Discrete spectrum of polynomially compact pseudodifferential operators and applications to the Neumann-Poincare operator in 3D Elasticity.

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Origin: The Neumann-Poincare operator in 3D elasticity: the 'double-layer potential.' A renewed interest - wave propagation in auxetics -materials with negative Poisson constant and plasmonic resonance. It was proved almost 30 years ago that it is a pseudodifferential operator of order 0 (a singular integral operator), with 3 points of essential spectrum. What can one say about eigenvalues converging to these points?

Pseudodifferential operators.

In R^d :

$$(Au)(x) = (2\pi)^{-d} \int a(x,\xi) e^{ix\xi} \left(\int e^{-iy\xi} u(y) dy \right) d\xi \equiv \mathcal{F}^{-1}_{\xi \to x} a(x,\xi)(\mathcal{F}_{y \to \xi} u(y)).$$

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$$\mathcal{F}^{-1}_{\xi \to x} a(x,\xi) (\mathcal{F}_{y \to \xi} u(y)).$$

Symbol: $a(x,\xi) \sim \sum a_{k-j}(x,\xi)$; $a_{k-j}(x,\xi)$ positively homogeneous of order $k-j$, $a_{k-j}(x,t\xi) = t^{k-j} a_{k-j}(x,\xi), t > 0$.

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For a manifold Γ - by means of decomposition of unity and gluing together of local representations. Symbol is defined on $T^*\Gamma$.
Vector case: $a(x,\xi)$ is a $N \times N$ matrix, the operator acts on N component vector-functions.

Part 1 Spectral theory for zero order (selfadjoint) pseudodifferential operators

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Until very recently, the only paper we could find: Malcolm Adams, JFA, 1983. For a zero order operator A on Γ (a smooth Riemannian closed manifold, dimension d) with principal symbol $a_0(x,\xi)$ ($(x,\xi) \in T^*\Gamma$) *the essential spectrum* $\mathbf{\Sigma}$ is the set of values of $a_0(x,\xi)$.

Discrete spectrum. Matrix case.

A an order 0 Ψdo in the space of N component vector-functions, $a_0(x,\xi)$ the principal symbol; $a_0(x,\xi) \in END(\mathbb{C}^N)$, $a_0 = a_0^*$. The essential spectrum: the set of EIGENVALUES of $a_0(x,\xi)$.

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What are we going to count???

If ω is a lower tip of $\mathbf{\Sigma}$, (this means $\mathbf{\Sigma}(A) \cap (\omega - \tau_-, \omega) = \emptyset$ for some $\tau_- > 0$), fix such τ_- and for $\tau \in (0, \tau_-)$ we denote $n_-(A; \omega, \tau)$ the number of eigenvalues of A in the interval $(\omega - \tau_-, \omega - \tau)$.

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Similarly, for an upper tip ω of $\mathbf{\Sigma}$ and a proper τ_+, denote $n_+(A; \omega, \tau)$ the number of eigenvalues of A in $(\omega + \tau, \omega + \tau_+)$ (A is can be omitted in this notation).

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If ω is an isolated point of $\mathbf{\Sigma}$, both $n_\pm(\omega, \tau)$ are defined and finite. **We are interested in estimates and asymptotics of $n_\pm(\omega, \tau)$ as $\tau \to 0$.** $n(\omega, \tau) = n_+(\omega, \tau) + n_-(\omega, \tau)$.

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If in $A - \omega$ is of actual order $-\kappa$, with 'principal' symbol $a_{-\kappa}(x, \xi)$; eigenvalue the asymptotics is

$$n_{\pm}(\omega, \tau) \sim \tau^{-d/\kappa} d^{-1} (2\pi)^{-d} \int_{|\xi|=1} \operatorname{Tr}\left([a_{-k}(x, \xi)]_{\pm}\right)^{d/\kappa} dx d\xi \tag{0.1}$$

M.Birman–M.Solomyak, 1977, 1979.

Polinomially compact: $p(A)$ compact

$\mathbf{\Sigma}(A)$ consists of several discrete points ω_l, $l = 1, \ldots, L$. We study the eigenvalues converging to ONE of these points, ω_s. Approach: We GUESS a polynomial, which distinguishes the eigenvalues of A near ω_s but almost 'destroys' eigenvalues near other $\omega_l, l \neq s$.

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The case $N = 3$ is of major interest and we will write formulas for this case, $d = 2$, Γ is a smooth compact surface, $a_0(x, \xi)$, is the principal symbol (in a fixed co-ordinate system and a frame), $a_{-1}(x, \xi)$ symbol of order -1.

$$\mathbf{p}_1(t) = (t - \omega_1)(t - \omega_2)^2(t - \omega_3)^2.$$

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$$
\begin{aligned}
\mathrm{sym}(\mathbf{p}_1(A)) \sim\, & a_{-1}(a_0-\omega_2)^2(a_0-\omega_3)^2 + (a_0-\omega_1)a_{-1}(a_0-\omega_2)(a_0-\omega_3)^2 + \\
& (a_0-\omega_1)(a_0-\omega_2)a_{-1}(a_0-\omega_3)^2 + (a_0-\omega_1)(a_0-\omega_2)^2 a_{-1}(a_0-\omega_3) + \\
& b_1 c_1(a_0-\omega_2)(a_0-\omega_3)^2 + b_1(a_0-\omega_2)c_1(a_0-\omega_3)^2 \\
& + b_1(a_0-\omega_2)^2 c_1(a_0-\omega_3) + b_1(a_0-\omega_2)^2(a_0-\omega_3)c_1 + \\
& (a_0-\omega_1)b_1 c_1(a_0-\omega_3)^2 + (a_0-\omega_1)b_1(a_0-\omega_2)c_1(a_0-\omega_3) + \\
& (a_0-\omega_1)b_1(a_0-\omega_2)(a_0-\omega_3)c_1 + (a_0-\omega_1)(a_0-\omega_2)b_1 c_1(a_0-\omega_3) + \\
& (a_0-\omega_1)(a_0-\omega_2)b_1(a_0-\omega_3)c_1 + (a_0-\omega_1)(a_0-\omega_2)^2 b_1 c_1 + \\
& b_2 c_2(a_0-\omega_2)(a_0-\omega_3)^2 + b_2(a_0-\omega_2)c_2(a_0-\omega_3)^2 + \\
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\end{aligned}
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For a concrete operator: calculate $\mathbf{p}_1(A)_{-1}$ apply the B-S formula for spectrum.

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For a concrete operator: calculate $\mathbf{p}_1(A)_{-1}$ apply the B-S formula for spectrum.Important: the algebraic structure is explicit. .

Part 2 The Neumann-Poincare operator $\equiv$ double layer potential

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$$\mathcal{K}f(x) = \int_\Gamma \frac{\partial}{\nu_y}\Phi(x,y)f(y)dS(y); x \in \Gamma;$$

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$$\mathcal{K}^* f(x) = \int_\Gamma \frac{\partial}{\nu_x}\Phi(x,y)f(y)dS(y); x \in \Gamma$$

$$\Gamma = \partial\Omega \subset \mathbb{R}^{d+1};$$

- bounded domain with smooth boundary, $\Phi(x,y)$- a fundamental solution for an elliptic operator $\mathcal{L}$ in $\mathbb{R}^d$, ν_y a properly defined 'conormal'.

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$\mathcal{L}$-Laplacian – C.Neumann, 1887; H. Poincare, 1897. Integral equations of potential theory: I.Fredholm, 1900–1903.
For Laplacian: $\mathcal{K}$ a compact operator, an order -1 PsDO.
Eigenvalue asymptotics is known.

Elasticity NP operator

Lamé equations, homogeneous isotropic material.

$$\mathcal{L}u \equiv \mathcal{L}_{\mu,\lambda} = \mathrm{div}\,(\mu \mathrm{grad}\, u) + \mathrm{grad}\,((\lambda+\mu)\mathrm{div}\, u) = 0, x = (x_1, x_2, x_3) \in \Omega,$$

μ, λ : Lamé parameters. $\varkappa = \frac{\lambda+\mu}{\mu} = (1-2\sigma)^{-1}$, σ - the Poisson constant, $u = (u_1, u_2, u_3)^{\perp}$

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Fundamental solution, 'Kelvin matrix':

$$\Phi_{jk}(x) = -\frac{\alpha}{4\pi}\delta_{jk}|x|^{-1} - \frac{\beta}{4\pi}x_j x_k |x|^{-3}, \ d = 3.$$

$\alpha = \frac{1}{2}(\mu^{-1} + (2\mu+\lambda)^{-1})$, $\beta = \frac{1}{2}(\mu^{-1} - (2\mu+\lambda)^{-1})$.
Conormal derivative: matrix 3×3 operator $\mathbf{T}$

$$\mathbf{T}_{jk} \equiv \mathbf{T}_{jk}(\partial_x, \nu_x) = \lambda\nu_x^j \partial_k(x) + \lambda\nu_x^k \partial_j + (\lambda+\mu)\delta_{jk}\partial_{\nu_x}.$$

$\mathcal{K}f(x) = \int_\Gamma \mathbf{T}_y \Phi(x-y) f(y) dS_y$. Even for a smooth boundary, the NP operator is not compact (off-diagonal terms).

Dim 3, M.Agranovich, B.Amosov, M.Levitin 1999; K. Ando, H.Kang, Y.Miyanishi 2017; Y.Miyanishi, G.Rozenblum 2021,

NP operator is a pseudodifferential operator on Γ with principal symbol

$$a_0(\xi') = \frac{i\kappa_0}{|\xi'|}\begin{pmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}.$$

$\kappa_0 = \frac{\mu}{2(2\mu+\lambda)}$, $\xi' = (\xi_1, \xi_2) \in T^*\Gamma$. Eigenvalues of the principal symbol: $0, \pm\kappa_0$.

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The only previously known case

A sphere, Deng, Li, Liu, 2019 (J.Spectr.Theory)

$$\Lambda_j^0(\mathcal{K}) = \frac{3}{2(2j+1)} \sim \frac{3}{4j}, \tag{0.2}$$

$$\Lambda_j^-(\mathcal{K}) = \frac{3\lambda - 2\mu(2j^2 - 2j - 3)}{2(\lambda + 2\mu)(4j^2 - 1)} \sim -\kappa_0 + \frac{2\mu}{(\lambda + 2\mu)j},$$

$$\Lambda_j^+(\mathcal{K}) = \frac{-3\lambda + 2\mu(2j^2 + 2j - 3)}{2(\lambda + 2\mu)(4j^2 - 1)}, \sim \kappa_0 + \frac{2\mu}{(\lambda + 2\mu)j},$$

all of them of multiplicity j, where λ, μ are Lamé constants.
pause all three sequences approach their limit points from above; there are no eigenvalues that approach these points from below.
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pause all three sequences approach their limit points from above; there are no eigenvalues that approach these points from below. Even this case is not easy, since variables do not separate.We investigate whether these properties remain for a general surface.

The main theorem.

1. Let χ, $\mathbf{W} = \int_\Gamma (k_1 + k_2)^2 dS$ be the Euler characteristic and the Willmore energy of Γ. Then

   $$n(\omega_s, \tau) : n_+(\omega_s, \tau) + n_-(\omega_s, \tau) \sim \tau^{-2}(\mathcal{A}(\omega_s, \lambda, \mu)\chi + \mathcal{B}(\omega_s, \lambda, \mu)\mathbf{W}),$$

   where $\mathcal{A}, \mathcal{B}$ are universal, depending only on λ, μ quadratically, coefficients.
2. Corollary. There are *always* infinitely many eigenvalues near each of the points ω_s;
3. The asymptotic formulas hold

   $$n_\pm(\omega_s, \tau) \sim \mathcal{C}_\pm(\omega_s, \lambda, \mu, \Gamma)\tau^{-2}.$$

   The coefficients are given by explicit terrible formulas.
4. Coefficients $\mathcal{C}_+(\omega_s, \lambda, \mu, \Gamma)$ are always positive; this means that there are infinitely many eigenvalues converging to ω_s *from above*.
5. If there exists a point where the surface Γ is strictly concave, then $\mathcal{C}_-(\omega_s, \lambda, \mu, \Gamma) > 0$, i.e., there are infinitely many eigenvalues of the NP operator converging to ω_s from below.

We repeat:

$b_\alpha = -i\partial_{\xi_\alpha} a_0, c_\alpha = \partial_{x_\alpha} a_0;$

$$\mathrm{symb}(\mathbf{p}_1(A))_{-1}$$

$$\sim a_{-1}(a_0-\omega_2)^2(a_0-\omega_3)^2 + (a_0-\omega_1)a_{-1}(a_0-\omega_2)(a_0-\omega_3)^2 +$$
$$(a_0-\omega_1)(a_0-\omega_2)a_{-1}(a_0-\omega_3)^2 + (a_0-\omega_1)(a_0-\omega_2)^2 a_{-1}(a_0-\omega_3) +$$
$$b_1c_1(a_0-\omega_2)(a_0-\omega_3)^2 + b_1(a_0-\omega_2)c_1(a_0-\omega_3)^2$$
$$+b_1(a_0-\omega_2)^2c_1(a_0-\omega_3) + b_1(a_0-\omega_2)^2(a_0-\omega_3)c_1 +$$
$$(a_0-\omega_1)b_1c_1(a_0-\omega_3)^2 + (a_0-\omega_1)b_1(a_0-\omega_2)c_1(a_0-\omega_3) +$$
$$(a_0-\omega_1)b_1(a_0-\omega_2)(a_0-\omega_3)c_1 + (a_0-\omega_1)(a_0-\omega_2)b_1c_1(a_0-\omega_3) +$$
$$(a_0-\omega_1)(a_0-\omega_2)b_1(a_0-\omega_3)c_1 + (a_0-\omega_1)(a_0-\omega_2)^2 b_1c_1 +$$
$$b_2c_2(a_0-\omega_2)(a_0-\omega_3)^2 + b_2(a_0-\omega_2)c_2(a_0-\omega_3)^2 +$$
$$b_2(a_0-\omega_2)^2c_2(a_0-\omega_3) + b_2(a_0-\omega_2)^2(a_0-\omega_3)c_2 +$$
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$$(a_0-\omega_1)(a_0-\omega_2)b_2(a_0-\omega_3)c_2 + (a_0-\omega_1)(a_0-\omega_2)^2 b_2c_2.$$

Order of action.

1. Express the NP operator $\mathcal{K}$ in convenient co-ordinate system and frame;
2. Study the subprincipal symbol of $\mathcal{K}$;
3. Study the derivatives of the principal symbol of $\mathcal{K}$
4. Analyze the formula for the symbol of $\mathbf{p}_s(\mathcal{K})$; determine the dependence on the geometry of Γ.
5. Introduce the model case containing all information on $\mathbf{p}_s(\mathcal{K})$;
6. Investigate the model case.

NP operator as integral operator

It has kernel (see Kupradze, 1979):

$$[\mathcal{K}(\mathbf{x},\mathbf{y})]_{p,q} = \mu(\lambda' - \mu')\frac{\nu_p(\mathbf{y})(x_q - y_q) - \nu_q(\mathbf{y})(x_p - y_p)}{|\mathbf{x}-\mathbf{y}|^3} +$$

$$\left(\mu(\mu' - \lambda')\delta_{p,q} - 6\mu\mu'\frac{(x_p - y_p)(x_q - y_q)}{|\mathbf{x}-\mathbf{y}|^2}\right)\sum_{l=1}^{3}\nu_l(\mathbf{y})\frac{x_l - y_l}{|\mathbf{x}-\mathbf{y}|^3}; \mathbf{x},\mathbf{y}\in\Gamma.$$

$$\lambda' = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)},\ \mu' = \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)},\ p,q = 1,2,3.$$

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Singularity: the diagonal $(p = q)$ terms in the kernel $\mathcal{K}(\mathbf{x},\mathbf{y})$ have a weak, $O(|\mathbf{x}-\mathbf{y}|^{-1})$ singularity as $|\mathbf{x}-\mathbf{y}| \to 0$, so they define weakly polar, and, therefore, compact, integral operators. The off-diagonal $(p \neq q)$ terms have singularity of order $|\mathbf{x}-\mathbf{y}|^{-2}$, odd in $\mathbf{x}-\mathbf{y}$, therefore they define bounded singular integral operators.

We choose a convenient co-ordinate system: $\mathbf{x}^0 \in \Gamma$; x_1, x_2 axes along some orthogonal tangential directions on Γ, x_3 along the exterior normal. To find the symbol of the NP operator, we expand the kernel at $\mathbf{x}^0$ in the Taylor series, in powers of $x - \mathbf{x}^0$.

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Crucial observation: The leading, order -2, term in this expansion does not depend on Γ. The second, order -1 term depends linearly on $k_1(\mathbf{x}^0), k_2(\mathbf{x}^0)$ with universal (depending only on λ, μ) coefficients.

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Passage to the symbol of $\mathcal{K}$: the Fourier transform of the kernel.

Main technical result. The symbols a_0, $\partial_{\xi_\alpha} a_0$ of the NP operator $\mathcal{K}$ are universal (described above.) The symbols a_{-1}, $b_\alpha = \partial_{x_\alpha} a_0$ are linear functions of the principal curvatures $k_1(\mathbf{x}^0), k_2(\mathbf{x}^0)$. The (matrix) coefficients at $k_1(\mathbf{x}^0), k_2(\mathbf{x}^0)$ in these functions are universal expressions depending only on λ, μ.

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Recall that the symbol of $\mathfrak{M} = \mathbf{p}_1(\mathcal{K})_{-1}$, the one which determines the eigenvalue asymptotics, consists of 24 product terms, but each product contains only 1 term depending on the curvatures. Therefore, their sum is also a linear function of principal curvatures.

The structural theorem

The symbol of $\mathfrak{M} = \mathbf{p}_1(\mathcal{K})_{-1}$, (expressed in the special co-ordinates system, determined by the curvature lines and the corresponding frame) has the form

$$\mathfrak{m}(x,\xi) = k_1(x)M_1(\xi) + k_2(x)M_2(\xi)$$

Where $M_1(\xi), M_2(\xi)$ are matrices determined only by the Lamé constants λ, μ. They also depend on the point ω_s of the essential spectrum.

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Remark: since the normal is exterior, k_1, k_2 are negative at the points where the surface Γ is convex.

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1. Γ is the sphere, $k_1 = k_2 = -1$: $M_1(\xi), M_2(\xi) \leq 0$.

Since there exists at least one point on Γ where at least one of curvatures is negative and the other one is nonpositive (the body is convex), there are infinitely many eigenvalues converging to ω_s from above.

If there exists at least one point where both curvatures are positive (a point of concavity,) there are infinitely many eigenvalues converging to the ω_s from below.

2. The dependence of $M_k(\xi)$ on λ, μ, ξ is hard to find, but still it is a rational task (done by means of a symbolic manipulation program.) The leading case is a cylinder $k_2 \equiv 0$, and instead of 24 terms we need to calculate only 14. But even here to find the asymptotics of eigenvalues is, probably, impossible, since one needs to find explicitly eigenvalues of a matrix 3×3 depending on parameters, and then integrate in x, ξ.
But: in the convex case, the symbol $\mathfrak{m}(x, \xi)$ is nonnegative, and here, fortunately, the Birman-Solomyak coefficient is in dimension 2 a rational expression:
$\operatorname{tr}(\mathfrak{m}(x, \xi)^2)$ is the sum of squares of absolute values of the entries, $\operatorname{tr}(\mathfrak{m}(x, \xi)^2) = \sum_{p,q} |\mathfrak{m}(x, \xi)_{p,q}|^2$, it is a quadratic function of curvatures, and this gives the asymptotics of eigenvalues.

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Since the matrix $\mathfrak{m}(x, \xi)$ is linear in curvatures, its square is a quadratic form in curvatures, with universal coefficients. Integration of this form gives a linear expression in the Euler characteristics and the Willmore energy.

Sources

G. Rozenblum. Eigenvalue asymptotics for polynomially compact pseudodifferential operators. Algebra i Analiz 33 (2021), no. 2, 215–232.(St.Petersburg Math.Journal, **33** (2) 2022); `arXiv:2006.10568`

G. Rozenblum. Asymptotics of eigenvalues of the Neumann–Poincaré operator in $3D$ elasticity. `arXiv:2112.07710`