equation in one dimension (laboratory coordinates):

$$u_{tt} - u_{xx} + \sin u = 0, \qquad x \in \mathbb{R}, \ \ t > 0.$$

Applications:

- Surfaces with negative Gaussian curvature (Eisenhart, 1909)
- Propagation of crystal dislocations (Frenkel and Kontorova, 1939)
- Elementary particles (Perring and Skyrme, 1962)
- Propagation of magnetic flux on a Josephson line (Scott, 1969)
- Dynamics of fermions in the Thirring model (Coleman, 1975)
- Oscillations of a rigid pendulum attached to a stretched rubber band (Drazin, 1983)

Superconductivity and quantum-tunneling

Josephson won the 1973 Nobel Prize in Physics for his discovery of the Josephson effect, describing the emergence of a supercurrent through a Josephson junction. The phase difference of wave functions of electrons in the super-conductors satisfy the sine-Gordon equation.

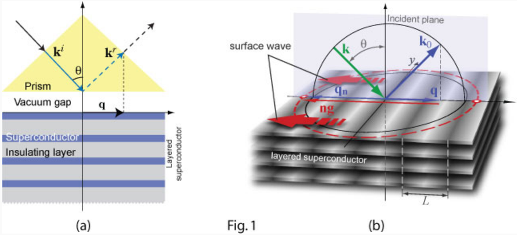

Figure 1: Two dimensional Josephson junction: infinite plates of superconductors separated by a thin dielectric barrier (image credit: AIST-NT, California, USA.)

PDEs on graphs

- Metric graph is a network-shaped structure of edges E_j which are assigned a metric (length).
- Edges are connected at vertices according to boundary conditions which determine the dynamics on the network.
- One can pose a system on PDEs on each edge. Graphs are not manifolds, though, so *the coupling is given exclusively through the boundary conditions at the vertices, known as the the "topology of the graph".*
- Many models are posed on branched domains that resemble a thin neighborhood of a graph, such as Josephson junction networks (Nakajima et al. 1976, 1978), electric circuits (Backhaus, Cherktov, 2013), unidirectional shallow water flow in a network (Bona, Cascaval, 2008), or nerve impulses in complex arrays of neurons (Scott, 2003).

Josephson tricrystal junctions

- The sine-Gordon equation was first conceived on a $\mathscr{Y}$-shaped Josephson junction by Nakajima *et al.* (1976, 1978) as a prototype for logic circuits.
- A $\mathscr{Y}$ junction consists of three long (semi-infinite) Josephson junctions coupled at one single common vertex, a structure known as a tricrystal junction.
- A $\mathscr{Y}$-junction of the first type (or type I) consists of one incoming (or parent) edge, $E_1 = (-\infty, 0)$, meeting at one single vertex at the origin, $v = 0$, with other two outgoing (children) edges, $E_j = (0, \infty)$, $j = 2, 3$.

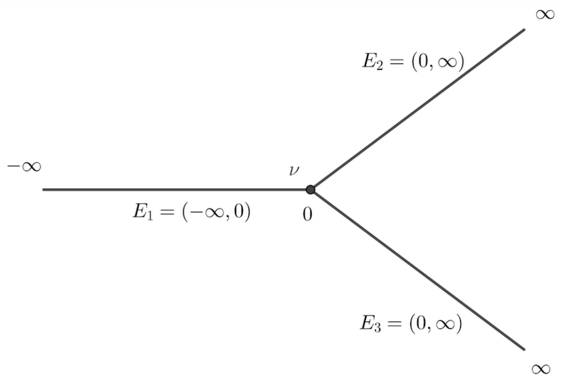

Figure 2: $\mathscr{Y}$-junction of the first type with $E_1 = (-\infty, 0)$ and $E_j = (0, \infty)$, $j = 2, 3$.

sine-Gordon model on a $\mathscr{Y}$-graph

The sine-Gordon model on a $\mathscr{Y}$-graph of type I reads

$$\partial_t^2 u_j - c_j^2 \partial_x^2 u_j + \sin u_j = 0, \qquad x \in E_j,\ t > 0,\ 1 \leqq j \leqq 3, \tag{SGg}$$

endowed with boundary conditions of either δ-type,

$$\begin{aligned} u_1(0-) = u_2(0+) = u_3(0+), \\ -c_1^2 u_1'(0-) + \sum_{j=2}^{3} c_j^2 u_j'(0+) = Z u_1(0-), \end{aligned} \tag{δ}$$

or of δ'-type,

$$\begin{aligned} c_1 u_1'(0-) = c_2 u_2'(0+) = c_3 u_3'(0+), \\ -c_1 u_1(0-) + \sum_{j=2}^{3} c_j u_j(0+) = \lambda c_1 u_1'(0-). \end{aligned} \tag{δ'}$$

Interpretation

- $\lambda, Z \in \mathbb{R}$ are given physical parameters. Measure the intensity of the interaction at the vertex.
- BCs of δ-type refers to continuity of the wave functions and a balance flux relation for the derivatives of the wave functions at the vertex.
- BCs of δ'-type imply continuity of the fluxes (surface current density is the same in all three thin films at the intersection) and a Kirchhoff-type rule for the self-induced magnetic flux.
- When $Z = 0$ we recover Kirchhoff's rule (Nakajima *et al.* 1978)

$$-c_1^2 \partial_x u_1(0-) + \sum_{j=2}^{3} c_j^2 \partial_x u_j(0+) = 0,$$

equivalent to charge conservation at the vertex.

- When $\lambda = 0$ we obtain a Kirchhoff-type rule for the self-induced magnetic flux (Nakajima *et al.* 1976)

$$-c_1 u_1(0-) + \sum_{j=2}^{3} c_j u_j(0+) = 0$$

- Both sets of boundary conditions correspond to all the self-adjoint extensions of the formal operator

$$\mathscr{F}\mathbf{u} = \left\{ \left(-c_j^2 \frac{d^2}{dx^2} \right) u_j \right\}_{j=1}^{3}, \qquad \mathbf{u} = (u_j)_{j=1}^{3},$$

on a star graph $\mathscr{Y}$ (for all values of λ and Z).

Stationary profiles

Equations (SGg) can be recast as a first order system,

$$\begin{cases} \partial_t u_j = v_j \\ \partial_t v_j = c_j^2 \partial_x^2 u_j - \sin u_j, \end{cases} \qquad x \in E_j,\ t > 0,\ 1 \leqq j \leqq 3. \qquad \text{(SGg')}$$

We are interested on the flow of (SGg') around solutions of stationary type,

$$u_j(x,t) = \phi_j(x), \qquad v_j(x,t) = 0,$$

$j = 1,2,3,\ x \in E_j,\ t > 0$, where

$$-c_j^2 \phi_j'' + \sin \phi_j = 0,$$

and the boundary conditions (δ) or (δ') at the vertex $\nu = 0$.

Preliminaries

Notations

- A closed, densely defined, symmetric operator in a Hilbert space H.
- The domain of A is denoted by $D(A)$.
- The deficiency indices of A are denoted by $n_{\pm}(A) := \dim\ker(A^* \mp iI)$ where A^* is the adjoint operator.
- The number of negative eigenvalues counting multiplicities (Morse index) of A is denoted by $n(A)$.
- For $-\infty \leq a < b \leq \infty$, $L^2(a,b)$ is Hilbert space equipped with the inner product

$$(u,v) = \int_a^b u(x)\overline{v(x)}dx.$$

- $H^m(a,b)$ we denote the classical Sobolev spaces on $(a,b) \subseteq \mathbb{R}$ with the usual norm.
- $\mathscr{Y}$ the junction graph parametrized by the edges $E_1 = (-\infty, 0)$, $E_j = (0, \infty)$, $j = 2,3$, attached to a common vertex $\nu = 0$.
- Sobolev and Lebesgue spaces on $\mathscr{Y}$ are defined as

$$H^m(\mathscr{Y}) = \oplus_{j=1}^{3} H^m(E_j) = H^m(-\infty, 0) \oplus H^m(0,\infty) \oplus H^m(0,\infty),$$
$$L^p(\mathscr{Y}) = \oplus_{j=1}^{3} L^p(E_j) = L^p(-\infty, 0) \oplus L^p(0,\infty) \oplus L^p(0,\infty),$$

Classical results on extension theory of symmetric operators (i)

Theorem (von Neumann decomposition)

Let A be a closed, symmetric operator, then

$$D(A^*) = D(A) \oplus \mathscr{N}_{-i} \oplus \mathscr{N}_{+i}.$$

with $\mathscr{N}_{\pm i} = \ker(A^* \mp iI)$. Therefore, for $u \in D(A^*)$ and $u = x + y + z \in D(A) \oplus \mathscr{N}_{-i} \oplus \mathscr{N}_{+i}$,

$$A^* u = Ax + (-i)y + iz.$$

Ref: Reed and Simon, vol.II, p. 138.

Classical results on extension theory of symmetric operators (ii)

Proposition 1

Let A be a densely defined, lower semi-bounded symmetric operator (that is, $A \geq mI$) with finite deficiency indices, $n_{\pm}(A) = k < \infty$, in the Hilbert space $\mathscr{H}$. Let $\widehat{A}$ be a self-adjoint extension of A. Then the spectrum of $\widehat{A}$ in $(-\infty, m)$ is discrete and consists of, at most, k eigenvalues counting multiplicities.

Ref.: Reed and Simon, vol.II, chapter X.

Classical results on extension theory of symmetric operators (iii)

Proposition 2

Let A be a densely defined, closed, symmetric operator in some Hilbert space H with deficiency indices equal $n_{\pm}(A) = 1$. All self-adjoint extensions A_θ of A may be parametrized by a real parameter $\theta \in [0, 2\pi)$ where

$$D(A_\theta) = \{x + c\phi_+ + \zeta e^{i\theta}\phi_- : x \in D(A), \zeta \in \mathbb{C}\},$$
$$A_\theta(x + \zeta\phi_+ + \zeta e^{i\theta}\phi_-) = Ax + i\zeta\phi_+ - i\zeta e^{i\theta}\phi_-,$$

with $A^*\phi_\pm = \pm i\phi_\pm$, and $\|\phi_+\| = \|\phi_-\|$.

Ref.: Reed and Simon, vol.II, chapter X.

Classical results on extension theory of symmetric operators (iv)

Proposition 3

All self-adjoint extensions of a closed, symmetric operator which has equal and finite deficiency indices have one and the same continuous spectrum.

Ref: Naimark, vol.II, p. 38.

Useful extension result

Proposition 4

Consider the closed symmetric operator, $(\mathscr{M}, D(\mathscr{M}))$, densely defined on $L^2(\mathscr{Y})$ by

$$\mathscr{M} = \Big(\Big(-c_j^2 \frac{d^2}{dx^2}\Big)\delta_{j,k}\Big),\ 1 \leqq j,k \leqq 3,$$

$$D(\mathscr{M}) = \Big\{(v_j)_{j=1}^3 \in H^2(\mathscr{Y}) : v_1(0-) = v_2(0+) = v_3(0+) = 0,\ \sum_{j=2}^3 c_j^2 v_j'(0+) - c_1^2 v_1'(0-) = 0\Big\},$$

with $\delta_{j,k}$ being the Kronecker symbol. Then the deficiency indices are $n_\pm(\mathscr{M}) = 1$. Moreover, we have that all the self-adjoint extensions of $(\mathscr{M}, D(\mathscr{M}))$, namely, $(\mathscr{J}_Z, D(\mathscr{J}_Z))$, $Z \in \mathbb{R}$, are defined by $\mathscr{J}_Z \equiv \mathscr{M}$ and $D(\mathscr{J}_Z)$ by

$$D(\mathscr{L}_Z) = \Big\{\mathbf{v} = (v_j)_{j=1}^3 \in H^2(\mathscr{Y}) : v_1(0-) = v_2(0+) = v_3(0+),\ \sum_{j=2}^3 c_j^2 v_j'(0+) - c_1^2 v_1'(0-) = Z v_1(0-)\Big\},$$

See Angulo Pava, P, J. Nonlinear Sci. (2021)

Instability theory with δ-interaction

Energy spaces

The energy space associated to (SGg') with δ-interaction is

$$\mathscr{E}(\mathscr{Y}) := \{(v_j)_{j=1}^3 \in H^1(\mathscr{Y}) : v_1(0-) = v_2(0+) = v_3(0+)\}.$$

First we verify that the Cauchy problem associated to (SGg') is well-posed in the energy space $\mathscr{E}(\mathscr{Y}) \times L^2(\mathscr{Y})$.

Local well-posedness (i)

Theorem (local well-posedness with δ-interaction)

For any $\Psi \in \mathscr{E}(\mathscr{Y}) \times L^2(\mathscr{Y})$ there exists $T > 0$ such that the sine-Gordon equation (SGg') has a unique solution $\mathbf{w} \in C([0,T]; \mathscr{E}(\mathscr{Y}) \times L^2(\mathscr{Y}))$ satisfying $\mathbf{w}(0) = \Psi$. For each $T_0 \in (0,T)$ the mapping

$$\Psi \in \mathscr{E}(\mathscr{Y}) \times L^2(\mathscr{Y}) \to \mathbf{w} \in C([0,T_0]; \mathscr{E}(\mathscr{Y}) \times L^2(\mathscr{Y})),$$

is at least of class C^2. Moreover, for all $t \in (0,T]$, $\mathbf{w}(t) \in \mathscr{E}(\mathscr{Y}) \times \mathscr{C}(\mathscr{Y})$, where $\mathscr{C}(\mathscr{Y}) = \{(v_j)_{j=1}^3 \in L^2(\mathscr{Y}) : v_1(0-) = v_2(0+) = v_3(0+)\}$.

Local well-posedness (ii)

Proof sketch: Not so tough, but delicate. Need explicit resolvent estimates and to apply Lumer-Phillips to obtain the semigroup.

1. Recast system in vector form

$$\mathbf{w}_t = JE\mathbf{w} + F(\mathbf{w})$$

where $\mathbf{w} = (u, v)^\top$, $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $u_1, v_1 : (-\infty, 0) \to \mathbb{R}$, $u_j, v_j : (0, +\infty) \to \mathbb{R}$, $j = 2, 3$,

$$J = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} \mathscr{F} & 0 \\ 0 & I_3 \end{pmatrix}, \quad F(\mathbf{w}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sin(u_1) \\ -\sin(u_2) \\ -\sin(u_3) \end{pmatrix}$$

where $I_3 =$ identity matrix and $\mathscr{F}$ is diagonal-matrix linear operator

$$\mathscr{F} = \Big(\Big(-c_j^2 \frac{d^2}{dx^2} \Big) \delta_{j,k} \Big), \quad 1 \leqq j, k \leqq 3.$$

Local well-posedness (ii)

2. Consider the operator $\mathscr{F}_Z \equiv \mathscr{F}$ defined on the δ-interaction domain

$$D(\mathscr{F}_Z) = \Big\{ (v_j)_{j=1}^3 \in H^2(\mathscr{Y}) : v_1(0-) = v_2(0+) = v_3(0+),\ \sum_{j=2}^{3} c_j^2 v_j'(0+) - c_1^2 v_1'(0-) = Z v_1(0-) \Big\}.$$

3. Perform the spectral analysis of $(\mathscr{F}_Z, D(\mathscr{F}_Z))$: For all $Z \in \mathbb{R}$, $\sigma_{\mathrm{ess}}(\mathscr{F}_Z) = \sigma_{\mathrm{ac}}(\mathscr{F}_Z) = [0, \infty)$. If $Z < 0$, $\mathscr{F}_Z$ has precisely one negative, simple eigenvalue,

$$\sigma_{\mathrm{pt}}(\mathscr{F}_Z) = \Big\{ -\frac{Z^2}{(\sum_{j=1}^3 c_j)^2} \Big\},$$

with eigenfunction $\Phi_Z = (e^{\alpha x}, e^{-\alpha x}, e^{-\alpha x})$, $\alpha = -Z/\sum_{j=1}^3 c_j > 0$. If $Z \geqq 0$, $\mathscr{F}_Z$ has no eigenvalues, $\sigma_{\mathrm{pt}}(\mathscr{F}_Z) = \varnothing$.

4. Explicit characterization of the resolvent of the operator $\mathscr{A} = JE$. Let $Z \in \mathbb{R}$. For $\lambda \in \mathbb{C}$ with $-\lambda^2 \in \rho(\mathscr{F}_Z)$, we have that $\lambda \in \rho(A)$ with $D(\mathscr{A}) = D(\mathscr{F}_Z) \times L^2(\mathscr{Y})$ and

$$R(\lambda : \mathscr{A}) = (\lambda I - \mathscr{A})^{-1} : H^1(\mathscr{Y}) \times L^2(\mathscr{Y}) \to D(\mathscr{A})$$

has the representation for $\Psi = (\mathbf{u}, \mathbf{v})$

$$R(\lambda : \mathscr{A})\Psi = \begin{pmatrix} -R(-\lambda^2 : \mathscr{F}_Z)(\lambda \mathbf{u} + \mathbf{v}) \\ -\lambda R(-\lambda^2 : \mathscr{F}_Z)(\lambda \mathbf{u} + \mathbf{v}) - \mathbf{u} \end{pmatrix},$$

where $R(-\lambda^2 : \mathscr{F}_Z) = (-\lambda^2 I_3 - \mathscr{F}_Z)^{-1} : L^2(\mathscr{Y}) \to D(\mathscr{F}_Z)$.

Local well-posedness (iv)

5. Apply Lumer-Phillips theorem to conclude the existence of a C_0-semigroup: $\mathscr{A} \equiv JE$ with $D(\mathscr{A}) = D(\mathscr{F}_Z) \times \mathscr{E}(\mathscr{Y})$ is the infinitesimal generator of a C_0-semigroup $\{W(t)\}_{t \geqq 0}$ on $H^1(\mathscr{Y}) \times L^2(\mathscr{Y})$. For any $\Psi \in H^1(\mathscr{Y}) \times L^2(\mathscr{Y})$ and $\theta > \beta + 1$, $\beta = \frac{Z^2}{(\sum_{j=1}^{3} c_j)^2}$, we have the representation formula

$$W(t)\Psi = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} e^{\lambda t} R(\lambda : \mathscr{A}) \Psi d\lambda$$

where $\lambda \in \rho(\mathscr{A})$ with $\operatorname{Re} \lambda = \theta$ and $R(\lambda : \mathscr{A}) = (\lambda I - \mathscr{A})^{-1}$, and for every $\delta > 0$, the integral converges uniformly in t for every $t \in [\delta, 1/\delta]$.

6. The local existence result follows from standard arguments (Banach fixed point theorem). Since the nonlinear term $F(z)$ is smooth, then the Implicit Function Theorem implies the smoothness property of the mapping data-solution.

□

Static solutions of kink type

For a given physical parameter $Z \in \mathbb{R}$ and the graph $\mathscr{Y} = (-\infty, 0) \cup (0, \infty) \cup (0, \infty)$. W.l.o.g. assume $c_j > 0$. Define static solutions of the form $\Phi = (\phi_j)_{j=1}^{3} \in H^2(\mathscr{Y})$, which are of kink type

$$\begin{aligned} \phi_1(x) &= 4 \arctan\left(e^{\frac{1}{c_1}(x-a_1)}\right), \quad x < 0, \\ \phi_i(x) &= 4 \arctan\left(e^{-\frac{1}{c_i}(x-a_i)}\right), \quad x > 0, \;\; i = 2, 3. \end{aligned} \tag{K-K}$$

Φ is subject to the boundary conditions (δ).

This yields the conditions:

$$\frac{1}{c_1}a_1 = -\frac{1}{c_2}a_2 = -\frac{1}{c_3}a_3$$

$$-\frac{e^{-\frac{a_1}{c_1}}}{1+e^{-\frac{2a_1}{c_1}}}\sum_{j=1}^{3} c_j = Z \arctan\left(e^{-\frac{a_1}{c_1}}\right).$$

for $Z \in \mathbb{R}$.

These conditions imply that, necessarily, $Z \in (-\sum_{j=1}^{3} c_j, 0)$. (Take a look at $f(y) = \frac{1+y^2}{y}\arctan(y)$ for $y \geqq 0$.)

(a) for $Z \in (-\sum_{j=1}^{3} c_j, -\frac{2}{\pi}\sum_{j=1}^{3} c_j)$ we obtain $a_1 > 0$, $a_2, a_3 < 0$, $\phi_i'' > 0$ for every i, and $\phi_1' > 0$, $\phi_j' < 0$ $(j = 1, 2)$. Thus, the profile is of tail-type. Moreover, $\phi_i \in (0, \pi)$, $i = 1, 2, 3$.

(b) the case $Z = -\frac{2}{\pi}\sum_{j=1}^{3} c_j$ implies $a_1 = 0 = a_2 = a_3$; therefore, $\phi_i(0) = \pi$ and $\phi_i''(0) = 0$, $i = 1, 2, 3$. In this case, we have a "smooth" profile around the vertex $\nu = 0$.

(c) if $Z \in (-\frac{2}{\pi}\sum_{j=1}^{3} c_j, 0)$ we get $a_1 < 0$; therefore $a_2, a_3 > 0$, and $\phi_i''(a_i) = 0$, $i = 1, 2, 3$. We also have $\phi_1' > 0$, $\phi_i' < 0$ $(i = 1, 2)$. Thus, the profile is of bump-type. Moreover, $\phi_i \in (0, \eta_0)$, $i = 1, 2, 3$, $\eta_0 = 4\arctan\left(e^{-\frac{1}{c_1}a_1}\right) > \pi$.

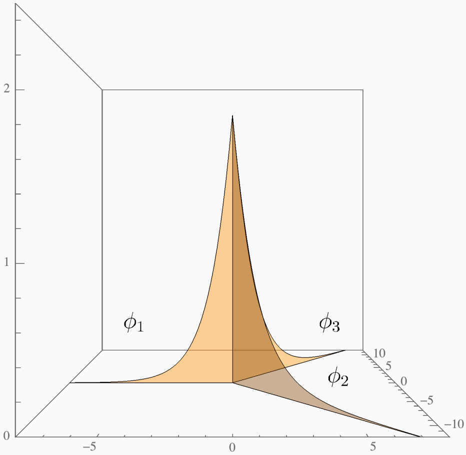

Figure 3: (a) “Tail” configuration for $Z \in (-\sum_{j=1}^{3} c_j, -\frac{2}{\pi}\sum_{j=1}^{3} c_j)$.

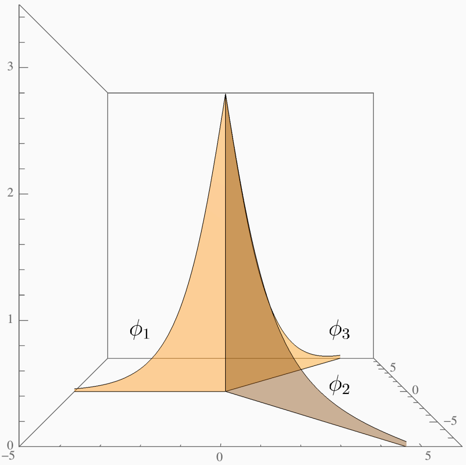

Figure 4: (b) "Smooth" profile solutions when $Z = -\frac{2}{\pi}\sum_{j=1}^{3} c_j$.

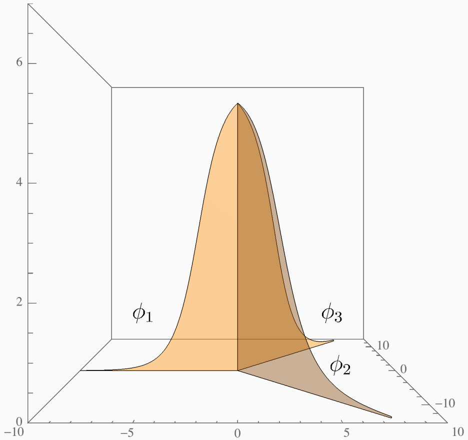

Figure 5: (c) Profiles of "bump" type for $Z \in (-\frac{2}{\pi}\sum_{j=1}^{3} c_j, 0)$.

Linearized problem around Φ

Rewrite system (SGg') as

$$\mathbf{w}_t = JE\mathbf{w} + F(\mathbf{w})$$

where $\mathbf{w} = (u, v)^\top$, with $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$,
$u_1, v_1 : (-\infty, 0) \to \mathbb{R}$, $u_j, v_j : (0, +\infty) \to \mathbb{R}$, $j = 2, 3$,

$$J = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} \mathscr{F} & 0 \\ 0 & I_3 \end{pmatrix}, \quad F(\mathbf{w}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sin(u_1) \\ -\sin(u_2) \\ -\sin(u_3) \end{pmatrix}$$

where $\mathscr{F}$ the diagonal-matrix linear operator

$$\mathscr{F} = \left(\left(-c_j^2 \frac{d^2}{dx^2} \right) \delta_{j,k} \right), \quad 1 \leqq j, k \leqq 3.$$

Here we will consider the operator $\mathscr{F}_Z \equiv \mathscr{F}$ defined on the δ-interaction domain

$$D(\mathscr{F}_Z) = \Big\{ \mathbf{v} \in H^2(\mathscr{Y}) : v_1(0-) = v_2(0+) = v_3(0+),\ \sum_{j=2}^{3} c_j^2 v_j'(0+) - c_1^2 v_1'(0-) = Z v_1(0-) \Big\},$$

Define the perturbation,

$$\mathbf{v} \equiv \mathbf{w} - \Phi.$$

and linearize around Φ,

$$\mathbf{v}_t = J\mathscr{E}\mathbf{v},$$

where $\mathscr{E}$ being the 6×6 diagonal-matrix $\mathscr{E} = \begin{pmatrix} \mathscr{L} & 0 \\ 0 & I_3 \end{pmatrix}$, and

$$\mathscr{L} = \Big(\Big(-c_j^2 \frac{d^2}{dx^2} + \cos(\phi_j) \Big) \delta_{j,k} \Big), \qquad 1 \leqq j,k \leqq 3. \qquad (**)$$

Here $D(\mathscr{L}) \equiv D(\mathscr{F}_Z)$ and hence $D(J\mathscr{E}) = D(\mathscr{E}) = D(\mathscr{F}_Z) \times L^2(\mathscr{Y})$.

Note that $\Phi \in D(\mathscr{F}_Z)$.

Spectral instability

For instability analysis we find a growing mode solution $\mathbf{v} = e^{\mu t}\Psi$ and $\operatorname{Re}\mu > 0$. Thus,

$$J\mathscr{E}\Psi = \mu\Psi,$$

with $\Psi \in D(J\mathscr{E})$.

Definition

The stationary vector solution $(\Phi, 0) \in D(\mathscr{E})$ is said to be spectrally stable for model sine-Gordon if the spectrum of $J\mathscr{E}$ satisfies $\sigma(J\mathscr{E}) \subset i\mathbb{R}$. Otherwise, the stationary solution is said to be spectrally unstable.

Observations

- It is standard to show that $\sigma_{\mathrm{pt}}(J\mathscr{E})$ is symmetric with respect to both the real and imaginary axes and $\sigma_{\mathrm{ess}}(J\mathscr{E}) \subset i\mathbb{R}$ by supposing J skew-symmetric and $\mathscr{E}$ self-adjoint.
- Hence it is equivalent to say that $\Phi \in D(J\mathscr{E})$ is spectrally stable if $\sigma_{\mathrm{pt}}(J\mathscr{E}) \subset i\mathbb{R}$, and it is spectrally unstable if $\sigma_{\mathrm{pt}}(J\mathscr{E})$ contains point λ with $\operatorname{Re}\lambda > 0$.
- The eigenvalue problem to solve is now reduced to,

$$J\mathscr{E}\Psi = \lambda\Psi, \quad \operatorname{Re}\lambda > 0, \;\; \Psi \in D(\mathscr{E}). \tag{EvP}$$

Linear instability criterion for the sine-Gordon model on a $\mathscr{Y}$-junction

Assumptions:

(S_1) $J\mathscr{E}$ is the generator of a C_0-semigroup $\{S(t)\}_{t\geqq 0}$.

(S_2) Let $\mathscr{L}$ be the matrix-operator in (**), defined on a domain $D(\mathscr{L}) \subset L^2(\mathscr{Y})$ on which $\mathscr{L}$ is self-adjoint.

(S_3) Suppose $\mathscr{L} : D(\mathscr{L}) \to L^2(\mathscr{Y})$ is invertible with Morse index $n(\mathscr{L}) = 1$ and such that $\sigma(\mathscr{L}) = \{\lambda_0\} \cup J_0$ with $J_0 \subset [r_0, \infty)$, for $r_0 > 0$, and $\lambda_0 < 0$,

Linear instability criterion for the sine-Gordon model on a $\mathscr{Y}$-junction

Assumptions:

(S_1) $J\mathscr{E}$ is the generator of a C_0-semigroup $\{S(t)\}_{t\geqq 0}$.

(S_2) Let $\mathscr{L}$ be the matrix-operator in (**), defined on a domain $D(\mathscr{L}) \subset L^2(\mathscr{Y})$ on which $\mathscr{L}$ is self-adjoint.

(S_3) Suppose $\mathscr{L} : D(\mathscr{L}) \to L^2(\mathscr{Y})$ is invertible with Morse index $n(\mathscr{L}) = 1$ and such that $\sigma(\mathscr{L}) = \{\lambda_0\} \cup J_0$ with $J_0 \subset [r_0, \infty)$, for $r_0 > 0$, and $\lambda_0 < 0$,

Theorem

Suppose assumptions (S_1) - (S_3) hold. Then the operator $J\mathscr{E}$ has a real positive and a real negative eigenvalue.

Proof sketch: The proof of the criterion is based on the work by Lopes (2002) and from the following result on closed convex cones by Krasnoselskiĭ (1964) (chapter 2, section 2.2.6):

Theorem

Let K be a closed convex cone of a Hilbert space $(X, \|\cdot\|)$ such that there are a continuous linear functional Π and a constant $a > 0$ such that $\Pi(u) \geqq a\|u\|$ for any $u \in K$. If $T : X \to X$ is a bounded linear operator that leaves K invariant, then T has an eigenvector in K associated to a nonnegative eigenvalue.

Apply the theorem to the non-empty closed convex cone

$$K_0 = \{z \in D(\mathscr{E}) : \langle \mathscr{E} z, z\rangle \leqq 0, \text{ and } \langle z, \Psi_0\rangle \geqq 0\}.$$

Then you show that the eigenvalue must be $\zeta \neq 0$. By symmetry, both $-\zeta$ and ζ are eigenvalues. □

Main instability result

Theorem

Let $Z \in (-\sum_{j=1}^{3} c_j, 0)$. Then the smooth family of stationary profiles of kink type $Z \mapsto \Phi_Z$ defined above is spectrally unstable for the sine-Gordon model (SGg').

Proof sketch (i)

The proof goes on a case-by-case basis and calculation of the Morse index (the number of negative eigenvalues counting multiplicities).

Lemma 1

Let $Z \in (-\sum_{j=1}^{3} c_j, 0)$. Then $\ker(\mathscr{L}_Z) = \{\mathbf{0}\}$.

Proof sketch: Follows from Sturm-Liouville theory on half-lines. Let $\mathbf{u} = (u_1, u_2, u_3) \in D(\mathscr{L}_Z)$ and $\mathscr{L}_Z \mathbf{u} = \mathbf{0}$. Since $-c_j^2 \frac{d^2}{dx^2}\phi_j' + \cos(\phi_j)\phi_j' = 0$, $j = 1, 2, 3$, we obtain

$$u_1(x) = \alpha_1 \phi_1'(x),\ \ x < 0, \quad u_j(x) = \alpha_j \phi_j'(x),\ \ x > 0,\ \ j = 2, 3,$$

with $\alpha_j \in \mathbb{R}$. Since $\phi_j'(0+) = -\phi_1'(0-)\frac{c_1}{c_j}$, $j = 2, 3$, we get from the conditions of $D(\mathscr{L}_Z)$

$$\alpha_1 = -\alpha_2 \tfrac{c_1}{c_2} = -\alpha_3 \tfrac{c_1}{c_3}, \quad \sum_{j=2}^{3} \alpha_j c_j^2 \phi_j''(0+) - \alpha_1 c_1^2 \phi_1''(0-) = Z\alpha_1 \phi_1'(0-).$$

Proof sketch (ii)

Cases:

1) Let $Z = -\frac{2}{\pi}\sum_{j=1}^{3} c_j$. Then, from $\phi_j''(0) = 0$ for all j we obtain $\alpha_1\phi_1'(0-) = 0$. Since $\phi_1'(0-) \neq 0$ we have $\alpha_1 = 0$ and so $\alpha_2 = \alpha_3 = 0$. Hence $\mathbf{u} = \mathbf{0}$.
2) Let $Z \in (-\frac{2}{\pi}\sum_{j=1}^{3} c_j, 0)$. From continuity we have

$$-c_j^2\phi_j''(0+) = -\sin(\phi_j(0+)) = -\sin(\phi_1(0-)) = -c_1^2\phi_1''(0-).$$

Then

$$-\alpha_1 c_1 \phi_1''(0-)\sum_{j=1}^{3} c_j = Z\alpha_1\phi_1'(0-).$$

Suppose $\alpha_1 \neq 0$. Then, since $\phi_1''(0-) < 0$ and $\phi_1'(0-) > 0$ we obtain a contradiction. Hence, $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Proof sketch (iii)

3) Let $Z \in (-\sum_{j=1}^{3} c_j, -\frac{2}{\pi}\sum_{j=1}^{3} c_j)$. Suppose $\alpha_1 \neq 0$. Then, since

$$\phi_1''(0) = 4\frac{e^{-\frac{a_1}{c_1}} - e^{-\frac{3a_1}{c_1}}}{[1+e^{-\frac{2a_1}{c_1}}]^2 c_1^2}, \quad \phi_1'(0) = \frac{4e^{-\frac{a_1}{c_1}}}{[1+e^{-\frac{2a_1}{c_1}}]c_1},$$

we obtain

$$(1-y^2)\arctan y = y, \qquad y = e^{-\frac{a_1}{c_1}}.$$

Since $a_1 > 0$ we obtain $y \in (0,1)$ and so the function $h(x) = (1-x^2)\arctan x - x$ has a zero for $x \in (0,1)$. Since $h(0) = 0, h(1) = -1$ and $h'(x) < 0$ on $(0,1)$, we obtain a contradiction. Hence, $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

□

Proof sketch (iv)

Lemma 2

Let $Z \in \left(-\sum_{j=1}^{3} c_j, -\frac{2}{\pi}\sum_{j=1}^{3} c_j\right]$. Then $n(\mathscr{L}_Z) = 1$.

Proof sketch. Follows from extension theory for symmetric operators. From **Proposition 4** the family $(\mathscr{L}_Z, D(\mathscr{L}_Z))$ represents all the self-adjoint extensions of the closed symmetric operator $(\mathscr{M}_0, D(\mathscr{M}_0))$ where

$$\mathscr{M}_0 = \left(\left(-c_j^2 \frac{d^2}{dx^2} + \cos(\phi_j)\right)\delta_{j,k}\right),\ 1 \leqq j,k \leqq 3,$$

$$D(\mathscr{M}_0) = \left\{(v_j)_{j=1}^3 \in H^2(\mathscr{G}) : v_1(0-) = v_2(0+) = v_3(0+) = 0,\ \sum_{j=2}^{3} c_j^2 v_j'(0+) - c_1^2 v_1'(0-) = 0\right\},$$

where $n_\pm(\mathscr{M}_0) = 1$. It can be shown that $\mathscr{M}_0 \geqq 0$. Let $L_j = -c_j^2 \frac{d^2}{dx^2} + \cos(\phi_j)$, therefore

$$L_j \psi = -\frac{1}{\phi_j'} \frac{d}{dx}\left[c_j^2 (\phi_j')^2 \frac{d}{dx}\left(\frac{\psi}{\phi_j'}\right)\right].$$

Note that $\phi_j' \neq 0$.

Proof sketch (v)

For $\Psi = (\psi_j) \in D(\mathscr{M}_0)$ we get

$$\langle \mathscr{M}_0\Psi, \Psi\rangle = \int_{-\infty}^{0} c_1^2(\phi_1')^2 \Big|\frac{d}{dx}\Big(\frac{\psi_1}{\phi_1'}\Big)\Big|^2 dx + \sum_{j=2}^{3}\int_0^{+\infty} c_j^2(\phi_j')^2\Big|\frac{d}{dx}\Big(\frac{\psi_j}{\phi_j'}\Big)\Big|^2 dx$$
$$- c_1^2\psi_1(0)\Big[\frac{\psi_1'(0)\phi_1'(0) - \psi_1(0)\phi_1''(0)}{\phi_1'(0)}\Big] + \sum_{j=2}^{3} c_j^2\psi_j(0)\Big[\frac{\psi_j'(0)\phi_j'(0) - \psi_j(0)\phi_j''(0)}{\phi_j'(0)}\Big]$$

The integral terms are non-negative and equal zero if and only if $\Psi \equiv 0$. Due to the conditions $\psi_1(0-) = \psi_2(0+) = \psi_3(0+) = 0$ the non-integral term vanishes and we get $\mathscr{M}_0 \geqq 0$.

Due to **Proposition 1** we have all the self-adjoint extensions $\mathscr{L}_Z$ of $\mathscr{M}_0$ satisfies $n(\mathscr{L}_Z) \leqq 1$. Next, for $\Phi = (\phi_1, \phi_2, \phi_3) \in D(\mathscr{L}_Z)$, it follows from the relations $L_j\phi_j = -\sin(\phi_j) + \cos(\phi_j)\phi_j$ that

$$\langle \mathscr{L}_Z\Phi, \Phi\rangle = \textstyle\int_{-\infty}^{0}[-\sin(\phi_1) + \cos(\phi_1)\phi_1]\phi_1 dx + \sum_{j=2}^{3}\int_0^{+\infty}[-\sin(\phi_j) + \cos(\phi_j)\phi_j]\phi_j dx < 0,$$

because of $0 < \phi_j(x) \leqq \pi$ for every $Z \in \big(-\sum_{j=1}^3 c_j, -\frac{2}{\pi}\sum_{j=1}^3 c_j\big]$ and $\theta\cos\theta \leqq \sin\theta$ for all $\theta \in [0, \pi]$. Then from minimax principle we arrive at $n(\mathscr{L}_Z) = 1$. This finishes the proof. $\square$

Lemma 3

Let $Z \in \left(-\frac{2}{\pi}\sum_{j=1}^{3} c_j, 0\right)$. Then $n(\mathscr{L}_Z) = 1$.

Proof sketch. Follows from analytic perturbation theory. From **Lemma 1**, for $Z^* := -\frac{2}{\pi}\sum_{j=1}^{3} c_j$ there holds $n(\mathscr{L}_{Z^*}) = 1$. Now, from the relation between Z and the $c_j's$ we have the continuous mapping function $Z \in \left(-\sum_{j=1}^{3} c_j, 0\right) \to a_1(Z)$ such that

$$a_1(Z) = \begin{cases} < 0, & \text{for } Z^* < Z < 0, \\ = 0, & \text{for } Z = Z^*, \\ > 0, & \text{for } -\sum_{j=1}^{3} c_j < Z < Z^*. \end{cases}$$

Proof sketch (vii)

- First, it can be proved that $\mathscr{L}_Z$ converges to $\mathscr{L}_{Z^*}$ as $Z \to Z^*$ in the generalized sense.
- Then, it can be shown that $n(\mathscr{L}_Z) = 1$ for $Z \in [Z^* - \delta_1, Z^* + \delta_1]$ for $0 < \delta_1 \ll 1$ sufficiently small.
- Finally, use a classical continuation argument based on the Riesz-projector to show that $n(\mathscr{L}_Z) = 1$ for all $Z \in (Z^*, 0)$. Define

  $$\omega = \sup\{\eta : \eta \in (Z^*, 0) \text{ such that } n(\mathscr{L}_Z) = 1 \text{ for all } Z \in (Z^*, \eta)\}.$$

  It can be proved (using Riesz projectors) that $\omega = 0$.

$\square$

Upon application of Lemmata 1, 2 and 3, as well as the linear instability criterion, we obtain the result.

Static solutions of kink/anti-kink type

For simplicity assume $c_j \equiv 1$ and define kink/anti-kink solutions,

$$\begin{aligned}
\phi_1(x) &= 4\arctan\left(e^{-(x-a_1)}\right), \quad x<0, \quad \lim_{x\to-\infty}\phi_1(x) = 2\pi \\
\phi_i(x) &= 4\arctan\left(e^{-(x-a_i)}\right), \quad x>0, \quad \lim_{x\to+\infty}\phi_i(x) = 0, \ \ i=2,3,
\end{aligned} \tag{K-aK}$$

subject to the boundary conditions (δ).

By a similar analysis we end up with $Z \in (-1,0)$.

Notice however, that *the kink/anti-kink stationary profiles* (K-aK) *do not belong to the energy space $H^2(\mathscr{Y})$.*

(a) for $Z \in (-1, -\frac{2}{\pi})$ we obtain $a_1 < 0$, $\phi_i'' > 0$, $i = 2, 3$, and $\phi_1''(a_1) = 0$. Thus, the profile looks like two left-translated anti-kinks on all the line. Moreover, $\phi_i(0) \in (\eta, \pi)$, $i = 1, 2, 3$, $\eta > 0$,

(b) the case $Z = -\frac{2}{\pi}$ implies $a_1 = a_2 = a_3 = 0$; therefore, $\phi_i(0) = \pi$ and $\phi_i''(0) = 0$, $i = 1, 2, 3$. In this case, we have two-classical anti-kink profile around the vertex $\nu = 0$

(c) for $Z \in (-\frac{2}{\pi}, 0)$ we obtain $a_1 > 0$, $\phi_i''(a_1) = 0$, $i = 2, 3$. The profile looks like two right-translated anti-kinks on all the line. Moreover, $\phi_i(0) \in (\pi, 2\pi)$, $i = 1, 2, 3$,

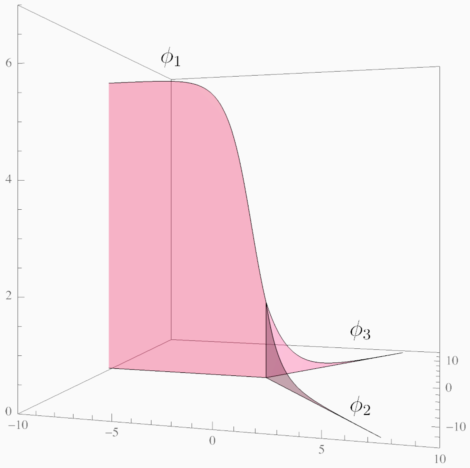

Figure 6: (a) Profile solutions of anti-kink/kink type when $Z \in (-1, -2/\pi)$, $c_j \equiv 1$ (left-translated anti-kink configuration).

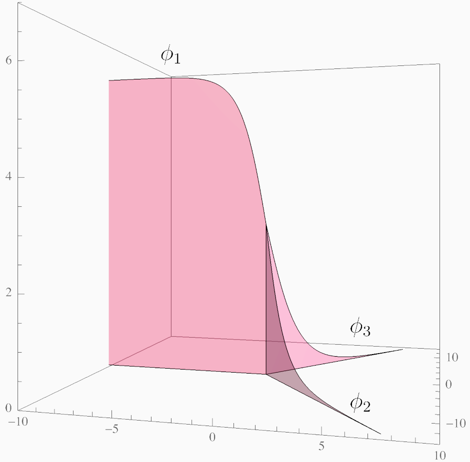

Figure 7: (b) Solution of anti-kink/kink type when $Z = -2/\pi$, $c_j \equiv 1$.

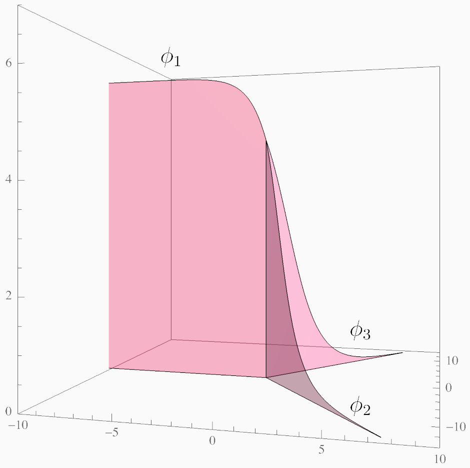

Figure 8: (c) Solution of anti-kink/kink type when $Z \in (-2/\pi, 0)$, $c_j \equiv 1$ (right-translated anti-kink).

Main instability result

Using the same techniques one can prove:

Theorem

Let $Z \in (-1, 0)$, $c_i \equiv 1$. Then the smooth family of stationary profiles of kink/anti-kink type $Z \mapsto \Phi_Z$ defined above is spectrally unstable for the sine-Gordon model (SGg').

Instability theory with δ'-interaction

Local well-posedness

The energy space associated to (SGg') with δ'-interaction is simply $H^1(\mathscr{Y}) \times L^2(\mathscr{Y})$. Similar arguments lead to

Local well-posedness

The energy space associated to (SGg') with δ'-interaction is simply $H^1(\mathscr{Y}) \times L^2(\mathscr{Y})$. Similar arguments lead to

Theorem (local well-posedness with δ'-interaction)

For any $\Psi \in H^1(\mathscr{Y}) \times L^2(\mathscr{Y})$ there exists $T > 0$ such that the sine-Gordon equation (SGg') has a unique solution $\mathbf{w} \in C([0,T]; H^1(\mathscr{Y}) \times L^2(\mathscr{Y}))$ satisfying $\mathbf{w}(0) = \Psi$. For each $T_0 \in (0,T)$ the mapping data-solution

$$\Psi \in H^1(\mathscr{Y}) \times L^2(\mathscr{Y}) \to \mathbf{w} \in C([0,T_0]; H^1(\mathscr{Y}) \times L^2(\mathscr{Y})),$$

is at least of class C^2.

Stationary solutions of kink type

Once again we propose the family $\Psi_{\lambda,\delta'} = \{\phi_i\}_{i=1}^3$ where

$$\begin{cases} \phi_1(x) = 4\arctan\left(e^{(x-a_1)/c_1}\right), & x \in (-\infty, 0), \\ \phi_j(x) = 4\arctan\left(e^{-(x-a_j)/c_j}\right), & x \in (0, \infty),\ j = 2, 3. \end{cases} \tag{K-K'}$$

Similar considerations over the boundary conditions (δ') imply $\lambda \in \left(-\infty, -\sum_{j=1}^3 c_j\right)$.

(a) for $\lambda \in (-\infty, -\frac{\pi}{2}\sum_{j=1}^{3} c_j)$ we obtain $a_2 > 0$, $a_3 > 0$, $a_1 < 0$, $\phi_i' < 0$ and $\phi_i''(a_i) = 0$, for $i = 1, 2, 3$. Moreover, $\phi_i \in (0, \eta)$, $i = 2, 3$, $-\phi_1 \in (-\eta, 0)$, with $\eta = 4\arctan\left(e^{a_2/c_2}\right) > \pi$. Thus, the profile is of bump-type.

(b) the case $\lambda = -\frac{\pi}{2}\sum_{j=1}^{3} c_j$ implies $a_1 = a_2 = a_3 = 0$, $\phi_1(0) = \phi_2(0) = \phi_3(0) = \pi$. Moreover, $\phi_i''(0) = 0$, $i = 1, 2, 3$.

(c) for $\lambda \in (-\frac{\pi}{2}\sum_{j=1}^{3} c_j, -\sum_{j=1}^{3} c_j)$ we obtain $a_2 < 0$, $a_3 < 0$, $a_1 > 0$, $\phi_i' < 0$ and $\phi_i'' > 0$ for $i = 2, 3$, $\phi_1' > 0$ and $\phi_1'' > 0$. $\phi_j \in (0, \pi)$ for every j. Thus, the profile is of tail-type.

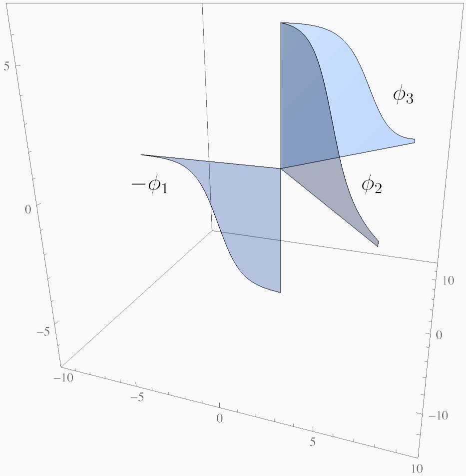

Figure 9: (a) "Bump-type" configuration for $\lambda \in (-\infty, -\frac{\pi}{2}\sum_{j=1}^{3} c_j)$.

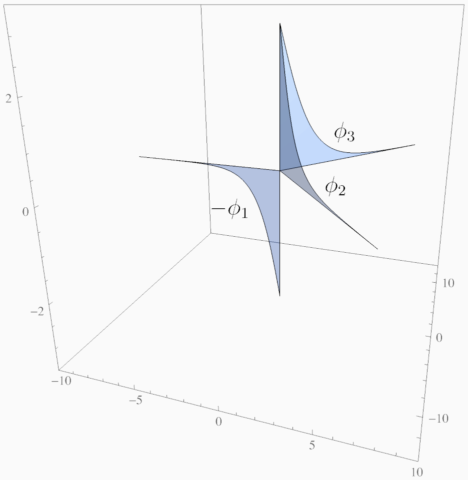

Figure 10: (b) Profile solution when $\lambda = -\frac{\pi}{2}\sum_{j=1}^{3} c_j$.

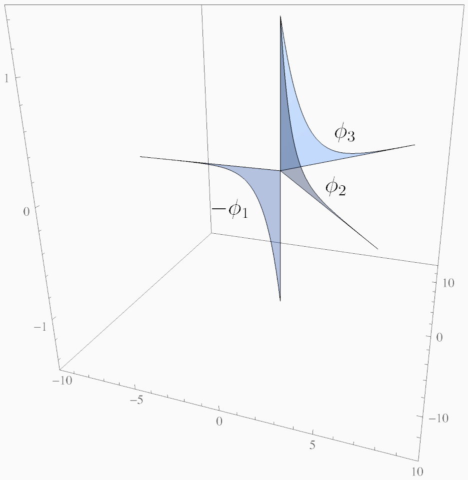

Figure 11: (c) “Tail-type” configuration for $\lambda \in (-\frac{\pi}{2}\sum_{j=1}^{3} c_j, -\sum_{j=1}^{3} c_j)$.

Spectral instability result

Similar spectral techniques can be applied to obtain

Theorem

Let $\lambda \in (-\infty, -\sum_{j=1}^{3} c_j)$, $c_j > 0$. Then the smooth family of stationary profiles of kink type, $\lambda \mapsto \Psi_{\lambda,\delta'}$ determined above is spectrally unastable in the following cases:

(i) for $\lambda \in (-\frac{\pi}{2}\sum_{j=1}^{3} c_j, -\sum_{j=1}^{3} c_j)$ and the constants a_i and c_i satisfying

$$a_3 = \frac{c_3}{c_2} a_2, \quad a_1 = -\frac{c_1}{c_2} a_2, \quad c_i > 0,$$

(ii) for $\lambda \in (-\infty, -\frac{\pi}{2}\sum_{j=1}^{3} c_j]$ with same conditions as in (i) plus $c_1 = c_2 = c_3$.

Same story for profiles of kink/anti-kink type $\Pi_{\lambda,\delta'}$. For simplicity $c_1 = c_2 = c_3 = 1$ and

(a) for $a_1 < 0$, $\lambda \in (-\infty, -\frac{\pi}{2})$.

(b) for $a_1 = 0$, $\lambda = -\frac{\pi}{2}$.

(c) for $a_1 > 0$, $\lambda \in (-\frac{\pi}{2}, +\infty)$.

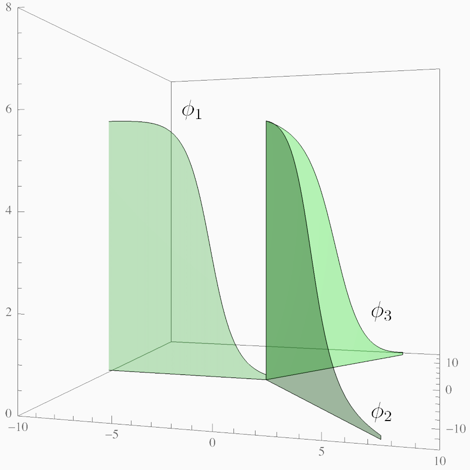

Figure 12: Solution of anti-kink/kink type when $\lambda \in (-\infty, -\frac{\pi}{2})$, $c_j \equiv 1$ ("bump-type" configuration).

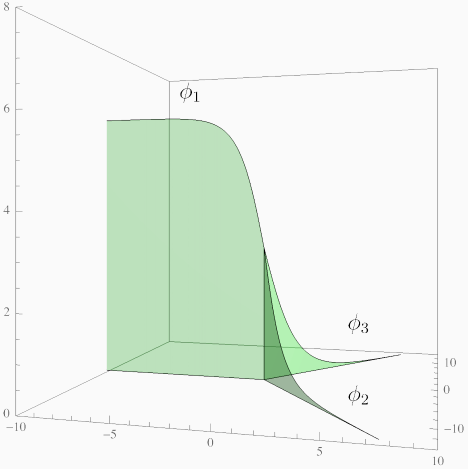

Figure 13: Solution of anti-kink/kink type when $\lambda = -\frac{\pi}{2}$, $c_j \equiv 1$.

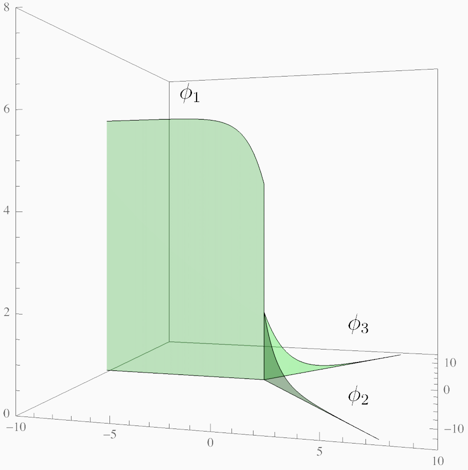

Figure 14: Solution of anti-kink/kink type when $\lambda \in (-\frac{\pi}{2}, \infty)$, $c_j \equiv 1$ ("tail-type" configuration).

Partial instability result

The spectral study yields

Theorem

Let $\lambda \in (-\frac{\pi}{2}, \infty)$. Then the smooth family of stationary kink/anti-kink profiles $\lambda \mapsto \Pi_{\lambda,\delta'}$ is spectrally unstable.

Discussion

Open questions

- Our results constitute a thorough (nonlinear) instability theory for static solutions to the sG equation on a $\mathscr{Y}$-junction graph.
- Are there stable structures? Susanto and van Gils (2005) (Kogan *et al.*, 2000) considered a δ'-interaction with $\lambda = 0$.
- They proposed a shifted profile which is apparently stable (numerics).
- Conjecture 1: that structure is a minimizer of a certain energy for $\lambda = 0$. (Work in progress.)
- Conjecture 2: there are particular structures that minimize Z- and λ-energies. (Work in progress.)

References

- **Angulo Pava, P**, J. Nonlinear Sci. 31 (2021), art. no. 50, 1–32.
- **Angulo Pava, P**, Phys. D 427 (2021), 133020, 1–12.
- **Angulo Pava, P**, Math. Z. 300 (2022), no. 3, 2885–2915.

Thanks!