Non-local abstract integro-differential equations

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Outline

1. Part I: Motivation
   - The classical heat equation
   - An interesting and important feature
2. Part II: Non-local in time partial differential equations
   - Non-local heat type equation
   - Explicit solution operator
   - Evolutionary integro-differential equations
3. Part III: Another direction (operators which are non-densely defined)
   - Current problems

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Heat equation in $\mathbb{R}^n$

Let us study the following equation in $\mathbb{R}^n$:

$$\partial_t u(t,x) - \Delta u(t,x) = 0, \quad u(0,x) = u_0(x), \tag{1}$$

where $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ is the Laplacian operator in $\mathbb{R}^n$.

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Recall that

$$\mathcal{F}(u)(\xi) := \widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{+2\pi x i \cdot \xi} \mathrm{d}x, \quad \xi \in \mathbb{R}^n;$$

and

$$\mathcal{F}^{-1}(\widehat{u})(x) := u(x) = \int_{\mathbb{R}^n} \widehat{u}(\xi) e^{2\pi x i \cdot \xi} \mathrm{d}\xi, \quad x \in \mathbb{R}^n.$$

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Therefore, the solution of equation (1) is given by

$$u(t,x) = \mathcal{F}^{-1}\big(\mathcal{F}(u_0)(\xi) e^{-|\xi|^2 t}\big)(x).$$

since $\widehat{\Delta u}(\xi) = -|\xi|^2 \widehat{u}(\xi)$.

Remarks:

1. Questions about well-posedness (regularity of the solution), i.e.

$$\|u\|_X \leqslant C\|u_0\|_Y!,$$

2. Blow-up results, in which spaces !,
3. Other properties of the solutions.

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All these questions are closely related with the considered spaces and the nature of space-operator.

$\mathbb{R}^n$ is a very good space and hence we can sometimes work with the explicit solution to get optimal and sharp estimates.

Another form to express the solution

We have that

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In this case, $e^{t\Delta}$ generates a C_0-semigroup, i.e., it is a map $T(t) : [0,+\infty) \to \mathcal{B}(X)$ (X a Banach space) such that

- $T(0) = I$.
- For all $t, r \geqslant 0$, it follows that $T(t+r) = T(t)T(r)$.
- For all $x_0 \in X$, we have $\lim_{t\to 0^+} T(t)x_0 = x_0$.

Thus, for our example, $T(t) = e^{t\Delta}$.

Which operators generate a C_0-semigroup

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Which operators generate a C_0-semigroup

The Hille–Yosida theorem characterizes the generators of strongly continuous semigroups of linear operators on Banach spaces.

Requests operators to be closed, densely defined, and

$$\|(\lambda I - A)^n\| \leqslant \frac{C}{(\lambda - \omega)^n}, \quad \lambda > \omega, \quad n \in \mathbb{N}, \quad \lambda \in \rho(A).$$

for all positive integers n and any λ in the resolvent set $\rho(A)$.

$\mathcal{L}$-Heat equation (General scenario)

Let us consider the following $\mathcal{L}$-heat equation:

$$\partial_t u(t,x) + \mathcal{L}u(t,x) = 0, \quad u(0,x) = u_0(x),$$

where $\mathcal{L}$ is a "good enough" (maybe unbounded) operator.

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where $\mathcal{L}$ is a "good enough" (maybe unbounded) operator.

For $t > 0$, we can use the functional calculus [1] (when it will be possible) to obtain

$$u(t,x) = \underbrace{e^{-t\mathcal{L}}}_{S(t)-\text{Solution operator}} u_0.$$

So, u satisfies the considered equation along with its initial condition.

[1] W. Arveson. A Short Course on Spectral Theory, vol. 209 (2006).

Classical questions

So, at this stage, we can ask about the existence, uniqueness, asymptotic behavior, norm estimates, properties, etc; of a solution of a partial differential equation.

[2] R. Akylzhanov, M. Ruzhansky. $L^p - L^q$ multipliers on locally compact groups. J. Funct. Anal. (2020).

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In particular, if we want to study $L^p - L^q$-norm estimates, we mainly would like to have

$$\|u(t,\cdot)\|_{L^q} = \|e^{-t\mathcal{L}}u_0\|_{L^q} \lesssim \|e^{-t\mathcal{L}}\|_{\mathcal{G}}\|u_0\|_{L^p},$$

on a certain space $\mathcal{G}$, where somehow we can just handle the propagator.

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These questions are related with the study of Spectral and Fourier multipliers ! A lot of works can be found in the literature, specially on spectral ones.

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Here we mention the recent general work [2]. See the next slide!

[2] R. Akylzhanov, M. Ruzhansky. $L^p - L^q$ multipliers on locally compact groups. J. Funct. Anal. (2020).

Work on locally compact groups (2020)

It was shown on a locally compact separable unimodular group [3]:

1. $\mathcal{G}$ is the noncommutative Lorentz space [4], i.e.

$$\|\phi(|\mathcal{L}|^5)\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} \phi(s)\big(\tau(E_{(0,s)}(|\mathcal{L}|))\big)^{\frac{1}{p}-\frac{1}{q}},$$

   for $1 < p \leqslant 2 \leqslant q < +\infty$, where ϕ is a monotonically decreasing continuous function on $[0,+\infty)$ with $\lim\limits_{v\to+\infty} \phi(v) = 0$, and $\mathcal{L}$ is a left invariant operator[6] on G.

[3] R. Akylzhanov, M. Ruzhansky. $L^p - L^q$ multipliers on locally compact groups. J. Funct. Anal. (2020).

[4] H. Kosaki. Non-commutative Lorentz spaces associated with a semi-finite von Neumann algebra and applications. Proc. Japan Acad. Ser. A Math. Sci., (1981).

[5] $\mathcal{L} = U|\mathcal{L}|$ with U an isometry and $|\mathcal{L}|$ is a nonnegative self-adjoint operator.

[6] $\mathcal{L}\pi_L(g) = \pi_L(g)\mathcal{L}$ for all $g \in G$ where $\pi_L(g)f(x) = f(g^{-1}x)$.

Direct application

Note that $\phi(v) = e^{-tv}$ is decreasing on $v \in [0, +\infty)$ such that $\phi(0) = 1$ and $\lim\limits_{v \to +\infty} \phi(v) = 0$.

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$$\|e^{-t\mathcal{L}}\|_{L^p(G) \to L^q(G)} \lesssim \sup_{s>0} e^{-st} \left(\tau(E_{(0,s)}(\mathcal{L}))\right)^{\frac{1}{p} - \frac{1}{q}}.$$

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Hence

Theorem (The $\mathcal{L}$-heat equation)

Let G be a locally compact unimodular separable group and let $\mathcal{L}$ be a positive left invariant operator (maybe unbounded) such that for some α we have

$$\tau(E_{(0,v)}(\mathcal{L})) \lesssim v^{\alpha}, \quad v \to +\infty.$$

Then for any $1 < p \leqslant 2 \leqslant q < +\infty$ it follows

$$\|e^{-t\mathcal{L}}\|_{L^p(G) \to L^q(G)} \leqslant C_{\alpha,p,q} t^{-\alpha\left(\frac{1}{p} - \frac{1}{q}\right)}, \quad t > 0.$$

First questions

Can we refine the class of functions to prove $L^p - L^q$ boundedness for general propagators?

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From which equations are coming these new classes of propagators?

Other type of propagators

```
Plot[{MittagLefflerE[1.95, 2, -x], MittagLefflerE[1.95, 1, -x],
  270/(1+x)}, {x, 0, 1000}, PlotLegends → "Expressions"]
```

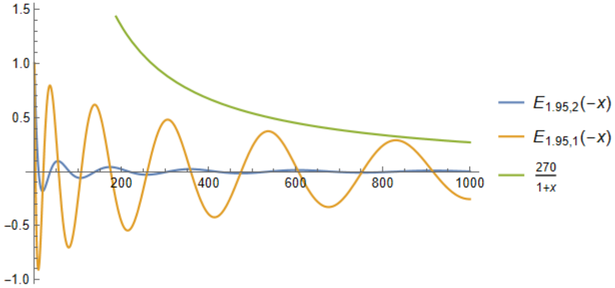

Figure: Fractional wave propagators functions for $\alpha = 1.95$ bounded uniformly.

Other type of propagators

```
Plot[{MittagLefflerE[1.9, 2, -x], MittagLefflerE[2, 2, -x],
  MittagLefflerE[2, 1, -x]}, {x, 0, 1000},
 PlotLegends → "Expressions", PlotRange → {-1.1, 1.1}]
```

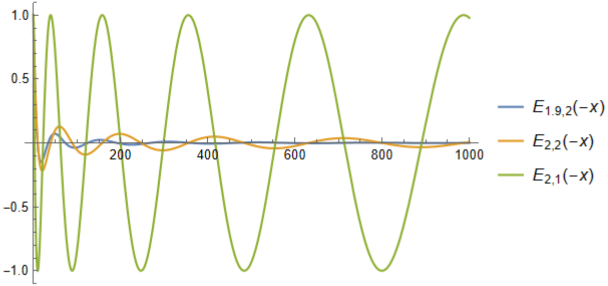

Figure: Classical wave propagator function (green) is not uniformily bounded by a decreasing vanishing at infinity function.

Other type of propagators

```
Plot[{MittagLefflerE[1.9, 0.1, -x], 1350/ (1 + x)},
 {x, 0, 1000}, PlotLegends → "Expressions"]
```

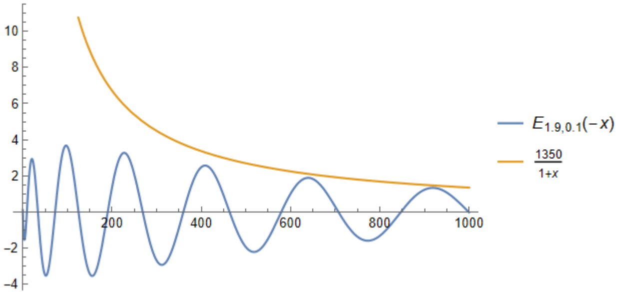

Figure: Mittag-Leffler functions with one small parameter.

Outline

Boundedness of some other type of propagators

Theorem

Let ϕ be a Borel measurable function on $Sp(|\mathcal{L}|)$. Suppose ψ is a monotonically decreasing continuous function on $[0, +\infty)$ such that $0 < \psi(0) < +\infty$, $\lim_{v \to +\infty} \psi(v) = 0$ and $\phi(v) \leqslant \psi(v)$ for all $v \in [0, +\infty)$. Then

$$\|\phi(|\mathcal{L}|)\|_{L^p(G) \to L^q(G)} \leqslant \sup_{v>0} \psi(v) \big(\tau(E_{(0,v)}(|\mathcal{L}|))\big)^{\frac{1}{p} - \frac{1}{q}},$$

for $1 < p \leqslant 2 \leqslant q < +\infty$. [a] [b]

[a] S. Gómez Cobos, J.E. Restrepo, M. Ruzhansky. $L^p - L^q$ estimates for non-local heat and wave type equations on locally compact groups. C. R. Acad. Sci. Paris, (2024).

[b] S. Gómez Cobos, J.E. Restrepo, M. Ruzhansky. Heat-wave-Schrödinger type equations on locally compact groups. arXiv:2302.00721, (2023).

Outline

Non-local heat type equation

Suppose that $t > 0$, $x \in \mathbb{R}^n$, $0 < \beta < 1$ and

$$\underbrace{\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \partial_s u(s,x) \mathrm{d}s}_{{}^C\partial_t^\beta u(t,x)} + \Delta u(t,x) = 0,$$

$$u(t,x)\big|_{t=0} = u_0(x).$$

[7] H. Pollard. The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. Bull. Amer. Math. Soc. 54, (1948), 1115–1116.

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$$u(t,x)\big|_{t=0} = u_0(x).$$

The solution is given by

$$u(t,x) = \mathcal{F}^{-1}\big(\mathcal{F}(u_0)(\xi) E_\beta(-|\xi|^2 t^\beta)\big)(x),$$

where the Mittag-Leffler function E_β is defined as

$$E_\beta(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\beta k + 1)}, \quad z \in \mathbb{C}.$$

For $0 < \beta < 1$, the function $E_\alpha(-x)$ is completely monotonic [7].

[7] H. Pollard. The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. Bull. Amer. Math. Soc. 54, (1948), 1115–1116.

Non-local heat type equation (General case)

We study the following equation:

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\begin{gathered}
{}^{C}\partial_t^{\beta} u(t) - \mathcal{L}u(t) = 0, \;\; t > 0, \;\; 0 < \beta < 1, \\
u(t)\big|_{t=0} = u_0,
\end{gathered}
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where $\mathcal{L} : \mathcal{D} \subset X \to X$ is a **closed linear operator densely defined** in a complex Banach space X.

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where $\mathcal{L} : \mathcal{D} \subset X \to X$ is a **closed linear operator densely defined** in a complex Banach space X. The solution can be given by

$$
u(t) = E_{\beta}(-t^{\beta}\mathcal{L})u_0, \quad t > 0.
$$

Properties of the solution operator $E_\beta(-t^\beta \mathcal{L})$

Definition

A family $\{E_\beta(-t^\beta \mathcal{L})\}_{t \geqslant 0} \subset \mathcal{B}(X)$ is called a solution operator of the abstract fractional equation if the following conditions are satisfied:

1. $E_\beta(-t^\beta \mathcal{L})$ is strongly continuous for $t \geqslant 0$ and $E_\beta(0) = I$;
2. $E_\beta(-t^\beta \mathcal{L})\mathcal{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{L})$ and $\mathcal{L}E_\beta(-t^\beta \mathcal{L})w = E_\beta(-t^\beta \mathcal{L})\mathcal{L}w$ for any $w \in \mathcal{D}(\mathcal{L})$, $t \geqslant 0$;
3. $E_\beta(-t^\beta \mathcal{L})w$ is a solution of the abstract fractional equation for any $w \in \mathcal{D}(\mathcal{L})$, $t \geqslant 0$.

Remark on densely defined operators

Suppose that $\mathcal{L} : \text{Dom}(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ is a positive linear operator densely defined in a separable Hilbert space $\mathcal{H}$. Notice that the assumption on densely defined operator allows us to think about operators of unbounded type. In fact, if we consider an operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ defined in the whole space (assume complex Hilbert space) and positive, then $\mathcal{L}$ is bounded (Hellinger-Toeplitz Theorem).

Outline

Explicit form of the solution operator

Let $\beta \in (0,2)$. The integral form of the solution operator can be written as (see Prüss or Bajlekova)

$$E_\beta(-t^\beta \mathcal{L}) = \frac{1}{2\pi i} \int_H e^{\gamma t} \gamma^{\beta-1} (\gamma^\beta + \mathcal{L})^{-1} d\gamma, \quad t \geqslant 0,$$

for $H \subset \rho(-\mathcal{L})$ (the resolvent set) and H is a suitable hankel's path.

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The above notation is consistent (convenient) since the integral representation of the Mittag-Leffler function $E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$ for $z \in \mathbb{C}$ and $\Re(\alpha) > 0$ is given by

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{\mathcal{H}} e^{\gamma}\gamma^{\alpha-1}(\gamma^\alpha - z)^{-1}\mathrm{d}\gamma,$$

where $\mathcal{H}$ is a suitable Hankel path.

Solution operator by means of the classical heat propagator

The fractional propagator can also be rewritten as follows

$$E_\alpha(-t^\alpha \mathscr{L}) = \int_0^{+\infty} M_\alpha(s) e^{-st^\alpha \mathscr{L}} \mathrm{d}s, \quad t \geqslant 0,$$

where $\{e^{-t\mathscr{L}}\}_{t\geqslant 0}$ is the C_0- semigroup generated by $-\mathscr{L}$, and

$$M_\alpha(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)}, \quad z \in \mathbb{C}, \quad 0 \leqslant \alpha < 1,$$

is the Wright-type function which is convergent in the whole z-complex plane.

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is the Wright-type function which is convergent in the whole z-complex plane. Some of the basic properties of this function are:

$$M_\alpha(x) \geqslant 0 \quad \text{for all} \quad x \in (0, +\infty), \quad \int_0^{+\infty} M_\alpha(s)\mathrm{d}s = 1,$$

and

$$\int_0^{+\infty} s^\gamma M_\alpha(s)\mathrm{d}s = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma\alpha+1)}, \quad \gamma > -1, \quad 0 \leqslant \alpha < 1.$$

Outline

$\mathcal{L}$-evolutionary differential equations

For $k \in \mathcal{PC}$, we now study the following equation:

$$\begin{aligned} \partial_t\big(k * [w(s) - w_0]\big)(t) + \mathcal{L}w(t) = 0, \quad t > 0, \\ w(t)\big|_{t=0} = w_0. \end{aligned} \tag{2}$$

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Now, if we if we think about strong (differentiable) solution, we can then rewrite equation (2) as

$$\int_0^t k(t-s)\partial_s w(s)\mathrm{d}s + \mathcal{L}w(t) = 0$$

or $(k * \partial_t w)(t) + \mathscr{L}w(t) = 0$.

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or $(k * \partial_t w)(t) + \mathscr{L}w(t) = 0$. Let us do the convolution with $\mathscr{K}$, and use the associativity of this operation along with $(\mathscr{K} * k)(t) = 1$, then

$$\int_0^t \partial_s w(s)\mathrm{d}s + \mathscr{K} * \mathscr{L}w(t) = 0$$

which implies

$$w(t) - w_0 + \int_0^t \mathscr{K}(t-s)\mathscr{L}w(s)\mathrm{d}s = 0.$$

Evolutionary integral equation of scalar type

$\mathfrak{L}$-evolutionary integral equation of scalar type ($\mathfrak{L} = k(t)\mathcal{L}$)

$$w(t) = h(t) + \int_0^t k(t-s)\mathcal{L}w(s)\mathrm{d}s, \quad t \geqslant 0,$$

where $\mathcal{L}$ is a closed linear unbounded operator in X and $k \in L^1_{loc}(\mathbb{R}_+)$.

Development of the theory in e.g. "J. Prüss. Evolutionary Integral Equations and Applications, Monogr. Math. 87, Birkhäuser, Basel, 1993."

Outline

Current problems with almost sectorial operators

Let $S_\mu := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leqslant \mu\} \cup \{0\}$. We consider the function arg with values in $(-\pi, \pi]$.

Definition (Almost sectorial operator)

Let $-1 < \gamma < 0$ and $0 \leqslant \omega < \pi$. By $\Theta_\omega^\gamma(X)$ we denote the set of all closed linear operators $A : D(A) \subset X \to X$ which satisfy

a. $\sigma(A) \subset S_\omega$.

b. For any $\omega < \mu < \pi$, there exists a positive constant C_μ such that

$$\|(z - A)^{-1}\| \leqslant C_\mu |z|^\gamma, \quad \text{for any} \quad z \notin S_\mu.$$

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It is important to recall that operators in the class Θ_ω^γ have the possibility of having non-dense domain and/or range.

Examples of these operators

The minus Laplacian in a bounded domain Ω is sectorial (this means $\gamma = 1$) under some suitable boundary conditions in $L^p(\Omega)$.

Moreover, it is also sectorial in the spaces of bounded or continuous functions.

[8] A. Lunardi. Analytic semigroups and optimal regularity in parabolic problems. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1995.

Examples of these operators

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Moreover, it is also sectorial in the spaces of bounded or continuous functions.

While, in the space of Hölder continuous functions, the minus Laplacian is almost sectorial, see e.g. Example 3.1.33 of [8]. **Therefore, it does not generate a C_0-semigroup.**

[8] A. Lunardi. Analytic semigroups and optimal regularity in parabolic problems. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1995.

Some of the references in this field

1. G. Da Prato. Semigruppi di crescenca *n*. Ann. Scuola Norm. Sup. Pisa, 20(3), (1966), 753–782.
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4. F. Periago, B. Straub. A functional calculus for almost sectorial operators and applications to abstract evolution equations. J. Evol. Equ., 2, (2002), 41–68.

First try: Non-local in time heat type equations

1. We want to study the existence and uniqueness of solutions of

$$^{C}\partial_t^{\beta} u(t) + Au(t) = f(t, u(t)), \quad t \in [0, T], \quad T > 0, \quad 0 < \beta < 1,$$

where $u(0) = u_0 \in X$, X is a Banach space, $A : \mathcal{D}(A) \subset X \to X$ is a closed, non-densely defined, almost sectorial operator and $f : (0, T] \times X \to X$ is continuous with respect to t and it satisfies certain conditions.

[9] R-N. Wang, D-H. Chen, T-J. Xiao. Abstract fractional Cauchy problems with almost sectorial operators. J. Differ. Equ. 252(1), (2012), 202–235.

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Non-local in time wave type equation

1. We study the existence and uniqueness of solutions (mild, classical, etc) of

$$^{C}\partial_t^{\beta} u(t) + Au(t) = f(t, u(t)), \quad t \in [0, T], \quad T > 0, \quad 1 < \beta < 2,$$

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Again, the most interesting part is how to construct the solution operator by using the functional calculus for a.s.o. of Periago and Straub (2002).

Direct abstract Cauchy problem with a time-dependent variable coefficient

J.E. Restrepo. Direct and inverse abstract Cauchy problems with fractional powers of almost sectorial operators. arXiv:2408.00240, (2024).

Direct abstract Cauchy problem with a time-dependent variable coefficient

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We consider the abstract Cauchy problem in a complex Banach space X with the coefficient $\phi(t)$ ($\phi(t) > 0$ for $t > 0$) and $A \in \Theta_\omega^\gamma$ $\left(0 < \omega < \pi/2\right)$ as follows:

$$
\begin{aligned}
&u_t(t) = \phi(t) A^\alpha u(t), \quad 0 < t < T, \quad 0 < \alpha < \frac{\pi}{2\omega}, \\
&u(0) = u_0 \in X.
\end{aligned}
\tag{3}
$$

The solution operator

Assume also that $A \in \Theta_\omega^\gamma$. So, we have that

$$\mathscr{T}_\alpha(t) = \frac{1}{2\pi} \int_{\Gamma_\theta} e^{-tz^\alpha} (z - A)^{-1} \mathrm{d}z, \quad \omega < \theta < \mu,$$

is a bounded linear operator in X. Note that the above expression for $\mathscr{T}_\alpha(t)$ could be also denoted by e^{-tA^α}.

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This operator is an analytic semigroup of growth order $\frac{\gamma+1}{\alpha}$, i.e.

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This operator is an analytic semigroup of growth order $\frac{\gamma+1}{\alpha}$, i.e.

1. $\mathscr{T}_\alpha(t+s) = \mathscr{T}_\alpha(t)\mathscr{T}_\alpha(s)$ for any $t, s \in S_{\frac{\pi}{2}-\alpha\omega}$.
2. There exists a positive constant $C(\gamma,\alpha)$ such that

$$\|\mathscr{T}_\alpha(t)\| \leqslant Ct^{-\frac{\gamma+1}{\alpha}}, \quad \text{for any} \quad t > 0.$$

3. The function $t \to \mathscr{T}_\alpha(t)$ is analytic in $S^0_{\frac{\pi}{2}-\alpha\omega}$ and

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathscr{T}_\alpha(t) = (-1)^k A^{k\alpha}\mathscr{T}_\alpha(t), \quad \text{for all} \quad t \in S^0_{\frac{\pi}{2}-\alpha\omega}.$$

Solution operator

Below we show that the solution operator $\mathscr{T}_{\alpha,\phi}(t)$ of problem (3) has an explicit representation in terms of the coefficient $\phi(t)$ and the analytic semigroup $\mathscr{T}_{\alpha}(t)$ of growth order $\frac{\gamma+1}{\alpha}$.

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Theorem

Let X be a complex Banach space. Then the solution operator of problem (3) is given by:

$$\mathscr{T}_{\alpha,\phi}(t) = \mathscr{T}_{\alpha}\left\{\left(\int_0^t \phi(s)\mathrm{d}s\right)\right\}, \quad 0 < t < T.$$

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Basically, we have that

$$\mathscr{T}_{\alpha,\phi}(t) = e^{-\left(\int_0^t \phi(s)\mathrm{d}s\right)A^\alpha}, \quad 0 < t < T.$$

Some relate works

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Thank you!