Norming constants of embedded bound states and bounded positon solutions of the Korteweg-de Vries equation

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Introduction

We are concerned with the inverse scattering problem for the full line Schrodinger operator $\mathbb{L}_q = -\partial_x^2 + q(x)$ in the presence of embedded eigenvalues (i.e. positive eigenvalues in the continuous spectrum) and understanding how such eigenvalues affect solutions to the initial value problem for the Korteweg-de Vries (KdV) equation

$$\begin{aligned} \partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad -\infty < x < \infty,\ t \geq 0, \\ u(x,0) = q(x). \end{aligned} \tag{1}$$

Derived in 1895 by Korteweg-de Vries, it was dormant till the earlier 1950s when Fermi-Ulam-Pasta had to deal with its discrete version. But would not yield a lot of information. The breakthrough came in 1967 from the fundamental Gardner–Greene–Kruskal–Miura discovery of what we now call the inverse scattering transform (IST) for the KdV equation giving rise to Soliton Theory.

IST is similar to the Fourier transform and consists of three steps:

1. (direct transform) Associate with $q(x)$ the Schrödinger operator $\mathbb{L}_q = -\partial_x^2 + q(x)$ on $L^2(\mathbb{R})$ and solve the direct scattering/spectral problem

$$q(x) \longrightarrow S_q,$$

where S_q is the scattering/spectral data.

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2. (time evolution) Use S_q as a a new set of variables which turns KdV into a simple first order linear ODE for $S_q(t)$ with the initial condition $S_q(0) = S_q$:
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3. (inverse transform) Solve the inverse scattering/spectra problem for the data $S_q(t)$:

$$S_q(t) \longrightarrow q(x,t).$$

The new potential $q(x,t)$ solves KdV.

This machinery runs smoothly if $q(x) = O(|x|)^{-2-\varepsilon}$ (short-range) or periodic (standard case) and main results were obtained in the 1970s .
Owing to IST, a tremendous amount has been learned about KdV and other integrable systems. In fact, KdV knew soliton and cnoidal solutions.
Question: What is still unknown about KdV after over 50 years? It is fair to say that everything is known if q is classical but, surprisingly enough, disproportionally less otherwise.
In fact, Vladimir Zakharov repeatedly asks "In spite of all these brilliant achievements, the theory of the KdV equation is not yet developed to a level which would satisfy a pragmatic physicist, who may ask the following question: What happens if the initial data in the KdV equation is neither decaying at infinity nor periodic? Suppose that the initial data is a bounded function

$$q(x) = u(x, 0), \quad |q(x)| < c.$$

Can we extend the IST to this case, which has great practical importance?"
Similar questions have been posed by many other founding fathers. But whole hell breaks loose if we are outside standard classes.

Major problems occur with all three steps in IST:

- Complicated spectrum of $\mathbb{L}_q = -\partial_x^2 + q(x)$ causes existential problems to Scattering Theory as we know it. Standard scattering data are no longer data.

These issues make any prediction about the IST (and WP too) for nonstandard data very difficult.
This prompts a question:
To what extent is the KdV equation more integrable than any other generic PDE if we are outside of standard classes? It is important to understand quite precisely what conditions should be imposed on q so that the KdV is WP and completely integrable (i.e. solvable via a suitable IST) For a long time it remained a virgin land.

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- Inverse Scattering Theory is not developed outside of standard cases.
- Last but not least, no known wellposedness (WP) results are available outside L^2 data (1993 Bourgain's famous result).

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Our specific objectives

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- In 2010 Vladimir Matveev wrote: "A very interesting unsolved problem is to study the large time behavior of the solutions to the KdV equation corresponding to the smooth initial data like $cx^{-1}\sin 2kx$, $c \in \mathbb{R}$", "The related inverse scattering problem is not yet solved and the study of the related large times evolution is a very challenging problem".

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- Such a potential is called Wigner-von Neumann (WvN) type and it may produce an embedded bound states at k^2 (and no other positive singular spectrum.). It is important that WvN potentials are in L^2 and due to the seminal Bourgain's results remains well-posed.

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- If $|c/\omega| > 1/\sqrt{2}$ then the negative spectrum (necessarily discrete) is infinite in general (Klaus 1982) but finite otherwise.

Our framework and main ingredients

1. Weyl m-function. $q(x)$ is Weyl limit point at $\pm\infty$ if

$$\mathbb{L}_q u := -u'' + q(x)\, u = \lambda u, \quad x \in \mathbb{R},$$

has a unique (up to a multiplicative constant) solution, Weyl solution, $\psi_{\pm}(x,\lambda) \in L^2(\pm\infty)$, $\lambda \in \mathbb{C}^+$. Only sufficient conditions are available. E.g., if q is essentially bounded below.

Lemma (On Weyl solution)

Let $\mathbb{L}_q u = Eu$ have a real solution $u_0 \in L^2(+\infty)$ for some $E > 0$ and let $\psi(x,\lambda)$ be a Weyl solution at $+\infty$. If $\psi(x, E + i0)$ exists then $\psi(x, E + i0) = Cu_0(x)$.

$$m_{\pm}(\lambda, x) = \pm\psi'_{\pm}(x,\lambda)\,/\,\psi_{\pm}(x,\lambda)\,, \quad \lambda \in \mathbb{C}^+$$

is called the right/left Weyl m-function, or just m-function (1D Dirichlet-to-Neumann map).

2. Reflection coefficient. Assume the following basic conditions:

1. $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ and real;

Condition (3) assumes some decay at $+\infty$ and implies:

It is well-defined for a.e. k and $R(-k) = \overline{R(k)}$, $|R(k)| \leq 1$.

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2. $\mathbb{L}_q \geq -c^2 > -\infty$;
3. $\mathbb{L}_q u = k^2 u$ has the right Jost solution $\psi(x,k)$

$$\psi(x,k) \sim \mathrm{e}^{ikx}, \psi'(x,k) \sim ik\mathrm{e}^{ikx},\ x \to +\infty.$$

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Condition (3) assumes some decay at $+\infty$ and implies:

- The pair $\{\psi, \overline{\psi}\}$ forms a fundamental set for the Schrodinger eq. and

$$W(\overline{\psi(x,k)}, \psi(x,k)) = 2\mathrm{i}k.$$

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- The left Weyl solution φ can be normalized for a.e. real k by

$$\varphi(x,k) = \overline{\psi(x,k)} + R(k)\psi(x,k), \quad \text{(basic scattering identity)}$$

with some $R(k)$ called the (right) reflection coefficient.

It is well-defined for a.e. k and $R(-k) = \overline{R(k)}$, $|R(k)| \leq 1$.

3. Diagonal Green's function. If $q \in L^1(+\infty)$ then the Jost solution exists for any $k \neq 0$. Slower decay may give rise to spectral singularities of $\psi(x,k)$. The adequate object to deal with such singularities is the diagonal Green's function of $\mathbb{L}_q$

$$g\left(k^2,x\right)=\frac{\psi_+\left(x,k\right)\psi_-\left(x,k\right)}{W\left(\psi_+\left(x,k\right),\psi_-\left(x,k\right)\right)}=-\frac{\varphi\left(x,k\right)\psi\left(x,k\right)}{2\mathrm{i}k}.$$

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- it is analytic in k^2 from $\mathbb{C}^+$ to $\mathbb{C}^+$;
- its poles (necessarily real), both isolated and embedded, are eigenvalues of $\mathbb{L}_q$;
- the potential $q(x)$ can be found from

$$G\left(-\kappa^2, x\right) \sim 1 - q(x)/2\kappa^2, \quad \kappa \to +\infty.$$

4. Norming constants of negative bound states. If the Schrodinge eq. also has a left Jost solution ψ_{-} then $\varphi(x,k) = T(k)\,\psi_{-}(x,k)$ where $T(k)$ is the transmission coefficient. From basic scattering identity:

$$T(k) = 2\mathrm{i}k / W(\psi_{-}, \psi_{+})$$

and hence $T(k)$ is meromorphic in $\mathbb{C}^{+}$ with simple poles (if any) $\{\mathrm{i}\kappa_n\}$, $\kappa_n > 0$, and $k^2 = -\kappa_n^2$ are the isolated poles of $g(k^2, x)$, i.e. negative bound states of $\mathbb{L}_q$.
Since $R(k)$ in general is only defined on $\mathbb{R}$, one needs to include pole information in the set of scattering data. It can be done via

$$\operatorname*{Res}_{k=\mathrm{i}\kappa_n} \varphi(x,k) = \mathrm{i}c_n^2 \psi(x, \mathrm{i}\kappa_n), \text{ (isolated pole condition)} \tag{2}$$

where positive c_n^2, called norming constants of bound states, must be specified. Also $c_n = \|\psi(\cdot, \mathrm{i}\kappa_n)\|^{-1}$.

5. Norming constants of positive bound states. $q \notin L^1(+\infty)$ may give rise to resonances (aka spectral singularities), i.e. real points where $\psi(x,k)$ shows a blow up behavior. WnN resonances are the only type of resonance relatively well-understood (Klaus JMP, 1991). In general $\psi(x,k)$ may blow up to any order, we however restrict our attention to the case $\psi(x,k) = O(k-\omega)^{-1}$, $k \to \omega \in \mathbb{R}$, i.e. ω is an embedded simple pole. Since $g(k^2,x)$ may only have a simple embedded pole, $\varphi(x,\omega)$ is then well-defined. If $\varphi(x,\omega) \neq 0$ then ω^2 is an embedded bound state. We show in NON, 2021, that $R(k)$ alone can not tell a resonance from a bound state. Therefore an extra condition is required. Using (2) as a pattern to follow, we set

$$\operatorname*{Res}_{k=\omega_n} \psi(x,k) = \frac{i\alpha_n^2}{R(\omega_n)}\varphi(x,\omega_n) \ \ (\text{embedded pole condition}) \qquad (3)$$

with some $\alpha_n^2 > 0$ which we call the norming constant of embedded bound state ω_n^2. The reason for putting an extra $R(\omega_n)$ will be clear later. We shall see that (3) indeed works.

6. Gauge transformation. This is our last (but not least) ingredient.

Lemma (on gauge transformation)

If $\varphi(x,k)$ and $\psi(x,k)$ are related by the basic scattering identity then so are

$$\begin{aligned}\widetilde{\varphi}(x,k) &= \varphi(x,k) + \sum a_n(x)\, W(\varphi(x,k), f_n(x,k)) \\ \widetilde{\psi}(x,k) &= \psi(x,k) + \sum a_n(x)\, W(\psi(x,k), f_n(x,k))\end{aligned} \tag{4}$$

with the same $R(k)$ for any real $a_n(x)$, and $f_n(x,k)$ real for real k.

The proof is by a direct verification and completely trivial. We took the name 'gauge' from Bilman-Miller CPAM, 2019, where such transformations are crucially used in the context of matrix Riemann-Hilbert problem associated with the focusing NLS. We however learned about them from Grava-Minakov SIMA 2020, where it is used in a similar context but in the mKdV setting. Note (4) is different.

Turning resonances into embedded bound states

Theorem (Main theorem)

Assume basic conditions 1-3. Suppose that
1. (Resonance condition) for $\omega_n^2 > 0$, $1 \leq n \leq N < \infty$, $\mathbb{L}_q u = \omega_n^2 u$ *has a unique (up to a scalar multiple)* $L^2(-\infty)$ *solution.*
2. (Continuity condition) $\psi(x,k)$ *and* $R(k)$ *are continuous at* $k = \omega_n$.
Then

$$\phi_n(x) := 2\,\mathrm{Re}\left[R(\omega_n)^{1/2}\,\psi(x,\omega_n)\right] \in L^2(-\infty), \qquad (5)$$

where the root is chosen with a cut along $(-\infty, 0)$, *and the potential*

$$q_{+N}(x) = q(x) - 2\partial_x^2 \log\det(\mathbf{I} + \mathbf{G}_+(x)), \qquad (6)$$

$$\mathbf{G}_+(x) := \left(\alpha_m\alpha_n \int_{-\infty}^{x} \phi_m(s)\,\phi_n(s)\,ds\right), \quad \text{(Gram matrix)} \qquad (7)$$

supports embedded bound states ω_n^2, $1 \leq n \leq N$, *for any real* $\alpha_n \neq 0$.

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- Our approach is new and stems from the Riemann-Hilbert problem put forward in AR, SAM 21 and is more suited for the IST.
- The Gelfand-Levitan approach is revisited in the the half-line case by Eastham 1982. His formula coincides with (6) for $N = 1$ but no formula for $N > 1$ is given.

Discussions

- The associated (orthogonal in L^2) eigenfunctions $(y_n(x))$ can be uniquely found from the linear system

$$y_n(x) + \sum_{m=1}^{N} y_m(x) \int_{-\infty}^{x} \alpha_m \alpha_n \phi_m(s) \phi_n(s)\, ds = -\alpha_n \phi_n(x).$$

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- For $N = 1$ $\|y\| = 1$ and

$$q_{+1}(x) = q(x) - 2\partial_x^2 \log\left(1 + 4\alpha^2 \int_{-\infty}^{x} \mathrm{Re}^2\left[R(\omega)^{1/2} \psi(s,\omega)\right] ds\right).$$

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- $q(x) - q_{+N}(x)$ is continuous, in $L^1(-\infty)$, and $O(1/x)$, $x \to +\infty$. I.e., as expected $q_{+N}(x)$ is no longer short-range at $+\infty$ even if $q(x)$ is. More specifically, for some A_n, δ_n

$$q(x) - q_{+N}(x) \sim \sum_{n=1}^{N} \frac{A_n}{x} \sin(2\omega_n x + \delta_n),\ x \to +\infty.$$

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- Embedded bound states may not be created on a short-range background. Indeed we must have $|R(\omega)| = 1$ for some $\omega \neq 0$.

Updated scattering data

Corollary

Assume that $q(x)$ *is as in Main Theorem and*

$$xq(x) \in L^1(+\infty) \text{ (short-range at } +\infty\text{).}$$

If $S(q) = \left\{ R(k), \left(-\kappa_n^2, c_n^2\right) \right\}$ *is scattering data for* q *then* $S(q_{+N}) = S(q) \cup \left\{ \left(\omega_n^2, \alpha_n^2\right), 1 \leq n \leq N \right\}$ *is the scattering data for* q_{+N}.

Rowan Killip asked the author if embedded bound states require norming constants. Corollary answers his question in the affirmative: $\left(\alpha_n^2\right)$ play the role of norming constants of embedded bound states.

Bounded positon

Theorem (On bounded positons)

Assume that $q(x)$ *is as in Main Theorem and* $xq(x) \in L^1(+\infty)$. *Let*

$$\mathbf{G}_+(x,t) := \left(\alpha_m \alpha_n \int_{-\infty}^{x} \phi_m(s,t)\,\phi_n(s,t)\,ds \right), \quad \textit{(Gram matrix)}$$

$$\phi_n(x,t) = 2\,\mathrm{Re}\left[R(\omega_n)^{1/2} e^{4i\omega_n^3 t} \psi(x,t,\omega_n) \right].$$

If $q(x,t)$ *solves KdV with data* $S(q)$ *then*

$$q_{+N}(x,t) = q(x,t) - 2\partial_x^2 \log\det\left(\mathbf{I} + \mathbf{G}_+(x,t)\right) \tag{8}$$

solves KdV with data $S(q_{+N})$. *Moreover, embedded bound states* (ω_n^2) *are preserved under the KdV flow.*

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- Yes. The second log-derivative term of (8) says that propagation of the ensemble of positons is determined by $4\omega_n^3 t + \omega_n x$ which is in the direction of linear dispersive waves.

Analogy with solitons

Theorem (AR, SAPM 2021)

Let $q(x,t)$ solve KdV with short-range $q(x)$ and $S(q)$ its right scattering data. Fix $D = \{(\kappa_n, c_n)\}_{n=1}^{N}$ and introduce the $N \times N$ matrix function $\mathbf{A}(x,t)$ with entries

$$A_{mn}(x,t) = c_m c_n \mathrm{e}^{8\left(\kappa_m^3 + \kappa_n^3\right)t} \int_x^\infty \psi(s,t;\mathrm{i}\kappa_m)\,\psi(s,t;\mathrm{i}\kappa_n)\,ds.$$

Then

$$q_{+N}(x,t) = q(x,t) - 2\partial_x^2 \log\det\{I + \mathbf{A}(x,t)\} \qquad (9)$$

solves KdV with initial data associated with the scattering data $S_q \cup D$.

The striking similarity between (8) and (9) suggests that each soliton property has its positon counterpart. The main difference between the two is in-built in the profoundly different behavior of $\psi(x,t;\mathrm{i}\kappa_n)$ and $\psi(x,t,\omega_n)$: the former has finitely many zeros (n to be precise) while the latter has infinitely many zeros for any n.

Updated conservation laws

Corollary (On conservation laws)

If q also supports the left Jost solution for a.e. $\operatorname{Im} k = 0$ then the transmission coefficient $T(k)$ is well-defined and

$$T_{+N}(k) = T(k).$$

I.e., our BDT preserves both R and T and the conservation laws read

$$\int_{-\infty}^{\infty} q_{+N}(x,t)\,dx = \int_{-\infty}^{\infty} q(x,t)\,dx \text{ (momentum)},$$

$$\int_{-\infty}^{\infty} q_{+N}(x,t)^2\,dx = \int_{-\infty}^{\infty} q(x,t)^2\,dx \text{ (energy)}.$$

Note that the transmission coefficient does not tell resonances from embedded bound states.

Paring embedded bound states

Theorem

Assume basic conditions 1-3. Let D be the set of embedded bound states of $\mathbb{L}_q$ and $D_0 = \left\{\omega_n^2, 1 \leq n \leq N < \infty\right\}$ be its subset: ω_n^2 are simple and $(k - \omega_n)\,\psi\,(x, k)$ are continuous in $\operatorname{Im} k = 0$ at ω_n. If $\{\phi_n, 1 \leq n \leq N\}$ is an orthonormal set of real eigenfunctions then the set of embedded bound states of

$$q_{-N}(x) = q(x) - 2\partial_x^2 \log\det\left(\mathbf{I} - \mathbf{G}_-(x)\right),$$

where $\mathbf{G}_-$ is the Gram matrix with entries $\int_{-\infty}^{x} \phi_n(s)\,\phi_m(s)\,ds$, coincides with $D \diagdown D_0$.

As is well-known, embedded bound states are unstable and may turn into resonances under an arbitrarily small perturbation. This theorem offers an explicit perturbation that purges only targeted embedded bound states.

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$$\left|R\left(k\right)\right| = \left|\frac{m_-\left(k^2, x\right) + \overline{m_+\left(k^2, x\right)}}{m_-\left(k^2, x\right) + m_+\left(k^2, x\right)}\right|.$$

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$$|R(k)| = \left| \frac{m_-(k^2,x) + \overline{m_+(k^2,x)}}{m_-(k^2,x) + m_+(k^2,x)} \right| .$$

- Due to condition 2, it follows from basic scattering identity that

$$\begin{aligned} R(\omega_n)^{-1/2} \varphi(x,\omega_n) &= \overline{R(\omega_n)^{1/2} \psi(x,\omega_n)} + R(\omega_n)^{1/2} \psi(x,\omega_n) \\ &= 2\,\mathrm{Re}\, R(\omega_n)^{1/2} \psi(x,\omega_n) \in \mathbb{R}. \end{aligned}$$

Since $R(\omega_n)^{-1/2} \varphi(x,k)$ is a Weyl solution that has a finite boundary value at ω_n it follows from condition 1 by Lemma that

$$\phi_n(x) := R(\omega_n)^{-1/2} \varphi(x,\omega_n) = 2\,\mathrm{Re}\, R(\omega_n)^{1/2} \psi(x,\omega_n) \in L^2(-\infty) \tag{10}$$

is a real solution of $\mathbb{L}_q u = \omega_n^2 u$.

- Since $\psi(x, \omega_n) \sim e^{i\omega_n x}$ at $+\infty$ (10) also yields

$$\phi_n(x) \sim 2\cos\left(\omega_n x + \frac{1}{2}\arg R(\omega_n)\right), \quad x \to +\infty. \tag{11}$$

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- Form a new pair φ_{+N}, ψ_{+N} by taking in the gauge transformation

$$f_n(x,k) = \frac{\phi_n(x)}{k^2 - \omega_n^2}, \quad a_n(x) = \alpha_n y_n(x).$$

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- Satisflying the embedded pole condition yields

$$\mathbf{v}_{+N}(x, k) = \mathbf{v}(x, k) + \sum_{n=1}^{N} \alpha_n y_n(x) \frac{W(\mathbf{v}(x, k), \phi_n(x))}{k^2 - \omega_n^2},$$

where $\mathbf{v} := (\varphi, \psi)$ and real (y_n) solve the linear system

$$y_n(x) + \sum_{m=1}^{N} y_m(x) \underbrace{\int_{-\infty}^{x} (\alpha_m \phi_m)(\alpha_n \phi_n)}_{=: g_{mn},\ \text{Gram matrix}} = -\alpha_n \phi_n(x), \qquad (12)$$

the system (12) having a unique solution (y_n) for any real α_n and x.

- Show that $y_n \in L^2(\mathbb{R})$ by showing $y_n(x) \sim -\alpha_n \phi_n(x) \in L^2(-\infty)$ and $y_n(x) = O(1/x) \in L^2(+\infty)$.

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- The second log derivative formula follow for the Jocobi formula.

Open questions

- Pelinovsky also asked "Does the "embedded solitons" disperse away in the time evolution?". Addressing this question amounts to understanding the behavior of $\psi(x, t, \omega_n)$ in the asymptotic regime around the "positon characteristic" $x = -12\omega_n^2 t$ as $t \to \infty$. The main challenge is that $|R(\omega_n)| = 1$ and the powerful nonlinear steepest descent method due to Deift-Zhou needs a serious modification, which to the best of our knowledge is only available in the case when $|R(0)| = 1$ but less than 1 otherwise. Note that in the NLS context and by totally different from methods a treatment of the the case $|R(\omega)| = 1$ was recently offered by Budylin, AA 2020. A KdV adaptation of his techniques should yield the answer to the question if embedded solitons (bounded positons) will disperse away or not (i.e. present a KdV breather).

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- Efim Pelinovsky asks if bounded positons could be a model for rogue waves? We do hope.

Thank you!