Fenton's sum of translates approach for classical minimax questions of approximation theory

Bálint Farkas, Béla Nagy and Szilárd Gy. Révész

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Papers about the subject

- B. Farkas, B. Nagy, and Sz. Gy. Révész, *A minimax problem for sums of translates on the torus*, Trans. London Math. Soc. **5** (2018), no. 1, 1–46.
- —— , *A homeomorphism theorem for sums of translates*, Rev. Mat. Complut. **37** (2024), no. 2, 341–389.
- —— , *On intertwining of maxima of sum of translates functions with nonsingular kernels*, Tr. Inst. Mat. Mekh. **28** (2022), no. 4, 262–272.
- —— , *On the weighted Bojanov-Chebyshev extremal problem*, Tr. Inst. Mat. Mekh. **29** (2023), no. 4, 193–216.
- —— , *On the weighted Bojanov-Chebyshev problem and the sum of translates method of Fenton*, Sbornik Math. **214** (2023), no. 8, 1163–1190.
- —— , *Fenton type minimax problems for sum of translates functions*, J. Math. Anal. Appl. **543** (2025), no. 2, Paper No. 128931, 25 pp.
- Tatiana Nikiforova, *On the weighted Bojanov-Chebyshev problem on an infinite interval*, Preprint, arXiv:2405.08561.

A motivation: Bojanov's variant of the Chebyshev problem

Write $\|.\|$ for the sup norm over a given interval $[a, b]$.

Theorem (Bojanov, 1979[1])

Let $\nu_1, \nu_2, \ldots, \nu_n$ be positive integers. Given $[a, b]$ there exists a unique set of points $a \le x_1^* \le x_2^* \le \ldots \le x_n^* \le b$ such that

$$\|(x - x_1^*)^{\nu_1}(x - x_2^*)^{\nu_2} \ldots (x - x_n^*)^{\nu_n}\| \\ = \inf_{a \le x_1 \le x_2 \le \ldots \le x_n \le b} \|(x - x_1)^{\nu_1}(x - x_2)^{\nu_2} \ldots (x - x_n)^{\nu_n}\| .$$

Moreover, $a < x_1^* < x_2^* < \ldots < x_n^* < b$. The extremal polynomial $T(x) := (x - x_1^*)^{\nu_1}(x - x_2^*)^{\nu_2} \ldots (x - x_n^*)^{\nu_n}$ is characterized by the following equioscillation property: there exists an array of points $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$ such that

$$T(t_k) = (-1)^{\nu_{k+1} + \cdots + \nu_n} \|T\| \quad (k = 0, 1 \ldots, n).$$

---

[1] *A generalization of Chebyshev polynomials*, J. Approx. Theory **26** (1979), no. 4, 293–300.

Connection to the classical Chebyshev problem

This contains the classical Chebyshev problem – where all $\nu_j = 1$ – but here the occurring polynomials do not form a vector space[2].

[2] The linear span is $\mathcal{P}_N$ with $N = \sum_i \nu_i$.

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$$\log |(x - x_1)^{\nu_1}(x - x_2)^{\nu_2} \dots (x - x_n)^{\nu_n}| = \sum_{j=1}^{n} \nu_j \log |x - x_j|.$$

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Logarithmic version of Bojanov's extremal problem:

$$\textbf{minimize} \quad \sup_{[0,1]} \sum_{j=1}^{n} \nu_j \log |\cdot - x_j|.$$

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Weights, variable majorants and snake polynomials

Weighted case: given a weight $w(x) \geq 0$,

$$\textbf{minimize} \quad \|w(x)(x-x_1)^{\nu_1}(x-x_2)^{\nu_2}\dots(x-x_n)^{\nu_n}\|.$$

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Note that (denoting $\|p\|_w := \|pw\|$, as usual)

$$\|p\|_w = C \Leftrightarrow \frac{-C}{w(x)} \leq p(x) \leq \frac{C}{w(x)} \ (\forall x \in [a,b]), \& C \text{ is best.}$$

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If $\nu_1 = \dots = \nu_n = 1$, very general results are known, even for non-symmetric norms (when lower and upper bounds differ):

$$CU(x) \leq p(x) \leq CV(x).$$

Snake polynomials with asymmetric bounds / norms

Extremal polynomials equioscillate between these bounds[3].
(Such extremal polynomials are called "snake polynomials".)

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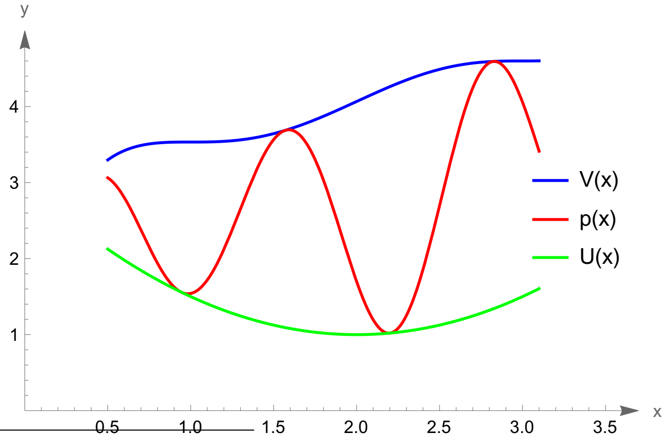
A snake polynomial with bounds $U \le p \le V$.



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The general weighted minimization problem

In this presentation we consider a vector of multiplicities $\nu := (\nu_1, \ldots, \nu_n)$ and look for optimal location of ordered nodes $0 =: x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} := 1$ with smallest possible $\| \cdot \|_w$. (We cannot deal with asymmetric weights.)

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We fix the order of the nodes with the given multiplicities. That is, $\mathbf{x} := (x_1, \ldots, x_n) \in \overline{S}$, with $S := \{\mathbf{x} \ : \ 0 < x_1 < \cdots < x_n < 1\}$ "the simplex".

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More generally, we consider the sum of translates function

$$F(\mathbf{x}, t) := J(t) + \sum_{i=1}^{n} K_i(t - x_i) \quad (\mathbf{x} := (x_1, \ldots x_n) \in \overline{S}).$$

where the K_i are kernel functions, while $J : [0,1] \to \underline{\mathbb{R}} := [-\infty, \infty)$ is an external field function. (In particular, $K_i = \nu_i K$ is possible.)

Generality of considerations

One goal is to keep $J(x) := \log w(x)$ as general as we can (within assuming $0 \leq w \leq C$, $\not\equiv 0$), but still allow $\log|\cdot|$ be replaced by more general K. A point, e.g., is to allow $w := \chi_E$ for $E \subset [0,1]$.

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In the first part we will only assume that J is bounded from above and $\neq -\infty$ at least on $n + 1$ points. (Here the endpoints $0, 1$ are counted with weight $1/2$ only!)
We will term this by saying that J is non-degenerate, "$J \not\equiv -\infty$".

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In the first part of the study we will keep also the kernels very general: we can allow totally different kernels K_i, satisfying some natural and minimal conditions (what $\nu_i \log|\cdot|$ does satisfy).

Our setup I - Properties of kernel functions

A function $K : (-1, 0) \cup (0, 1) \to \mathbb{R}$ is a kernel function if

(i) it is concave both on $(-1, 0)$ and on $(0, 1)$, and

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$$\lim_{t \downarrow 0} K(t) = \lim_{t \uparrow 0} K(t). \tag{1}$$

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A stronger condition is singularity, i.e. assuming about (1) that

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Our setup II - Further properties of kernel functions

A kernel function K is a strictly concave kernel function if it is strictly concave on both of the intervals $(-1, 0)$ and $(0, 1)$.

[4] *A min-max theorem for sums of translates of a function*, J. Math. Anal. Appl. **244** (2000), no. 1, 214–222.

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Another assumption – introduced by Fenton[4] – is the "cusp condition" that K (or, sometimes, J) satisfies

$$\lim_{t,x\uparrow 0} \frac{K(t)-K(x)}{t-x} = -\infty \text{ and } \lim_{t,x\downarrow 0} \frac{K(t)-K(x)}{t-x} = \infty. \qquad (\infty'_{\pm})$$

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Note that $r \log |\cdot|$ $(r > 0)$ satisfies all the above.

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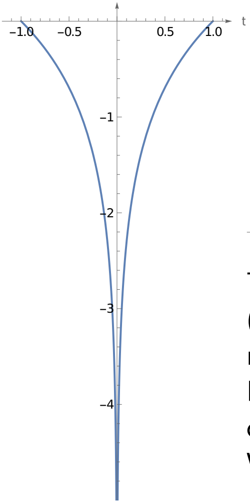
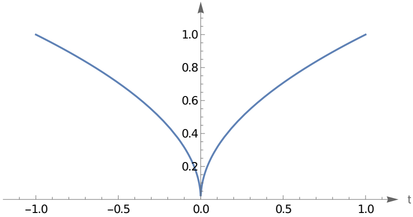
Examples for singular & nonsingular kernel functions





The singular logarithmic kernel (left) and the non-singular square root kernel (right).
Even the latter is subject to the cusp condition ($\infty'_{\pm}$).
We do not need evenness.

Our setup III - The sum of translates function F

Recall: the set $S := \{\mathbf{y} : \mathbf{y} \in [0,1]^n,\ 0 < y_1 < \cdots < y_n < 1\}$ is called the open simplex. The closed simplex is its closure:

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$$F(\mathbf{y}, t) := J(t) + \sum_{j=1}^{n} K_j(t - y_j) \quad (\mathbf{y} \in \overline{S},\ t \in [0,1]). \qquad (2)$$

(As said, J, K_j can attain only $-\infty$, but not $+\infty$, thus summing their translates leads to computable results.)

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$$F(\mathbf{y}, t) := J(t) + \sum_{j=1}^{n} K_j(t - y_j) \quad (\mathbf{y} \in \overline{S},\ t \in [0,1]). \tag{2}$$

(As said, J, K_j can attain only $-\infty$, but not $+\infty$, thus summing their translates leads to computable results.)

For K_j being concave, the non-degeneracy assumption "$J \not\equiv -\infty$" is in fact equivalent to that $F(\mathbf{y}, \cdot) \not\equiv -\infty$ ($\forall \mathbf{y} \in \overline{S}$), always.

[5] The Bojanov case is $K_j := \nu_j \log|\cdot|$. More generally, it is of special interest when the kernels are constant multiples of each other: $K_j = r_j K$.

Our setup IV - minimax and maximin goal functions

Set $y_0 := 0$ and $y_{n+1} := 1$. For $\mathbf{y} \in \overline{S}$ and $0 \leq j \leq n$ put

$$I_j(\mathbf{y}) := [y_j, y_{j+1}], \qquad m_j(\mathbf{y}) := \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t),$$

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We also aim at characterizing solution node systems (if any) and find, describe or approximate extremal values.

An illustration of our setup without any weight

An example for the graph of a sum of translates function $F(\mathbf{x}, \cdot)$.

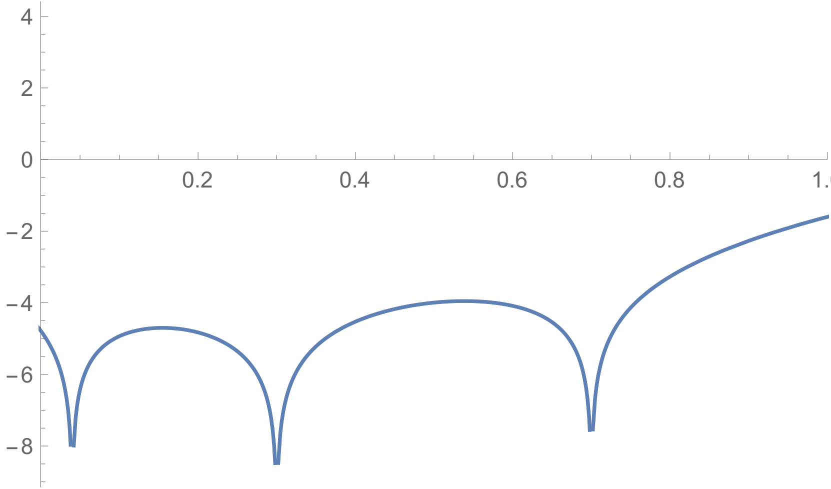
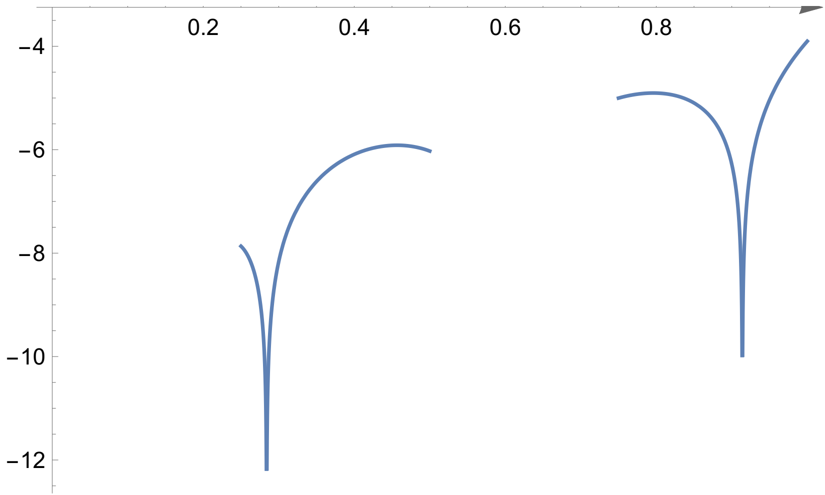


Illustration in case of a weighted situation

$F(\mathbf{x}, \cdot)$ in case of the weight w chosen as the indicator function χ_E of $E := [1/4, 1/2] \cup [3/4, 1]$, $J(t) := \log w(t)$.
We put $K(t) := \log |t|$, $x_1 = 1/3$, $x_2 = 2/3$.



Theorem of Fenton from 2000

Theorem (Fenton[6])

Let the field $J : (0,1) \to \mathbb{R}$ *be concave, and the kernel* K *be monotone* (M), *strictly concave and* C^2, *with* $K'' < 0$ *on* $(-1,0) \cup (0,1)$ *and satisfying the cusp condition* $(\infty'_{\pm})$, *too.*
Then for $F(\mathbf{y};t) := J(t) + \sum_{j=1}^{n} K(t - y_j)$ *there exists an extremal (minimax) node system* $\mathbf{w} := (w_1, \ldots, w_n)$ *in the open simplex* S*:*

$$M(\overline{S}) := \inf_{\mathbf{y}\in\overline{S}} \overline{m}(\mathbf{y}) := \inf_{\mathbf{y}\in\overline{S}} \sup_{[0,1]} F(\mathbf{y};\cdot) = \overline{m}(\mathbf{w}) := \sup_{[0,1]} F(\mathbf{w};\cdot). \quad (3)$$

Moreover, $F(\mathbf{w},\cdot)$ *equioscillates on the intervals* $I_j(\mathbf{w}) = [w_j, w_{j+1}]$*:*

$$m_0(\mathbf{w}) = m_1(\mathbf{w}) = \cdots = m_n(\mathbf{w}).$$

Furthermore, $\mathbf{w}$ *is the unique equioscillation node system, and it is the only maximin point:* $m(S) := \sup_{\mathbf{y}\in\overline{S}} \underline{m}(\mathbf{y}) = \underline{m}(\mathbf{w})$, *too.*

[6] *A min-max theorem for sums of translates of a function*, J. Math. Anal. Appl. **244** (2000), no. 1, 214–222.

Theorem of Fenton from 2000 - continued

An immediate consequence of the uniqueness and coincidence of the minimax=maximin=equioscillation point is that the so-called "Sandwich Property" $M(\overline{S}) = m(\overline{S})$ holds. Equivalently,

$$\underline{m}(\mathbf{x}) := \min_{j=0,\ldots,n} m_j(\mathbf{x}) \leq M(\overline{S}) \leq \overline{m}(\mathbf{x}) := \max_{j=0,\ldots,n} m_j(\mathbf{x}) \qquad (\forall \mathbf{x} \in \overline{S}).$$

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Moreover, it is seen that the local/interval maxima $m_j(\mathbf{x})$ of any node system approximate the extremum from both sides.

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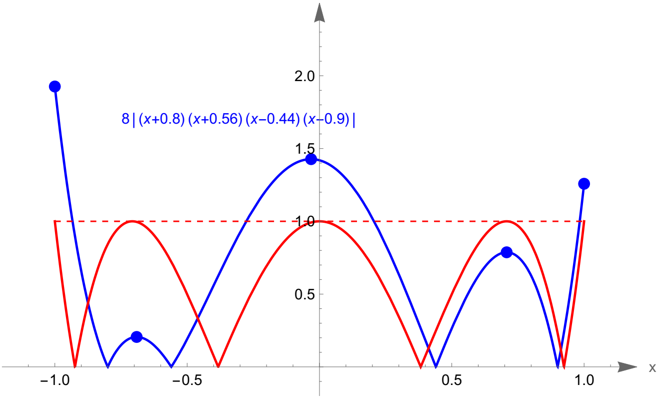
Two-sided approximation in the Chebyshev Problem



The torus setup – Phd thesis of Ambrus from 2009

Theorem (Ambrus, Ball, Erdélyi, 2013[9])

For any array $z_1, \dots, z_n$ of complex numbers of modulus 1, there exists a complex number z also of absolute value 1, such that

$$\sum_{j=1}^{n} \frac{1}{|z - z_j|^2} \le \frac{n^2}{4}.$$

The inequality is sharp, with equality if and only if the z_j are equidistant (nth roots) $z_j^n = c$ ($|c| = 1$) for all $j = 1, 2, \dots, n$.

[9] *Chebyshev constants for the unit circle*, Bull. Lond. Math. Soc. **45** (2013), no. 2, 236–248.

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If $z = e^{it}$ and $w = e^{is}$ $(t, s \in \mathbb{R})$, then $|z - w| = 2|\sin \frac{t-s}{2}|$, hence introducing $K(t) = -\frac{1}{4\sin^2(t/2)}$, the problem can be rewritten as

$$\min_{t_1,\ldots,t_n \in [-\pi,\pi]} \max_{t \in [-\pi,\pi]} \sum_{j=1}^{n} K(t - t_j) = ?$$

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The conjecture of Ambrus and Ball

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To cure this, let us fix one node at $y_0 := 0$; change $n \longrightarrow n+1$.

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Two-sided approximation of an equioscillating $\mathbf{w}$ means that no $\mathbf{x}$ would majorize it ($\mathbf{w}$ cannot majorize $\mathbf{x}$, for $\overline{m}(\mathbf{w})$ is minimal.)

Questions of existence and unicity on the unit circle

A natural question: Is $\Phi^{-1}(\mathbf{0})$ empty, one point, several points?

[11] Kilgore, *A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm*, J. Approx. Theory **24** (1978), no. 4, 273–288.

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Let the kernels K_j belong to $C^2(0, 2\pi)$, be strictly concave even with $K_j'' < 0$, and singular (∞) $(j = 0, \ldots, n)$.
Consider $F(\mathbf{y}, t) = K_0(t) + \sum_{j=1}^{n} K_j(t - y_j)$ for $\mathbf{y} \in S$.
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Inspiration was gained from the famous solution[11] of the Bernstein Conjecture on the optimal Lagrange interpolation.
We proved local homeomorphism via nonvanishing of the Jacobian, proved properness of the mapping, and finally applied general topology[12] to derive the global homeomorphism.

[11] Kilgore, *A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm*, J. Approx. Theory **24** (1978), no. 4, 273–288.

[12] Eilenberg, *Sur quelques propriétés des transformations localement homeomorphes*, Fundam. Math. **24** (1935), 35–42

Minimax results on the torus

Theorem (A global minimax theorem on the torus)

Suppose the kernel functions $K_0, K_1, \ldots, K_n$ *are strictly concave and either all satisfy* $(\infty'_{\pm})$, *or all belong to* $C^1(0, 2\pi)$.
Then there is a global minimax node system $\mathbf{w} \in \mathbb{T}^n$ *with*

$$M := \inf_{\mathbf{y} \in \mathbb{T}^n} \sup_{t \in \mathbb{T}} F(\mathbf{y}, t) = \sup_{t \in \mathbb{T}} F(\mathbf{w}, t).$$

Moreover, we have the following:

(a) $\mathbf{w}$ *is an equioscillation point:* $m_0(\mathbf{w}) = \cdots = m_n(\mathbf{w})$.

(b) $\mathbf{w} \in S := S_\sigma$ *for some open simplex (the nodes are different), and*

(c) *The Sandwich Property* $M(\overline{S}) = M = m(\overline{S})$ *holds on* S, *i.e.*

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The result holds only in this global version – there are counterexamples for specific individual simplexes.

Key ingredients

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Due to limit, unicity is lost. We have unicity only in 3.).

Remark

For approximation theory (i.e. logarithmic potential theory) and the strong polarization problem singularity is a natural condition. However, Fenton's Theorem extends to nonsingular kernels, where the Homeomorphism Theorem (surjectivity) necessarily fails. In the nonsingular case, continuity, existence of some equioscillating node system, unicity, intertwining all need to be proved independently of that result.

Goals in extending the results from $\mathbb{T}$ to the case of $[0, 1]$

Give a description of the interval case which allows for weights.

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Discuss non-majorization (intertwining) issues in general (not just for equiosillating systems).

Continuity of the interval maxima functions m_j

Proposition

Let $K_1, \ldots, K_n$ be arbitrary kernel functions and $J : [0,1] \to \underline{\mathbb{R}}$ be an extended continuous field function on $[0,1]$.
Then for each $j \in \{0, 1, \ldots, n\}$ the function $m_j : \overline{S} \to \underline{\mathbb{R}}$ is extended continuous.

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$$G(x) := \max_{y \in B} g(x, y) \text{ is continuous on } A.$$

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But what if J is not even u.s.c.?

The first miracles happen...

Proposition (Regularity of m_j for singular kernels)

Let $K_1, \ldots, K_n$ *be arbitrary singular kernel functions* (∞)*, and* $J : [0,1] \to \underline{\mathbb{R}}$ *be an arbitrary field function.*

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So, we have Lipschitz continuity at regular $\mathbf{y}$, i.e. if $\mathbf{y}$ belongs to

$$Y := \{\mathbf{y} \in \overline{S} \ : \ m_j(\mathbf{y}) > -\infty \ (j = 0, \ldots, n)\} = \bigcap_{j=0}^{n} \{m_j > -\infty\}.$$

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The next example illustrates that singularity of kernels is needed!

Example: K_j's are not singular, Φ is neither continuous nor surjective

Let $n = 1$ and set $\mathbf{y} = (y)$,

$$J(t) := \begin{cases} 0, & \text{if } 0 \leq t < 1/2, \\ 1, & \text{if } 1/2 \leq t \leq 1, \end{cases} \quad \text{and } K(t) := \sqrt{|t|}.$$

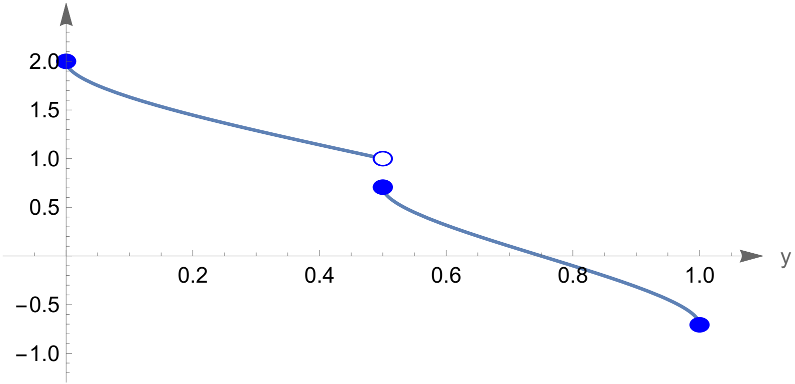
One can check

$$z_0(y) = \begin{cases} 0, & \text{if } 0 \leq y < 1/2, \\ 1/2, & \text{if } 1/2 \leq y \leq 1, \end{cases} \quad \text{and } z_1(y) = 1.$$

Therefore we can write

$$m_0(y) = \begin{cases} \sqrt{y}, & \text{if } 0 \leq y < 1/2, \\ \sqrt{y - 1/2} + 1, & \text{if } 1/2 \leq y \leq 1, \end{cases}$$
$$m_1(y) = 1 + \sqrt{1 - y}.$$

Example: Φ is neither continuous nor surjective – graph



The graph of the difference function $m_1(y) - m_0(y)$

Weakening the condition of monotonicity

Definition

K satisfies a periodized monotonicity condition with $c(\geq 0)$, if

$$K'(t) - K'(t-1) \geq c \quad \text{for a.a. } t \in (0,1). \qquad (PM_c)$$

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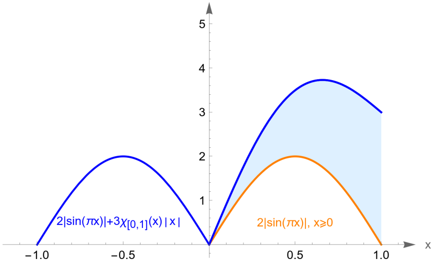
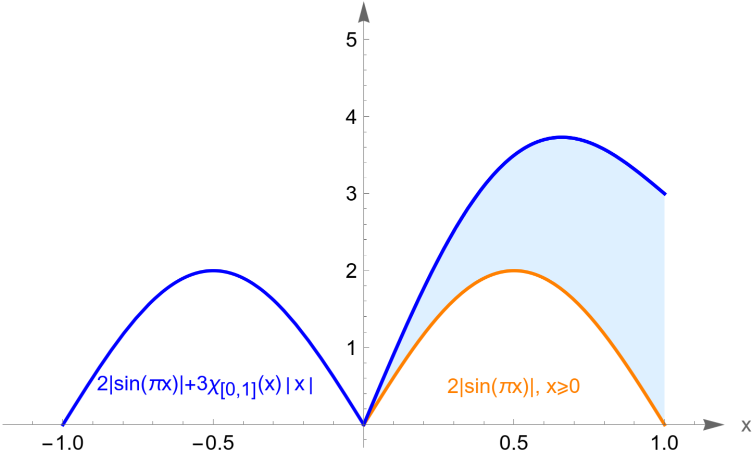


A typical kernel satisfying (PM_c)



Towards a Homeomorphism Theorem

How far the Homeomorphism Theorem may be extended?

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Theorem (Clarke[13])

Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be a locally Lipschitz function.
Further, let $\mathbf{x}_0 \in U$ be a point such that $D_{\mathrm{Clarke}} f(\mathbf{x}_0)$ has full rank.
Then f is a bi-Lipschitz homeomorphism in a neighbourhood of $\mathbf{x}_0$.

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With this we proved that Φ is a local homeomorphism.

To extend local to global, we needed two more facts:

(i) that Φ is proper (standard) and that

(ii) Y is connected (miraculously...).

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$$\Phi|_Y : Y \to \mathbb{R}^n, \quad \mathbf{x} \mapsto (m_1(\mathbf{x}) - m_0(\mathbf{x}), \ldots, m_n(\mathbf{x}) - m_{n-1}(\mathbf{x}))$$

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Moreover, the same remains in effect under only assuming (PM_0) on the kernels, provided J satisfies either a singularity condition or a cusp condition.

The last part is important as it allows to extend even to the periodic case! With almost arbitrary field functions!!

Towards minimax results

In the rest of the talk we will confine ourselves to "Bojanov-type" kernel systems, where $K_j = r_j K$ with one base kernel K. Also, we will restrict to singular kernels.

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Surprisingly, after getting through proofs which badly fail for not u.s.c. fields, at the end we were still able to remove this condition from several of our results.

A condition we badly need is monotonicity (M)–here we cannot get around monotonicity by some relaxed, periodized conditions. The main reason is the use of perturbation lemmas, which work for monotone kernels only.

A very precise perturbation lemma

Lemma (Strong Maximum Perturbation Lemma)

Let K be a kernel function subject to the monotonicity condition (M) *and $J : [0,1] \to \underline{\mathbb{R}}$ be an u.s.c. field.*
Further, let $\mathbf{w} \in S$ be a non-degenerate node system, and let $\mathcal{I} \cup \mathcal{J} = \{0, 1, \ldots, n\}$ be a non-trivial partition of $\{0, \ldots, n\}$.
Then, arbitrarily close to $\mathbf{w}$, there exists $\mathbf{w}' \in S \setminus \{\mathbf{w}\}$ with

$$F(\mathbf{w}', t) \leq F(\mathbf{w}, t) \ (\forall t \in I_i(\mathbf{w}')) \text{ and } I_i(\mathbf{w}') \subseteq I_i(\mathbf{w}) \ (i \in \mathcal{I}); \quad (5)$$

$$F(\mathbf{w}', t) \geq F(\mathbf{w}, t) \ (\forall t \in I_j(\mathbf{w})) \text{ and } I_j(\mathbf{w}') \supseteq I_j(\mathbf{w}) \ (j \in \mathcal{J}). \quad (6)$$

As a result, we also have

$$m_i(\mathbf{w}') \leq m_i(\mathbf{w}) \ (i \in \mathcal{I}) \quad \text{and} \quad m_j(\mathbf{w}') \geq m_j(\mathbf{w}) \ (j \in \mathcal{J}). \quad (7)$$

Moreover, if K is strictly concave and hence strictly monotone, then (5) and (6) are strict for all points where $J(t) \neq -\infty$. Furthermore, (7) is also strict for all indices k with non-singular $I_k(\mathbf{w})$; in particular, for all indices k in case $\mathbf{w} \in Y$.

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How do we obtain minimax and equioscillation theorems?

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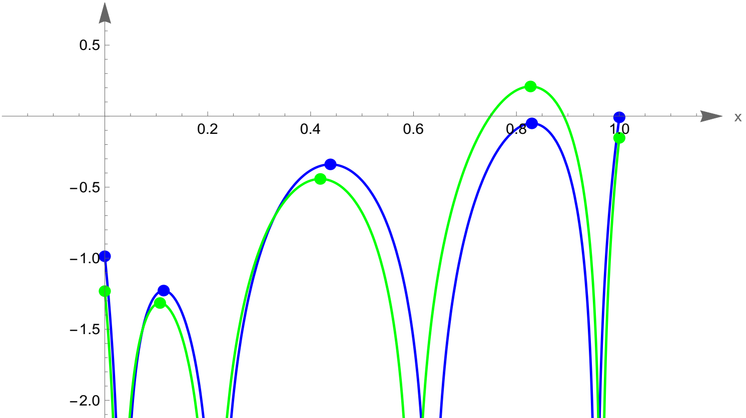

Use of the Perturbation Lemma

Even if there are n equal attainment of the maximum $\overline{m}(\mathbf{x})$, we can (according to the Perturbation Lemma) lower each of them simultaneously–at the expense of somewhat (continuity !) raising the (otherwise smaller) $n+1$st maximum.

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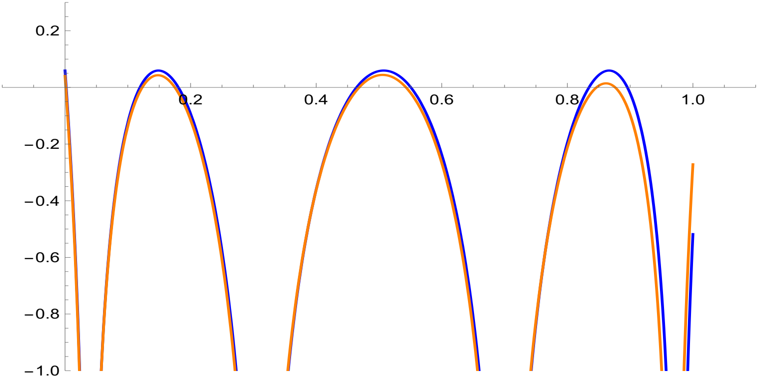

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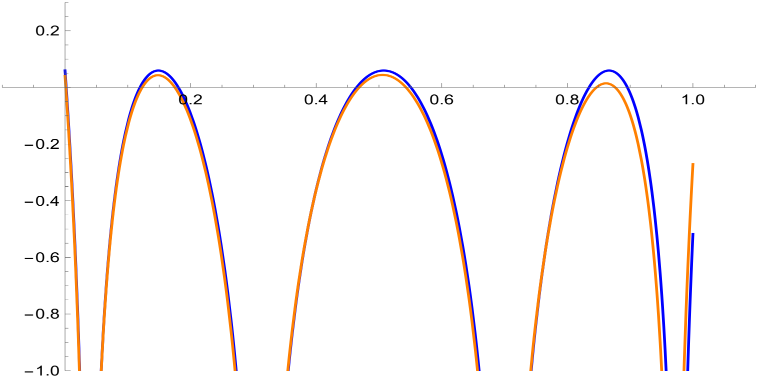

Therefore, an extremal node system must equioscillate.

The Minimax Theorem

Theorem (Minimax Equioscillation Theorem - Singular Case)

Let K be singular (∞) *and monotone* (M)*, and let J be u.s.c. arbitrary.*

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Let K be singular (∞) and monotone (M), and let J be u.s.c. arbitrary.
Then $M(\overline{S}) = m(\overline{S})$ and there exists some node system $\mathbf{w} \in \overline{S}$, also belonging to $Y (\subset S)$, with the three properties that it is an equioscillating point, it attains the simplex maximin and also it attains the simplex minimax: $\underline{m}(\mathbf{w}) = m(\overline{S}) = M(\overline{S}) = \overline{m}(\mathbf{w})$.

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We obtain both with weights, too.

Sandwich Property, Majorization and a Suprise

Theorem (Intertwining theorem)

Let K be a singular (∞), strictly concave and (strictly) monotone kernel and let $J : [0,1] \to \underline{\mathbb{R}}$ be an u.s.c. / arbitrary field function. Then for nodes $\mathbf{x}, \mathbf{y} \in Y$ majorization cannot hold, i.e., the coordinatewise inequality $\mathbf{m}(\mathbf{x}) \leq \mathbf{m}(\mathbf{y})$ can only hold if $\mathbf{x} = \mathbf{y}$.

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Consider the Chebyshev problem, where $K(t) := \log |t|$, $J(t) \equiv 0$, and so we have strict concavity and monotonicity. Then for any two node systems $\mathbf{x}, \mathbf{y} \in S$ $\exists 0 \le i \ne j \le n$ such that

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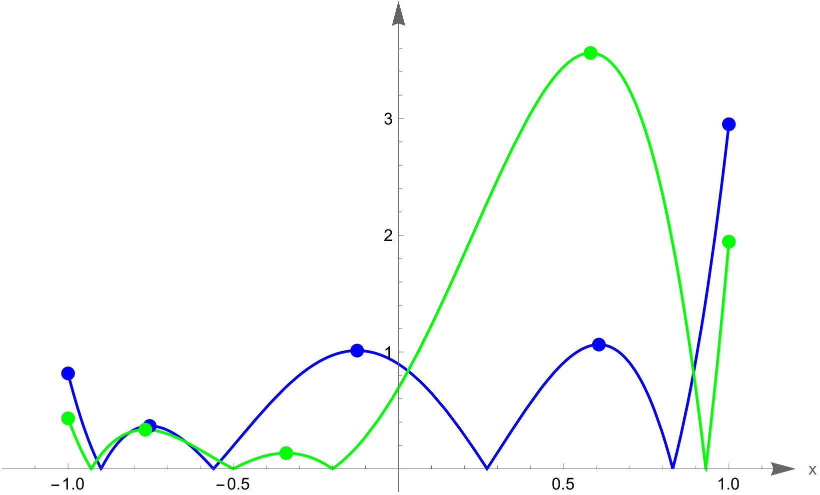
This seems folklore if one of the node systems is the extremal one.

Chebyshev's problem and intertwining of local maxima



Two arbitrary node systems generate intertwining local maxima vectors.

Intertwining of local maxima for non-singular kernels

Conjecture

Intertwining still holds for arbitrary (non-singular) strictly concave and (strictly) monotone kernel K and u.s.c. field function J.

[14] *On intertwining of maxima of sum of translates functions with nonsingular kernels*, Tr. Inst. Mat. Mekh. **28** (2022), no. 4, 262–272.

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However, it holds probably only under some monotonicity assumptions on K.

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Furthermore, strict majorization $m_i(\mathbf{x}) > m_i(\mathbf{y})\,(i = 0, \dots, n)$ cannot hold for any $\mathbf{x}, \mathbf{y} \in Y$.

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Apart from providing solutions or new proofs for classical problems, strong general principles, such as unicity (bi-Lipschitz homeomorphism) and intertwining (non-majorization) reveal themselves.

Applications I

- Fenton himself applied his method to the theory of entire functions. It is still to be studied if this direction of applications can furnish new proofs or even new results in that field.
- Ambrus, Ball and Erdélyi have dealt with the so-called "strong polarization problem", akin the "linear polarization constant problem" of functional analysis.
- Logarithmic potential theory is very closely connected to our study. Chebyshev constants and their estimates for various sets like union of k intervals can be studied by the method.
- Several problems of interpolation theory, most notably Hermite-Fejér interpolation on free nodes, can be studied by the method (Mycielski-Paszkowski result).

Applications II

- As explained, Chebyshev polynomials and Bojanov-Chebyshev polynomials can be studied by the method. Even for the classic Chebyshev polynomials, intertwining results seem to have remained unnoticed until this study.
- Currently, we with Béla Nagy are working towards a version where instead of univariate kernels K, which are translated as $K(\cdot - x)$, we allow bivariate kernels. In integral operators and general abstract potential theory such bivariate kernels occur, but the univariate specifications $K(\cdot, x)$ at different fixed nodes x can behave very differently for different x (as they are not just translates of each other).
  These can be further applied to study Blaschke products $\prod_{j=1}^{n} \left(\frac{x_j - t}{1 - x_j t} \right)$ whose factors usually depend on two variables, with one of the parameters determining the root or pole, and the other remaining a variable.

Applications III

If succeeding, the respective theory may be suitable to obtain further results for meromorphic functions or approximation with rational functions.

Open Questions I

- If K is nonsingular, the Homeomorphism Theorem fails badly. We saw that Φ is neither surjective, nor continuous in general. However, locally it is still one-to-one (as this is based on calculus with a Jacobian).
  Is "global injectivity" saved? Important for the unicity of equioscillating node systems e.g..
- Intertwining Conjecture. It relied on the Homeomorphism Theorem, but we have given a different proof even for nonsingular kernels when $n = 1, 2, 3$. Well, not the full power of the Homeomorphism Theorem is necessary for the general proof to go through. It seems that injectivity would as well suffice.

Open Questions II

- We obtained the unweighted trigonometric Bojanov Theorem in case of the classical log-trigonometric kernel $\log\sin|\pi t|$.
  In fact, for any field J, any $r_j > 0$, and any singular and monotone kernel K which is not necessarily periodic, but is subject to (PM_c).
  Furthermore, also for nonsingular kernels if J is either nil, or concave itself ("Fenton case") or satisfies some mild singularity or cusp condition.
  So, the case of fully general field J is still missing. One may try a limiting argument first: take $J_k(t) := J(t) + \log_{-}(kt)$.
  But the main problem is that a monotone and periodic function is necessarily a constant.
  So we failed to obtain a satisfactory common generalization of the periodic and interval case.
  Most probably the torus case needs to be reworked "directly" with the more general conditions and methods here.

Open Questions III

- Infinite intervals?
  As far as we know, till recently not even the 1978 Bojanov Theorem was carried through e.g. to Hermite polynomials. The above results were carried over by Tataiana Nikiforova. Here to not to loose sense some admissibility condition of $J(t) + rK(t) \to -\infty (|t| \to \infty)$ is necessary. Assuming that, the existence of a corresponding "Mhaskar-Rahmanov-Saff Number" can be proved. Nikiforova's method is to "transfer results" from the finite interval case to the real line by a linear mapping to the "MRS Interval".
- To me it seemed that this way the Homeomorphism Theorem cannot be transferred. However, very recently (in a paper not yet published) Tatiana Nikiforova succeeded in proving the Homeomorphism Theorem on infinite intervals by her method.

Open Questions IV

- Of course, we need not stop at assuming that the kernels are similar (proportional): we can try to deal with totally different kernels K_j. In fact, we have already done so regarding the Homeomorphism Theorem: there the kernels were arbitrary, (but all singular). The other main ingredient, the above Strong Perturbation Lemma does not seem to generalize. Yet, some less strong Perturbation Lemma was proved for $\mathbb{T}$. Does it suffice – for at least some results – to carry through? Maybe under some reasonable conditions connecting the K_j and J?

- We do have some initial / partial results for general and nonsingular $K_1, \ldots, K_n$, but technicalities are more involved. In particular, we need to harmonize conditions on the K_j and J, and use limiting arguments with perturbed systems (e.g. approximating with systems satisfying the singularity condition).

Open Questions V

- Our treatment of the weight / field is rather general. Still, it does not allow w to be unbounded, like e.g. Jacobi weights. In such a case of course the kernels K should have enough strong singularity properties to counterweight the growth of J. That version of Fenton's Theory still needs to be worked out. It is not clear what new phenomena may occur here.

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