On Hitchin-Thorpe inequality for 4-dimensional Ricci solitons[1]

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Seminar on Analysis, Differential Equations and Mathematical Physics - Russia

[1] Proc. Amer. Math. Soc. 2023

Outline:

- Gradient Ricci solitons
- Motivation & Background
- 4D gradient shrinking Ricci solitons
- The Hitchin-Thorpe inequality

Gradient Ricci solitons:

A gradient Ricci soliton is a Riemannian manifold (M^n, g) together with some (potential) function $f : M \to \mathbb{R}$ such that

$$Ric + Hess\, f = \lambda g, \tag{1}$$

for some $\lambda \in \mathbb{R}$.

- *shrinking:* $\lambda > 0$, (up to scaling $\lambda = \frac{1}{2}$),
- *steady:* $\lambda = 0$,
- *expanding:* $\lambda < 0$, (up to scaling $\lambda = -\frac{1}{2}$).

• In general,

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g.$$

Motivation: Gradient Ricci solitons

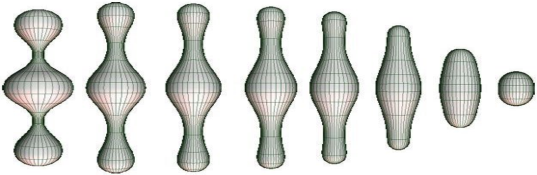
(M^n, g, f) satisfying

$$Ric + Hess\, f = \lambda g$$

- Natural extension of Einstein manifolds.
- Special solutions to the Ricci Flow.
- Model of singularities of the Ricci Flow.
- Critical points of a certain geometric functional (Perelman's ν-entropy functional).

The Ricci flow introduced by Hamilton (1982):

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -2Ric_{g(t)} \\ \\ g(0) = g_0 \end{cases} \tag{2}$$



Ricci solitons are special solutions to the RF

If (M^n, g, f) is a gradient Ricci soliton, i.e.,

$$Ric + Hess\, f = \lambda g,$$

then

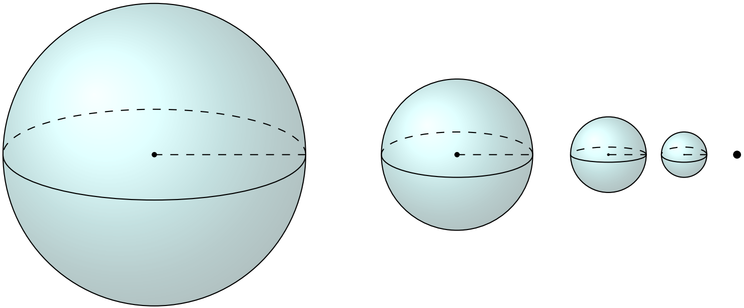
$$g(t) = (1 - 2\lambda t)\Phi_t^* g$$

is a self-similar solution to Hamilton's Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}.$$

Here, Φ_t is the 1-parameter family of diffeomorphisms generated by $\nabla f/(1 - 2\lambda t)$.

Sphere through the Ricci flow



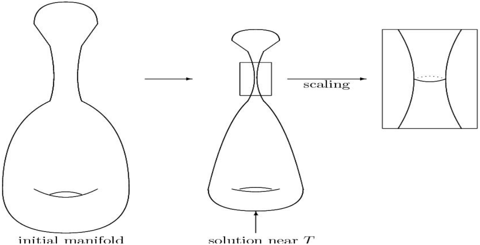
$$R(t) = \frac{1}{T - t}$$

... when $t \to T$, one sees that $R \nearrow \infty$

Model singularities of the Ricci flow

- Consider a solution $g(t)$ to RF on the maximal time interval $[0, T)$, where

$$0 < T < \infty \quad \text{and} \quad |Rm|_{max}(t) \to \infty \quad \text{as} \quad t \to T.$$



In this case, we say that $g(t)$ develops finite time singularities (as $t \to T$).

Model singularities of the Ricci flow

- Type I singularities:

$$\limsup_{t \to T} (T - t)|Rm|_{max}(t) < \infty.$$

By **Sesum, Naber, Enders, Buzano, Topping, Chen, Wang, Zhang, Hallgren and Bamler**.

Theorem

The blow-ups around a Type I singularity point of a Ricci flow converge to (nontrivial) gradient shrinking Ricci solitons (GSRS).

Some basic examples of **GSRS**

- $\mathbb{S}^n/\Gamma$, or more generally, any positive Einstein manifold M^n.
- The Gaussian solitons: $\mathbb{R}^n$ with the flat metric δ_{ij} and potential function $f(x) = \frac{1}{4}|x|^2$.
- Finite quotient of cylinders $N^k \times \mathbb{R}^{n-k}/\Gamma$ $(k \geq 2)$, where N^k is positive Einstein.

Gradient shrinking Ricci solitons

- **B.-L. Chen** (2009) showed that it has nonnegative scalar curvature ($R \geq 0$).

- **H.-D. Cao** and **D. Zhou** (2010) proved that

$$\frac{1}{4}\Big(r(x) - c\Big)^2 \leq f(x) \leq \frac{1}{4}\Big(r(x) + c\Big)^2, \qquad (3)$$

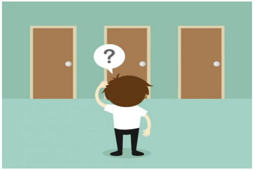
for all $r(x) \geq r_0$.

  - **Perelman** proved (3) by assuming that $|Ric| \leq C$.
  - (3) implies that

$$\int_M e^{-f} dV_g < \infty.$$

What to do? ...

- To classify, or to understand the geometry of Ricci solitons;
- To construct new examples of Ricci solitons which may give us new intuition and guidance.



Compact Ricci solitons

- **Perelman** (2002) proved that every compact Ricci soliton is a gradient Ricci soliton.
- **Hamilton** and **Ivey** (1993) showed that a compact gradient steady or expanding Ricci soliton is necessarily an Einstein metric.
  - Consequently, compact (non-Einstein) Ricci solitons must be **shrinking**.
- **B.-L. Chen** (2009) proved that a gradient shrinking Ricci soliton has positive scalar curvature (unless it is Ricci flat).

Classification 2D & 3D Ricci solitons

Theorem (Hamilton, 1988)

Any 2D compact gradient shrinking Ricci soliton is isometric to a quotient of the sphere $\mathbb{S}^2$.

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Any $2D$ compact gradient shrinking Ricci soliton is isometric to a quotient of the sphere $\mathbb{S}^2$.

Theorem (Ivey, Perelman)

Any $3D$ compact gradient shrinking Ricci soliton is isometric to a quotient of the sphere $\mathbb{S}^3$.

- Even the non-compact gradient shrinking Ricci soliton have been classified in 2 and 3 dimensions.


4D

Four-dimensional manifolds

- The bundle of 2-forms can be invariantly decomposed as a direct sum
$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \tag{4}$$
where $\Lambda^\pm$ is the (± 1)-eigenspace of Hodge star operator. This decomposition is conformally invariant.
- On 4-manifolds
$$\begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{2}\left(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\right) \\ &\quad - \frac{R}{6}\left(g_{jl}g_{ik} - g_{il}g_{jk}\right). \end{aligned} \tag{5}$$
- The Weyl tensor W is an endomorphism of Λ^2 such that
$$W = W^+ \oplus W^-.$$

Four-dimensional manifolds

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{R}{12} I & \mathring{Ric} \\ \hline \mathring{Ric}^* & W^- + \frac{R}{12} I \end{array} \right),$$

where $\mathring{Ric} = Ric - \frac{R}{4} g$.

"*dimension four seems to represent a sort of Goldilocks zone for the Einstein equation.*" (**C. LeBrun**)

- If M^4 is Kähler, then $|W^+|^2 = \frac{R^2}{24}$.

The Weyl tensor W is on target

Classifications involving the Weyl tensor W

By the works of **Eminenti-La Nave-Mantegazza**, **Ni-Wallach**, **Cao-Wang-Zhang**, **Zhang**, **Petersen-Wylie** & **Munteanu-Sesum**:

- Locally conformally flat (i.e. $W = 0$) 4D GSRS $\Longrightarrow$ a quotient of the sphere $\mathbb{S}^4$.

By **Chen-Wang**:

- Half-conformally flat (i.e. $W^+ = 0$ or $W^- = 0$) 4D GSRS $\Longrightarrow$ either $\mathbb{S}^4$ or $\mathbb{CP}^2$.

By **H.-D. Cao** & **Chen**, 2013:

- Bach-flat 4D GSRS $\Longrightarrow$ Einstein.

$$B_{ij} = \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R_{kl} W_i{}^k{}_j{}^l.$$

By **Munteanu-Sesum** and **Fernández-López** & **García-Río**:

- 4D GSRS with harmonic Weyl tensor (i.e. $\delta W = 0$[2]) $\Longrightarrow$ Einstein.

By **Wu**, **Wu** & **Wylie**, 2018:

- The same conclusion holds under the weaker condition of harmonic self-dual Weyl tensor (i.e. $\delta W^+ = 0$).

[2] The fourth-order vanishing condition $\mathrm{div}^4(W) = 0$ was also considered by Catino, Mastrolia and Monticelli in 2017.

Compact (non-Einstein) examples in 4D:

- (**Cao-Koiso**, 1991): The first example of (nontrivial) compact shrinking Ricci soliton: $\mathbb{CP}^2\#(-\mathbb{CP}^2)$.
- (**Wang-Zhu**, 2004): The second one: $\mathbb{CP}^2\#2(-\mathbb{CP}^2)$.
  - *In the compact case, a nontrivial Kähler-Ricci soliton is Fano (i.e., the first Chern class $C_1(M)$ is positive) and the Futaki-invariant is nonzero.*
  - *Moreover, by* **Tian** *and* **Zhu** *(2000), the soliton vector field is unique up to holomorphic automorphisms of the underlying complex manifold.*

Compact Ricci solitons

Problem (H.-D. Cao, 2006)

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It remains to be determined whether a compact non-Einstein gradient Ricci soliton is necessarily Kähler.

- Unlike the cases of dimensions 2 and 3, the classification of higher dimension gradient shrinking Ricci soliton is still incomplete.

Topology x Geometry (4D Ricci solitons)

- By Poincaré duality, the Euler characteristic and signature of M^4 are given by

$$\chi(M) = 2 - 2b_1(M) + b_2(M)$$

and

$$\tau(M) = b_+(M) - b_-(M),$$

where $b_1(M)$ and $b_2(M) = b_+ + b_-(M)$ are the first and second Betti numbers of M^4, respectively.

Topology x Geometry (4D compact Ricci solitons)

- By **Derdziński**, the first Betti number of a 4D compact Ricci soliton is $b_1(M) = 0$ and hence,

$$\chi(M) = 2 + b_2(M) > 0,$$

(i.e., Berger's inequality).

- Moreover, we have the inequality: $\chi(M) > |\tau(M)|$.
- By **Derdziński**, **Fernández-López and García-Río**, and **Wylie**, it is known that every compact Ricci soliton has finite fundamental group (i.e., $\pi_1(M) < \infty$).

On compact 4D-manifolds...

The classical **Gauss-Bonnet-Chern formula** asserts that

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(|W^+|^2 + |W^-|^2 + \frac{R^2}{24} - \frac{1}{2}|\mathring{Ric}|^2\right) dV_g \tag{6}$$

and the **Hirzebrush's theorem**

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W^+|^2 - |W^-|^2\right) dV_g. \tag{7}$$

Observe that ...

$$2\chi(M) \pm 3\tau(M) = \frac{1}{4\pi^2}\int_M \left(|W^\pm|^2 + \frac{R^2}{24} - \frac{1}{2}|\mathring{Ric}|^2 \right) dV_g .$$

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Theorem (Hitchin-Thorpe inequality, 1974)

Every 4D compact Einstein manifold M^4 satisfies

$$\chi(M) \geq \frac{3}{2}|\tau(M)|. \tag{8}$$

Moreover, equality holds if and only if M^4 is finitely covered by a torus or $K3$ surface.

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Theorem (LeBrun, 1996)

There are infinitely many compact simply connected smooth 4-manifolds which do not admit Einstein metrics, but nevertheless satisfy the strict Hitchin-Thorpe inequality.

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- Observe that

$$\Big(2\chi(M) \pm 3\tau(M)\Big)\Big(\mathbb{CP}^2\#(-\mathbb{CP}^2)\Big) = 8$$

$$\Big(2\chi(M) + 3\tau(M)\Big)\Big(\mathbb{CP}^2\#2(-\mathbb{CP}^2)\Big) = 7$$

Returning to the general case, we have

$$2\chi(M) \pm 3\tau(M) = \frac{1}{4\pi^2}\int_M \left(|W^\pm|^2 + \frac{R^2}{24} - \frac{1}{2}|\mathring{Ric}|^2 \right) dV_g.$$

- On a 4D compact Ricci soliton, we have

$$\int_M |\mathring{Ric}|^2 dV_g = \frac{1}{4}\int_M R^2 dV_g - Vol(M).$$

- Consequently, on a 4D compact Ricci soliton, one obtains that

$$2\chi(M) \pm 3\tau(M) = \frac{1}{48\pi^2} \int_M \left(24|W^{\pm}|^2 - R^2 + 6\right) dV_g.$$

- It is known that $|W^+|^2 = \frac{R^2}{24}$ for any compact Kähler manifold with the natural orientation from the complex structure.

• The strict inequality $2\chi(M) \pm 3\tau(M) > 0$ holds for any compact Kähler Ricci soliton.

A conjecture by H.-D. Cao (2006)

"*Does the Hitchin-Thorpe inequality hold for compact 4-dimensional gradient shrinking Ricci solitons*?"

- In the last years, some partial answers were obtained by **Fernández-López** and **García-Río**, **L. Ma**, **H. Tadano** and others.

Hitchin-Thorpe inequality and Ricci flow

Theorem (Fang-Zhang-Zhang, 2008)

If M^4 *is a compact manifold which the normalized Ricci flow exists for all* $t > 0$ *with uniformly bounded sectional curvature, then*

$$\chi(M) \geq \frac{3}{2}|\tau(M)|.$$

Theorem (Zhang-Zhang, 2010)

If M^4 *is a compact manifold with non-positive Yamabe invariant and admitting a long time solutions of the normalized Ricci flow with bounded scalar curvature, then*

$$\chi(M) \geq \frac{3}{2}|\tau(M)|.$$

Hitchin-Thorpe inequality for Ricci solitons

The Hitchin-Thorpe inequality holds for compact Ricci solitons under one of the following conditions:

- **L. Ma**, (2013): $\int_M R^2 dV_g \leq 6\mathrm{Vol}(M)$.
- **Fernández-López and García-Río**, (2010):

$$dia(M,g) \leq \max\left\{\sqrt{\frac{2}{\frac{1}{2}-c}}, \sqrt{\frac{2}{C-\frac{1}{2}}}, 2\sqrt{\frac{2}{C-c}}\right\},$$

where $C = \sup_{x\in M}\{Ric(v,v);\ v \in T_pM,\ |v| = 1\}$ and $c = \inf_{x\in M}\{Ric(v,v);\ v \in T_pM,\ |v| = 1\}$.

A similar result was obtained by H. Tadano (2018).

Hitchin-Thorpe inequality for Ricci solitons

$$Ric + Hess\, f = \frac{1}{2} g.$$

Theorem (Cheng, R.____, Zhou, 2023)

Let (M^4, g, f) be a 4D compact GSRS. Then

$$8\pi^2 \chi(M) \geq \int_M |W|^2 dV_g + \frac{1}{24} Vol(M)(5 - e^{f_{\max} - f_{\min}}). \qquad (9)$$

Moreover, equality holds if and only if g is an Einstein metric (in this case, f is constant).

Hitchin-Thorpe inequality for Ricci solitons

Corollary (Cheng, R.____, Zhou, 2023)

Let (M^4, g, f) *be a 4D compact GSRS. If* $f_{\max} - f_{\min} \leq \log 5$, *then the Hitchin-Thorpe inequality*

$$\chi(M) \geq \frac{3}{2}|\tau(M)| \tag{10}$$

holds on M.

Hitchin-Thorpe inequality for Ricci solitons

Corollary (Cheng, R.____, Zhou, 2023)

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holds on M.

Remark:

- This provides a partial answer to H.-D. Cao's conjecture;
- Notice that $f_{\max} - f_{\min} \leq \log 5 \approx 1.6$;
- on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ we have $f_{\max} - f_{\min} \approx 1.06$.

As an application, we deduce the following volume upper bounds.

Theorem (Cheng, R._____, Zhou, 2023)

Let (M^4, g, f) be a 4D compact GSRS. Then the following assertions hold:

1. $$Vol(M)\left(5 - e^{f_{\max} - f_{\min}}\right) \leq 384\pi^2.$$

   Equality holds if and only if (M, g) is a sphere $\mathbb{S}^4$ with the radius $\sqrt{6}$.

2. $$Vol(M)\left(5 - e^{f_{\max} - f_{\min}}\right) \leq \mathcal{Y}(M, [g])^2,$$

   where $\mathcal{Y}(M, [g])$ stands for the Yamabe invariant of (M^4, g). Moreover, equality holds if and only if g is an Einstein metric.

Sketch of the proof of Theorem 1

- We assume that f is not constant. Otherwise, the result is already true.
- We consider the sub-level set of the potential function:

$$D(t) = \{x \in M; \ \ f(x) < t\}.$$

Proposition

Let (M^n, g, f) be an n-dimensional complete (not necessarily compact) GSRS, where f is non-constant. Suppose that h is a bounded measurable function. Then we have

1. *the set of the critical points of f and each level set of f satisfy $\mathcal{H}^n(\{|\nabla f| = 0\}) = 0$ and $\mathcal{H}^n(\{f = c\}) = 0$, respectively.*
2. *$F(t) := \int_{D(t)} h dV_g$ is absolutely continuous and $F'(t) = \int_{f=t} \frac{g}{|\nabla f|}$ a.e.*

Sketch of the proof of Theorem 1

- Let $a = f_{\max}$ and $b = f_{\min}$ the maximum and minimum values of f on M^4, respectively. We consider the set The one obtains that

$$\int_{D(s)} \langle \nabla R, \nabla f \rangle dV_g = \int_a^s \int_{D(s)} \Big(R + \langle \nabla R, \nabla f \rangle - 2|Ric|^2 \Big) dV_g dt.$$

Sketch of the proof of Theorem 1

- Define the function $\phi(s)$ and $\psi(s)$ by

$$\phi(s) = \int_a^s \int_{D(t)} \langle \nabla R, \nabla f \rangle dV_g dt$$

and

$$\psi(s) = \int_a^s \int_{D(t)} \left(R - 2|Ric|^2\right) dV_g dt.$$

- Of which, we have

$$\phi'(s) = \phi(s) + \psi(s).$$

Sketch of the proof of Theorem 1

- Since $\phi(a) = 0$, one obtains that

$$\phi'(s) = e^s \int_a^s \psi'(t) e^{-t} dt.$$

- Consequently, we arrive at

$$\begin{aligned} \phi'(b) &\leq e^b \int_a^b \int_{D(t)} \left(\frac{1}{2} - \frac{1}{2}(R-1)^2 \right) dV_g e^{-t} dt \\ &\leq \frac{1}{2} Vol(M) \left(e^{a-b} - 1 \right). \end{aligned}$$

- Then, we obtain the asserted result

$$\begin{aligned} 4\pi^2 \left(2\chi(M) \pm 3\tau(M)\right) &\geq 2 \int_M |W^{\pm}|^2 dV_g \\ & \quad \frac{1}{24} Vol(M) \left(5 - e^{f_{max} - f_{min}} \right). \end{aligned}$$

□

Thank you for your attention!

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